Research Statement
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Overview

My research is situated at the interface of algebraic combinatorics, representation theory, algebraic geometry, and algebraic topology. It focuses on developing new combinatorial structures and methods for computation, particularly related to the representation theory of Lie algebras and modern Schubert calculus on flag manifolds. Thus, my work is part of the current trend in mathematics which emphasizes computation and the approach to it based on using combinatorial structures to encode complex objects and combinatorial methods to manipulate them (see, e.g., my surveys [16, 42]).

Often combinatorics plays another important role, namely it reveals interesting connections between various algebraic or geometric structures, leading to a unification of different perspectives. Thus, another goal of my research is to identify such connections. Computer experiments (based on software written by myself or my students, which is made publicly available, such as [63, 64, 65, Lub13b]) play an important role in all stages of my research: developing the combinatorial models, discovering and testing the corresponding formulas and algorithms, as well as proving the theorems.

An important part of my recent work is concentrated on the alcove model in the representation theory of symmetrizable Kac-Moody algebras, which I started in joint work with A. Postnikov [30, 32]. Among the applications of this model to which I contributed, and which revealed interesting connections between the mentioned areas, are the following:

- a uniform combinatorial model for (Kashiwara’s crystals of) irreducible integrable highest weight modules of symmetrizable Kac-Moody algebras;
- the Ram-Yip formula for Macdonald polynomials [RY11, OS17] and more efficient formulas implied by it;
- a uniform model for certain finite-dimensional representations of affine Lie algebras (namely, for tensor products of Kirillov-Reshetikhin crystals, including the corresponding energy function and combinatorial R-matrix), and the connection with Macdonald polynomials specialized at $t = 0$;
- multiplication formulas in the $K$-theory of flag varieties (of both finite and symmetrizable Kac-Moody type);
- multiplication formulas in the quantum $K$-theory of flag varieties.

Other applications to KR crystals and quantum $K$-theory were given in [JS10] and [BM11, BCMP16], respectively. Other recent applications [Mil16, MR16] involve the study of affine Deligne-Lusztig varieties and the Peterson “quantum=affine” isomorphism, relating the quantum cohomology of a flag variety to the homology of the affine Grassmannian.

Below I will give more details about this work, as well as about other work and ongoing projects at the interface of algebraic combinatorics, representation theory, Schubert calculus, and algebraic topology. In the past, I also did research in pattern recognition, mainly on mathematical aspects related to clustering and learning, plus related applications. This work, which will not be mentioned below, is contained in [51, 54, 55, 56, 57, 58, 59, 60, 61, 66], as well as other publications. Numbered citations are publications I authored or co-authored.

The work I have done since August 2004 has been partially supported by five 3-year grants from the National Science Foundation and a Simons Foundation grant; from these grants, I supported all my Ph.D. students as Research Assistants, and hired a Postdoctoral Associate. Recently, I also benefited from a one-year research grant from the Max Planck Institute in Bonn, Germany, enabling me to spend a very productive sabbatical year there. I have been the recipient of the SUNY Albany
I am working in several areas of combinatorial representation theory, as described below.

1.1. The alcove model for highest weight crystals. Crystal bases arose in the early 90s in the representation theory of quantum groups (the work of Kashiwara [Kas91] and Lusztig [Lus93]), as well as exactly solvable lattice models in statistical physics. Crystal bases can be viewed, intuitively, as bases for representations of a quantum group “at zero temperature”. In this limit of the quantum parameter \( q \to 0 \), such a basis has a simple combinatorial structure (a directed graph with colored edges), called a crystal, which captures the essential information of the representation. This information is sufficient for solving classical problems in representation theory, such as decomposing tensor products of representations and deriving branching rules.

