Research Statement

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Overview

My research is situated at the interface of algebraic combinatorics, representation theory, algebraic geometry, and algebraic topology. It focuses on developing new combinatorial structures and methods for computation, particularly related to the representation theory of Lie algebras and modern Schubert calculus on flag manifolds. Thus, my work is part of the current trend in mathematics which emphasizes computation and the approach to it based on using combinatorial structures to encode complex objects and combinatorial methods to manipulate them (see, e.g., my surveys [16, 42]).

Often combinatorics plays another important role, namely it reveals interesting connections between various algebraic or geometric structures, leading to a unification of different perspectives. Thus, another goal of my research is to identify such connections. Computer experiments (based on software written by myself or my students, which is made publicly available, such as [63, 64, 65, Lub13b]) play an important role in all stages of my research: developing the combinatorial models, discovering and testing the corresponding formulas and algorithms, as well as proving the theorems.

An important part of my recent work is concentrated on the *alcove model* in the representation theory of symmetrizable Kac-Moody algebras, which I started in joint work with A. Postnikov [30, 32]. Among the applications of this model to which I contributed, and which revealed interesting connections between the mentioned areas, are the following:

- a uniform combinatorial model for (Kashiwara's crystals of) irreducible integrable highest weight modules of symmetrizable Kac-Moody algebras;
- the Ram-Yip formula for Macdonald polynomials [RY11, OS17] and more efficient formulas implied by it;
- a uniform model for certain finite-dimensional representations of affine Lie algebras (namely, for tensor products of Kirillov-Reshetikhin crystals, including the corresponding energy function and combinatorial R-matrix), and the connection with Macdonald polynomials specialized at $t = 0$;
- multiplication formulas in the K-theory of flag varieties (of both finite and symmetrizable Kac-Moody type);
- multiplication formulas in the quantum K-theory of flag varieties.

Other applications to KR crystals and quantum K-theory were given in [JS10] and [BM11, BCMP16], respectively. Other recent applications [Mil16, MR16] involve the study of affine Deligne-Lusztig varieties and the Peterson "quantum=affine" isomorphism, relating the quantum cohomology of a flag variety to the homology of the affine Grassmannian.

Below I will give more details about this work, as well as about other work and ongoing projects at the interface of algebraic combinatorics, representation theory, Schubert calculus, and algebraic topology. In the past, I also did research in pattern recognition, mainly on mathematical aspects related to clustering and learning, plus related applications. This work, which will not be mentioned below, is contained in [51, 54, 55, 56, 57, 58, 59, 60, 61, 66], as well as other publications. Numbered citations are publications I authored or co-authored.

The work I have done since August 2004 has been partially supported by five 3-year grants from the National Science Foundation and a Simons Foundation grant; from these grants, I supported all my Ph.D. students as Research Assistants, and hired a Postdoctoral Associate. Recently, I also benefited from a one-year research grant from the Max Planck Institute in Bonn, Germany, enabling me to spend a very productive sabbatical year there. I have been the recipient of the SUNY Albany

President's Excellence Award for Research and Creative Activities, and of the SUNY Chancellor's Award for Excellence in Scholarship and Creative Activities.

1. Combinatorial representation theory

I am working in several areas of combinatorial representation theory, as described below.

1.1. The alcove model for highest weight crystals.

Crystal bases arose in the early 90s in the representation theory of quantum groups (the work of Kashiwara [Kas91] and Lusztig [Lus93]), as well as exactly solvable lattice models in statistical physics. Crystal bases can be viewed, intuitively, as bases for representations of a quantum group "at zero temperature". In this limit of the quantum parameter $q \to 0$, such a basis has a simple combinatorial structure (a directed graph with colored edges), called a crystal, yet which captures the essential information of the representation. This information is sufficient for solving classical problems in representation theory, such as decomposing tensor products of representations and deriving branching rules.

In order to work with crystals, we need effective realizations of them. In the joint work with Postnikov [30, 32], we introduced a combinatorial model for (highest weight) crystals of semisimple Lie algebras and, more generally, of symmetrizable Kac-Moody algebras. We called this model the *alcove model*, because, in the case of semisimple Lie algebras, it is in terms of the corresponding affine Weyl group and the related alcove picture. Our model has advantages due to its generality, simplicity, combinatorial nature, and diverse applications (see Sections 1.2, 1.3, 2.3, and 2.4 below). In particular, the related computations are very explicit and straightforward, since they only involve enumerating certain saturated chains in the Bruhat order on the corresponding Weyl group. A reviewer of my work had the following comment about the alcove model: "This beautiful model has the potential of successfully competing with the celebrated (but notoriously complicated) Littelmann's path model in the representation theory of semisimple Lie groups" (the Littelmann path model was developed in [Lit94, Lit95]).

In [33], I used the alcove model to give a type-independent combinatorial realization of *Lusztig's involution* [Lus93] on a crystal for a semisimple Lie algebra, which exhibits it as a self-dual poset; this is the first direct generalization of the *Schützenberger involution* on *semistandard Young tableaux* [Ful97]. In [31], I present a combinatorial description of the *commutor* in the category of crystals due to Henriques and Kamnitzer [HK06] (a commutor is an isomorphism between the crystals $X \otimes Y$ and $Y \otimes X$). In his thesis [Ada11], my student, W. Adamczak, relates the alcove model in the classical Lie types $A-D$ to the well-known model based on fillings known as *Kashiwara-Nakashima tableaux* [KN94]. Other researchers have also been using the alcove model (see the Overview above and Section 1.2).