In order to work with crystals, we need effective realizations of them. In the joint work with Postnikov [30, 32], we introduced a combinatorial model for (highest weight) crystals of semisimple Lie algebras and, more generally, of symmetrizable Kac-Moody algebras. We called this model the alcove model, because, in the case of semisimple Lie algebras, it is in terms of the corresponding affine Weyl group and the related alcove picture. Our model has advantages due to its generality, simplicity, combinatorial nature, and diverse applications (see Sections 1.2, 1.3, 2.3, and 2.4 below). In particular, the related computations are very explicit and straightforward, since they only involve enumerating certain saturated chains in the Bruhat order on the corresponding Weyl group. A reviewer of my work had the following comment about the alcove model: “This beautiful model has the potential of successfully competing with the celebrated (but notoriously complicated) Littelmann’s path model in the representation theory of semisimple Lie groups” (the Littelmann path model was developed in [Lit94, Lit95]).

In [33], I used the alcove model to give a type-independent combinatorial realization of Lusztig’s involution [Lus93] on a crystal for a semisimple Lie algebra, which exhibits it as a self-dual poset; this is the first direct generalization of the Schützenberger involution on semistandard Young tableaux [Ful97]. In [31], I present a combinatorial description of the commutor in the category of crystals due to Henriques and Kamnitzer [HK06] (a commutor is an isomorphism between the crystals \( X \otimes Y \) and \( Y \otimes X \)). In his thesis [Ada11], my student, W. Adamczak, relates the alcove model in the classical Lie types \( A − D \) to the well-known model based on fillings known as Kashiwara-Nakashima tableaux [KN94]. Other researchers have also been using the alcove model (see the Overview above and Section 1.2).

I have implemented the alcove model in Maple, and the package is available on my webpage [65]. A more recent implementation, due to Brant Jones [Jon08], is also available in the rapidly growing open-source system Sage [S+12]. A two-hour tutorial on the alcove model, which I gave at ICERM, is available in the video archive of the institute [62].

1.2. Macdonald and Hall-Littlewood polynomials. Macdonald [Mac03] defined a remarkable family of orthogonal symmetric polynomials \( P_\lambda(x; q, t) \) associated to a finite root system, which bear his name. They specialize to the Hall-Littlewood polynomials (spherical functions for a Chevalley group over a \( p \)-adic field) [Mac71] upon setting \( q = 0 \), and to the corresponding irreducible characters upon setting \( t = 0 \) too. The importance of Macdonald polynomials is due to their deep connections with: double affine Hecke algebras, affine Lie algebras, Hilbert schemes, statistical physics etc.

A combinatorial formula for the Hall-Littlewood polynomials \( P_\lambda(x; t) \) of arbitrary type was given in terms of the alcove model by Schwer [Sch06], cf. also [Ram06]. On another hand, in type \( A \), an
apparently unrelated formula for $P_\lambda(x; t)$ follows from the Haglund-Haiman-Loehr (HHL) formula for Macdonald polynomials [HHL05], being based on certain fillings of the Young diagram $\lambda$.

In [23] I show that the two formulas above are closely related. Namely, we can group the terms in Schwer’s formula into classes, such that the sum in each class is a term in the formula based on fillings. This compression phenomenon explains the way in which a certain intricate statistic on fillings in [HHL05] (called “inv”) follows naturally from more general concepts. I study the compression phenomenon for Macdonald polynomials in [29], and for Hall-Littlewood polynomials of type $B$ and $C$ in [24]; see also [27]. In the Macdonald case, I compress the recent Ram-Yip formula [RY11] and derive the HHL formula [HHL05], in terms of fillings. In types $B$ and $C$, the HHL-type formulas I derive are new; the compression is quite large, e.g. by a factor of 45 for $\lambda = (3, 2, 1, 0)$ in type $C_4$. The case of non-symmetric Macdonald polynomials is addressed in the joint work [15] with my student, K. Ramer.

The idea of compression was further developed by Gaussent-Littelmann and Klostermann (for Hall-Littlewood polynomials) in [GL11, Klo11], and by M. Yip [Yip11] (for the multiplication of Hall-Littlewood polynomials).

An older work of mine related to Macdonald polynomials involves the substitutional inverse of formal power series (also known as Lagrange inversion). In [44], I study the involution on the algebra of symmetric function defined by Macdonald [Mac95], which is based on Lagrange inversion. More precisely, I prove a positivity result related to the Schur function expansion of the images of skew Schur functions under the mentioned involution. I also give a $q$-analogue of this result, which is a special case of a conjecture in the theory of Macdonald polynomials related to the Bergeron-Garsia Nabla operator [BGHT99]. The results in my paper were generalized in [LW08].

### 1.3. Kirillov-Reshetikhin crystals, the energy function, and the quantum alcove model.

In the joint paper [18] with my former student A. Lubovsky (cf. also his thesis [Lub13a]), we define a generalization of the alcove model, which we call the quantum alcove model, as it is based on enumerating paths in the so-called quantum Bruhat graph of the corresponding finite Weyl group. This graph first appeared in connection with the quantum cohomology of flag varieties (see [FW04] and Section 2.4), and is obtained by adding extra down edges to the Hasse diagram of the Bruhat order on the Weyl group. If we restrict to paths with no down edges, we recover the classical alcove model. The main construction in [18], cf. also [14], is that of crystal operators in the quantum alcove model, both classical ones $f_i$, $i > 0$, and the affine one $f_0$. A. Lubovsky presented our work as a plenary talk at the 2012 international conference “Formal Power Series and Algebraic Combinatorics”.

My main result with S. Naito, D. Sagaki, A. Schilling, and M. Shimozono [9, 12] is that the quantum alcove model uniformly describes tensor products of column shape Kirillov-Reshetikhin (KR) crystals [KR90], for all untwisted affine types. (KR crystals are certain finite crystals for affine Lie algebras.) This is proved based on various properties of the parabolic analogue of the quantum Bruhat graph, which we study in [17]. A closely related model is based on so-called quantum Lakshmibai-Seshadri (LS) paths. Only some type-specific models existed before.

Our work highlights the connections between the objects in Figure 1. Ion [Ion03] showed that the specialized Macdonald polynomial $P_\mu(x; q, t = 0)$ is a Demazure character for an affine algebra when the root $\alpha_0$ is short. (Demazure modules $V_\mu(\Lambda)$ are submodules of highest weight ones $V(\Lambda)$ determined by a Borel subalgebra acting on an extremal weight vector.) Fourier-Littelmann [FL06] showed that the above Demazure character is the graded character $X_\mu(x; q)$ of a tensor product of one-column KR modules in simply laced types. The extension of both results to types $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ is problematic. However, a corollary of the above realization of KR crystals is that, for all untwisted affine root systems $A_{n-1}^{(1)} - G_2^{(1)}$, we have $P_\mu(x; q, 0) = X_\mu(x; q)$. In fact, the
quantum alcove model came up earlier, in [32], where we conjectured that it underlies the Chevalley multiplication formula in the quantum $K$-theory of flag varieties $G/B$, cf. Section 2.4. Beside experimental evidence, there is recent theoretical evidence for this due to Braverman-Finkelberg [BF11, BF12]. In simply laced types, they relate $P_\mu(x; q, 0)$ to the $J$-function in quantum $K$-theory (via their $q$-Whittaker functions), while, in principle, one can derive the structure constants in quantum $K$-theory from the $J$-function (although this is hard).

Two reviewers of our work pointed out the following. “It is this lack of understanding of the geometry that arguably makes the non-simply laced cases notoriously hard and, for the time being, the study of the structure of KR modules through the combinatorics of their crystal basis seems to be the only option... It is particularly significant that a number of combinatorial structures that appeared separately... are very clearly related and it is reasonable to expect that these connections will generate further research.” “This $P = X$ equality in the non simply laced cases is undoubtedly a strong result interesting for a large mathematical audience.”

I will now discuss some applications of our results. First, we define a statistic on the quantum alcove model called height, and in [9, 19] we show that it efficiently calculates the corresponding energy function (the latter comes from solvable lattice models, and it gives a natural grading on a tensor product of KR crystals [ST12]). In [21, 22] we translate the height statistic into a more transparent one on the tableau models in types $A$ and $C$ (in type $A$ we recover the Lascoux-Schützenberger charge statistic [LS79]); with my students, C. Briggs and A. Schultze, we have been addressing the type $B$ and $D$ cases. In [7] we give a uniform realization, based on some combinatorial moves called quantum Yang-Baxter moves, of the combinatorial $R$-matrix (this is the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals).

Concerning generalizations of the above results, in [8] we show that the specialization at $t = 0$ of a non-symmetric Macdonald polynomial equals the graded character of a Demazure-type submodule of the above tensor product of KR modules; we also extend our combinatorial models.

The quantum alcove and quantum LS paths models were implemented in Sage[S+12]. Using this, we carried out a computer verification in exceptional types of certain properties of the KR crystals $B^{r-1}$ [9], which were conjectured in [HKO+99], but were only known in classical types.

In [5] we extend the above work to arbitrary KR crystals, and a combinatorial interpretation of the cluster algebra relations satisfied by the KR characters is derived.
We conclude with some recent developments inspired by our work. Based on [9], Chari-Ion [CI15] showed that $P_\mu(x; q, 0)$ is essentially the character of a local Weyl module for a current algebra; they also derive a BGG reciprocity theorem for current algebras, which is relevant to the categorification of Macdonald polynomials in [Kho13]. A combinatorial realization of the crystal bases of level 0 extremal weight modules for affine algebras, and the corresponding Demazure modules, is given in [INS16] in terms of so-called semi-infinite LS paths. A crystal-theoretic interpretation of the relation between local and global Weyl modules for current algebras is given in [NS16], based on the above models. In [NNS17, FKM17] the specialization of a non-symmetric Macdonald polynomial at $t = \infty$ is expressed in terms of quantum LS paths, and a representation theoretic interpretation is given in a special case. An extension of the results in [9] to the twisted type $A^{(2)}_{2n}$ is given in [Nom16].

Finally, so-called generalized Weyl modules are defined and studied in our setup in [FM17, Nom17]; they are closely related to our Demazure-type submodules of KR modules in [8].

1.4. Crystal posets. In [11] we investigate the way in which well-known properties of the weak Bruhat order on a Weyl group can be lifted (or not) to a corresponding crystal graph, viewed as a partially ordered set (crystal poset); the latter projects to the weak order via the so-called key map [Lit94, 30]. First, a crystal theoretic analogue of the statement that any two reduced expressions for the same Coxeter group element are related by Coxeter moves is proven for all lower intervals $(\hat{0}, v)$ in a crystal graph. Second, the Möbius function of lower intervals is shown to always be 0 or ±1, and a precise description is given. Moreover, the corresponding order complex is proven always to be homotopy equivalent to a ball or to a sphere of some dimension, despite often not being shellable. The main tools used are the alcove model (see Section 1.1) and the Quillen fiber lemma. Several applications are given. All these results are shown to fail for arbitrary intervals in a crystal poset.

1.5. Kostka-Foulkes polynomials. Lusztig [Lus83] defined the Kostka-Foulkes polynomial $K_{\lambda \mu}(t)$ as a $t$-analogue of the multiplicity of a weight $\mu$ in the irreducible representation $V(\lambda)$ of highest weight $\lambda$ of a semisimple Lie algebra. This has many remarkable properties: for instance, it is a special affine Kazhdan-Lusztig polynomial. While there is a combinatorial formula for $K_{\lambda \mu}(t)$ in type $A$, finding such formulas beyond type $A$ was a long-standing problem. In [3] we give the first such formula, for $K_{\lambda \mu}(t)$ in type $C$; the special case $\mu = 0$ is, in fact, the most complex one. In [2] we study the so-called atomic decomposition of $K_{\lambda \mu}(t)$ and $t$-characters, which reveals the combinatorics of the geometric construction of representations via the Satake correspondence [MV07].

1.6. $q$-analogues of weight multiplicities for Lie superalgebras. In [28], we define and study a generalization of Lusztig’s $q$-analogue of weight multiplicities [Lus83] to the general linear Lie superalgebra $\mathfrak{gl}(n, m)$ and the orthosymplectic superalgebra $\mathfrak{osp}(2n, \mathbb{R})$. We prove the positivity property of this $q$-analogue, and exhibit a positive combinatorial formula in the case of $\mathfrak{gl}(n, m)$ by generalizing the charge statistic [LS79].

1.7. Combinatorial basis constructions for Lie algebra representations. Crystals only partially encode the structure of a representation (they encode what happens “at zero temperature”, as explained in Section 1.1). In order to completely recover the action of a Lie algebra on a basis, we need a more complex structure, known as a representation diagram; this is a graph on the same vertices as the crystal graph but containing more (colored) edges, plus complex numbers as edge labels. The theory of representation diagrams was developed in [Don03, Don97]; it is part of an emerging area known as “constructive representation theory”.

For $\mathfrak{sl}_n$, the celebrated Gelfand-Tsetlin basis [GC50] is the only known basis for which the Lie algebra action is given by explicit formulas. All the known proofs of the Gelfand-Tsetlin construction use sophisticated algebraic tools. Using the setup of representation diagrams, in the joint paper
with Hersh [26], we rederived the construction in a simple way, together with a certain minimality property of the corresponding graph.

1.8. Whittaker functions and Demazure characters. Whittaker functions on p-adic groups $G$ are a basic tool in the theory of automorphic forms, e.g., in the construction of $L$-functions. They are associated with an unramified principal series representation of $G$. The Casselman-Shalika formula expresses a spherical Whittaker function as the corresponding irreducible character times a scaling factor. I am interested in the recently discovered, deep connections of Whittaker functions to combinatorial representation theory and Schubert calculus on flag manifolds.

The Iwahori Whittaker functions generalize the spherical ones in the same way as the non-symmetric Macdonald polynomials generalize the symmetric ones. Brubaker, Bump, and Licata [BBL15] expressed the Iwahori Whittaker functions in terms of the Demazure-Lusztig operators, and observed intriguing connections with singularities of Schubert varieties. In [6] we investigate a generalization of the Casselman-Shalika formula, namely the expansion of an Iwahori Whittaker function in terms of Demazure characters. The related coefficients, which are analogues of the scaling factor in the Casselman-Shalika formula, are computed in some cases; not surprisingly (see above), these cases correspond to smooth Schubert varieties.

A surprising application of the above work involves the $K$-theory stable basis in Schubert calculus, which currently attracts considerable interest; namely, the transition matrix between the stable basis and the Schubert classes is closely related to the matrix considered above [SZZ17].

2. Modern Schubert calculus

The object of study in modern Schubert calculus is the generalized flag variety $G/B$, where $G$ is a connected, simply connected, semisimple complex Lie group, $T$ a maximal torus, $B \supseteq T$ a Borel subgroup, and $B^-$ the opposite Borel subgroup; more generally, we can consider a partial flag variety $G/P$, where $P$ is a parabolic subgroup of $G$. In type A, $SL_n/B$ is the variety $Fl_n$ of complete flags $(0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n)$ in $\mathbb{C}^n$. For each Weyl group element $w$, the subset $X_w := B^- wB/B$ is the corresponding Schubert cell, and its closure $X_w$ is the corresponding Schubert variety, of complex codimension $\ell(w)$. We denote by $\sigma_w$ the cohomology class of $X_w$, which is in $H^{2\ell(w)}(G/B)$. The collection of all these classes forms a basis of $H^*(G/B)$ (cf. [BGG73]).

The central problem in this area is understanding the multiplication of Schubert classes:

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{uv}^w \sigma_w,$$

where the (Schubert) structure constants $c_{uv}^w$ are nonnegative integers, since they count points in a suitable triple intersection of Schubert varieties. Finding a combinatorial interpretation for $c_{uv}^w$ (and, in particular, a proof of their nonnegativity which bypasses geometry) is known as the Schubert problem for the cohomology of $G/B$. The importance of this problem stems from the geometric significance of the Schubert structure constants, and from the fact that a combinatorial interpretation for them would facilitate a deeper study of their properties (such as their symmetries, vanishing etc.). The Schubert problem proved to be a very hard problem; for two-step (partial) flag varieties, one solution to it was given by Coskun in [Cos09]. This problem generalizes to $K$-theory, quantum cohomology, and $T$-equivariant cohomology/$K$-theory.

In order to address the Schubert problem, it is useful to have polynomial representatives for Schubert classes. In type A these are the Schubert polynomials (for cohomology) and Grothendieck polynomials (for $K$-theory) due to Lascoux and Schützenberger [LS82, Las90]. These polynomials are indexed by permutations in the symmetric group $S_n$. 
2.1. The cohomology of the classical flag variety. In [39], I investigate some of the connections not yet understood between several combinatorial structures for the construction of Schubert polynomials. In particular, I introduce a crystal-like structure (cf. Section 1.1) on one particular family of combinatorial objects in this area.

In [45, 46], I extend the work of Fomin and Greene on noncommutative Schur functions [FG98] (which is a useful tool for proving certain positivity results for symmetric functions) by defining noncommutative analogues of Schubert polynomials. Several applications are given.

In [40], we generalize skew Schur (symmetric) functions by defining skew Schubert polynomials. We show that our skew Schubert polynomials expand in the basis of Schubert polynomials with nonnegative integer coefficients that are precisely the type $A$ structure constants $c^w_{uv}$ in (1). The results in this paper were extended to quantum cohomology by Postnikov in [Pos05b].

The study of the action of the Steenrod algebra on the mod $p$ cohomology of spaces has many applications to the topological structure of those spaces. In [47], I present several combinatorial formulas for the action of Steenrod operations on the cohomology of Grassmannians. This paper was cited in [Woo98], which is the most comprehensive survey on the Steenrod algebra.

In [25] I propose a new approach to the Schubert problem, based on generalizing Fomin’s growth diagrams (which realize Schützenberger’s jeu de taquin [Ful97]) from chains in Young’s lattice of partitions [Sta99][Appendix 1] to chains in the Bruhat order on $S_n$. A general conjecture for the Schubert structure constants is given, and it is proved in special cases.

2.2. The $K$-theory of the classical flag variety. In [37], we give several new formulas for Grothendieck polynomials. One of our substitution formulas was used in [KY04] to attack the Schubert problem for Grothendieck polynomials.

In [34], we derive explicit formulas, with no cancellations, for expanding in the basis of Grothendieck polynomials the product of two such polynomials, one of which is indexed by an arbitrary permutation, and the other by a cycle of the form $(k-p+1, k-p+2, ..., k+1)$ or $(k+p, k+p-1, ..., k)$. These are Pieri-type formulas, generalizing my Monk-type formula in [41].

The Pieri-type formulas in [34] also generalize the corresponding formulas in the $K$-theory of Grassmannians that I previously obtained in [43]. On the other hand, the latter formulas were used in a crucial way in Buch’s work [Buc02a, Buc02b]. For instance, in [Buc02b], they are used in the proof of a combinatorial formula for multiplying any two Schubert classes in the $K$-theory of Grassmannians. Other applications of the results in [43] are given by Lam and Pylyavskyy [LP07].

2.3. Schubert calculus for generalized flag varieties. Let us now turn to the $K$-theory of generalized flag varieties, where $G$ is a complex semisimple Lie group. The Schubert classes, which form a basis of the $K$-theory of $G/B$, are the classes of structure sheaves $\mathcal{O}_w = \mathcal{O}_{X_w}$ of Schubert varieties $X_w$, for $w$ in the Weyl group.

In [32] we gave a very general Chevalley-type multiplication formula [Che94] in the $T$-equivariant $K$-theory $K_T(G/B)$, where $T$ is the corresponding torus. This is a formula for multiplying the class of $\mathcal{O}_w$ with the class of the line bundle $\mathcal{L}_\lambda$ associated to an arbitrary weight $\lambda$. Our formula is the natural generalization of my previous type $A$ formula in [41], and it is based on the alcove model (cf. Section 1.1). It is more general than the formulas in [PR99, GR04], which only work for dominant/antidominant weights.

Based on the $K$-Chevalley formula, in [36] we give a model for $K_T(G/B)$ in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. This result generalizes my earlier work [38] and belongs to the research initiated by Fomin and Kirillov [FK99] on the realization of the cohomology and $K$-theory of $G/B$ as commutative subalgebras of certain noncommutative algebras, cf. [Baz06, KM04, KM05a, KM05b].
In joint work with M. Shimozono [20], we extend the Chevalley formula in [32], as well as the formulas in terms of Lakshmibai-Seshadri (LS) paths in [PR99, GR04], from the finite case to Kac-Moody flag manifolds; in doing this, we address some gaps in [PR99, GR04].

2.4. Quantum cohomology and quantum K-theory. Let us now consider the quantum cohomology $QH^*(Fl_n)$ of the type $A$ flag variety. This is a certain quotient of $\mathbb{Z}[q, x] := \mathbb{Z}[q_1, \ldots, q_{n-1}] \otimes \mathbb{Z}[x_1, \ldots, x_n]$.

Fomin, Gelfand, and Postnikov [FGP97] defined polynomial representatives in $\mathbb{Z}[q, x]$, called quantum Schubert polynomials, for the quantum Schubert classes in $QH^*(Fl_n)$, for $w \in S_n$. They also gave a quantum Chevalley formula for multiplying an arbitrary quantum Schubert polynomial with one indexed by an adjacent transposition (simple reflection). The formula is in terms of the quantum Bruhat graph, see Section 1.3.

In [35], we extend the ideas in [FGP97] to the quantum K-theory $QK(Fl_n)$, which is a K-theory version of quantum cohomology (cf. [Lee04, GL03]). We define quantum Grothendieck polynomials by extending the quantization map approach to the construction of quantum Schubert polynomials in [FGP97]. Our quantization map is substantially more involved than its quantum cohomology counterpart; its construction is based on the presentation of $QK(Fl_n)$ stated in [KM05c], but only recently proved in [KPSZ17]. Using results in [34], we proved a “quantum K-Chevalley formula” for the quantum Grothendieck polynomials, which is the natural generalization of the corresponding formulas (mentioned above) in K-theory [32] and quantum cohomology [FGP97]; this formula is in terms of the quantum alcove model, which was described in Section 1.3. We conjecture that our quantum Grothendieck polynomials represent Schubert classes in $QK(Fl_n)$. This conjecture is supported by extensive tests performed by A. Buch [Buc10], by the result in [KPSZ17] mentioned above, as well as by the work of Braverman-Finkelberg discussed in Section 1.3.

In [32] we stated our “quantum K-Chevalley formula” in the arbitrary root system setup. This formula determines the multiplicative structure of $QK(G/B)$.

3. Combinatorics related to generalized cohomology theories

In this section, I describe older work of mine on the combinatorics of formal group laws, as well as recent work on generalized cohomology Schubert calculus.

A (one-dimensional, commutative) formal group law over a commutative ring $R$ is a formal power series $F(x, y)$ in $R[[x, y]]$ with the following properties: (1) $F(x, 0) = F(0, x) = x$; (2) $F(x, y) = F(y, x)$; (3) $F(x, F(y, z)) = F(F(x, y), z)$. By considering the pair $(R, F(x, y))$, one can define the category of such objects. The universal ring is called the Lazard ring, and this is isomorphic to $\mathbb{Z}[x_1, x_2, \ldots]$ by a celebrated theorem of Lazard [Haz78].

Formal group laws have a close connection to algebraic topology. Indeed, any complex oriented cohomology theory has a formal group law naturally associated with it. For instance, ordinary cohomology $H^*(\cdot)$ has the trivial formal group law $x + y$, K-theory has the multiplicative one $x + y - xy$, hyperbolic cohomology (which corresponds to a singular elliptic curve, and gives a stalk version of the elliptic cohomology of Ginzburg-Kapranov-Vasserot [GKV95]) has $(x + y - \mu_1 xy)/(1 + \mu_2 xy)$, and complex cobordism $MU^*(\cdot)$ has the universal formal group law.

3.1. Combinatorics of formal group laws. In [50, 52, 53], we studied the combinatorics related to the coefficients of the (iterated) universal formal group law. This work was based on the connection between formal group theory and the combinatorics related to incidence Hopf algebras [DRS72, Sch94] via umbral calculus [RT93].
In [48], I give a shorter proof of Lazard’s theorem than the classical one, based on a new approach, which involves the combinatorics of symmetric functions. I also defined and studied new polynomial generators for the Lazard ring.

Other combinatorial results related to formal group laws are contained in my paper [49]. This is devoted to the construction and study of a generalization of the necklace algebra defined by Metropolis and Rota [MR83]. The latter was introduced in order to simplify the construction of the universal ring of Witt vectors (associated with a commutative ring). My generalized necklace algebra corresponds to an arbitrary formal group law, whereas the one of Metropolis and Rota corresponds to the multiplicative formal group \( x + y - xy \). A \( q \)-deformation of the classical necklace algebra is defined by considering the formal group law \( x + y - qxy \). The constructions in [49] were rephrased in the language of profinite groups and categories in a series of papers by Y.-T. Oh [Oh05, Oh06, Oh07a, Oh07b, Oh07c, Oh08a, Oh08b].

3.2. **Schubert calculus in equivariant hyperbolic cohomology.** Modern Schubert calculus has been focusing on the cohomology and \( K \)-theory, as well as their \( T \)-equivariant versions, for generalized flag manifolds \( G/B \), where \( G \supset B \supset T \) are a connected complex semisimple Lie group, a Borel subgroup, and the corresponding torus. The main results in Schubert calculus for other cohomology theories have only been obtained recently, in [CPZ13, HHH05, HK11] etc. After the above theory has been developed, the next step is to give explicit formulas, thus generalizing well-known results in cohomology and \( K \)-theory, which are usually based on combinatorial structures. With this goal in mind, I started a long-term collaboration with Kirill Zainoulline.

In [10] we extend to equivariant hyperbolic cohomology the combinatorial formulas of Billey and Graham-Willems [Bi99, Gr02, Wil04] in equivariant cohomology and \( K \)-theory for the localization of Schubert classes at torus fixed points, which are based on the concept of a root polynomial.

The main difficulty in Schubert calculus beyond \( K \)-theory is the fact that the topologically defined cohomology classes corresponding to a Schubert variety depend on its chosen Bott-Samelson desingularization, and thus on a reduced word for the given Weyl group element (which is not the case in ordinary cohomology and \( K \)-theory). In [4, 13] we define Schubert classes independently of a reduced word in equivariant hyperbolic cohomology (for the complete flag variety \( G/B \) and the partial one \( G/P \)). Our construction is based on the Kazhdan-Lusztig basis of a corresponding Hecke algebra. In [13] we formulate two important conjectures about our Kazhdan-Lusztig (KL) Schubert classes: (1) a positivity conjecture; (2) the fact that they coincide with the topologically defined Schubert classes in the case of smooth Schubert varieties. We prove the second conjecture, called smoothness conjecture, in several cases. In [4] we also use the new geometric framework to reinterpret many classical results in Kazhdan-Lusztig theory, and also to obtain some new ones. In [1] we give a geometric interpretation of the KL Schubert classes for the type \( A \) Grassmannian, as the classes of the small resolutions of Schubert varieties constructed by Zelevinsky [BL00].

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