I have implemented the alcove model in `Maple`, and the package is available on my webpage [65]. A more recent implementation, due to Brant Jones [Jon08], is also available in the rapidly growing open-source system `Sage` [S^+12]. A two-hour tutorial on the alcove model, which I gave at ICERM, is available in the video archive of the institute [62].

1.2. Macdonald and Hall-Littlewood polynomials.

Macdonald [Mac03] defined a remarkable family of orthogonal symmetric polynomials $P_\lambda(x;q,t)$ associated to a finite root system, which bear his name. They specialize to the *Hall-Littlewood polynomials* (spherical functions for a Chevalley group over a p-adic field) [Mac71] upon setting $q=0$, and to the corresponding irreducible characters upon setting $t=0$ too. The importance of Macdonald polynomials is due to their deep connections with: double affine Hecke algebras, affine Lie algebras, Hilbert schemes, statistical physics etc.

A combinatorial formula for the Hall-Littlewood polynomials $P_\lambda(x;t)$ of arbitrary type was given in terms of the alcove model by Schwer [Sch06], cf. also [Ram06]. On another hand, in type A, an apparently unrelated formula for $P_\lambda(x;t)$ follows from the Haglund-Haiman-Loehr (HHL) formula for Macdonald polynomials [HHL05], being based on certain fillings of the Young diagram λ.

In [23] I show that the two formulas above are closely related. Namely, we can group the terms in Schwer's formula into classes, such that the sum in each class is a term in the formula based on fillings. This compression phenomenon explains the way in which a certain intricate statistic on fillings in [HHL05] (called "inv") follows naturally from more general concepts. I study the compression phenomenon for Macdonald polynomials in [29], and for Hall-Littlewood polynomials of type B and C in [24]; see also [27]. In the Macdonald case, I compress the recent Ram-Yip formula [RY11] and derive the HHL formula [HHL05], in terms of fillings. In types B and C, the HHL-type formulas I derive are new; the compression is quite large, e.g. by a factor of 45 for $\lambda = (3,2,1,0)$ in type C_4. The case of non-symmetric Macdonald polynomials is addressed in the joint work [15] with my student, K. Ramer.

The idea of compression was further developed by Gaussent-Littelmann and Klostermann (for Hall-Littlewood polynomials) in [GL11, Klo11], and by M. Yip [Yip11] (for the multiplication of Hall-Littlewood polynomials).

An older work of mine related to Macdonald polynomials involves the substitutional inverse of formal power series (also known as *Lagrange inversion*). In [44], I study the involution on the algebra of symmetric function defined by Macdonald [Mac95], which is based on Lagrange inversion. More precisely, I prove a positivity result related to the Schur function expansion of the images of skew Schur functions under the mentioned involution. I also give a q-analogue of this result, which is a special case of a conjecture in the theory of Macdonald polynomials related to the Bergeron-Garsia Nabla operator [BGHT99]. The results in my paper were generalized in [LW08].

1.3. **Kirillov-Reshetikhin crystals, the energy function, and the quantum alcove model.** In the joint paper [18] with my former student A. Lubovsky (cf. also his thesis [Lub13a]), we define a generalization of the alcove model, which we call the *quantum alcove model*, as it is based on enumerating paths in the so-called *quantum Bruhat graph* of the corresponding finite Weyl group. This graph first appeared in connection with the quantum cohomology of flag varieties (see [FW04] and Section 2.4), and is obtained by adding extra down edges to the Hasse diagram of the Bruhat order on the Weyl group. If we restrict to paths with no down edges, we recover the classical alcove model. The main construction in [18], cf. also [14], is that of crystal operators in the quantum alcove model, both classical ones f_i, $i > 0$, and the affine one f_0. A. Lubovsky presented our work as a plenary talk at the 2012 international conference "Formal Power Series and Algebraic Combinatorics".

My main result with S. Naito, D. Sagaki, A. Schilling, and M. Shimozono [9, 12] is that the quantum alcove model uniformly describes tensor products of column shape *Kirillov-Reshetikhin (KR) crystals* [KR90], for all untwisted affine types. (KR crystals are certain finite crystals for affine Lie algebras.) This is proved based on various properties of the parabolic analogue of the quantum Bruhat graph, which we study in [17]. A closely related model is based on so-called *quantum Lakshmibai-Seshadri (LS) paths.* Only some type-specific models existed before.

Our work highlights the connections between the objects in Figure 1. Ion [Ion03] showed that the specialized Macdonald polynomial $P_\mu(x;q,t=0)$ is a *Demazure character* for an affine algebra when the root α_0 is short. (Demazure modules $V_w(\Lambda)$ are submodules of highest weight ones $V(\Lambda)$ determined by a Borel subalgebra acting on an extremal weight vector.) Fourier-Littelmann [FL06] showed that the above Demazure character is the graded character $X_\mu(x;q)$ of a tensor product of one-column KR modules in simply laced types. The extension of both results to types $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ is problematic. However, a corollary of the above realization of KR crystals is that, for all untwisted affine root systems $A_{n-1}^{(1)}-G_2^{(1)}$, we have $P_\mu(x;q,0) = X_\mu(x;q)$. In fact, the

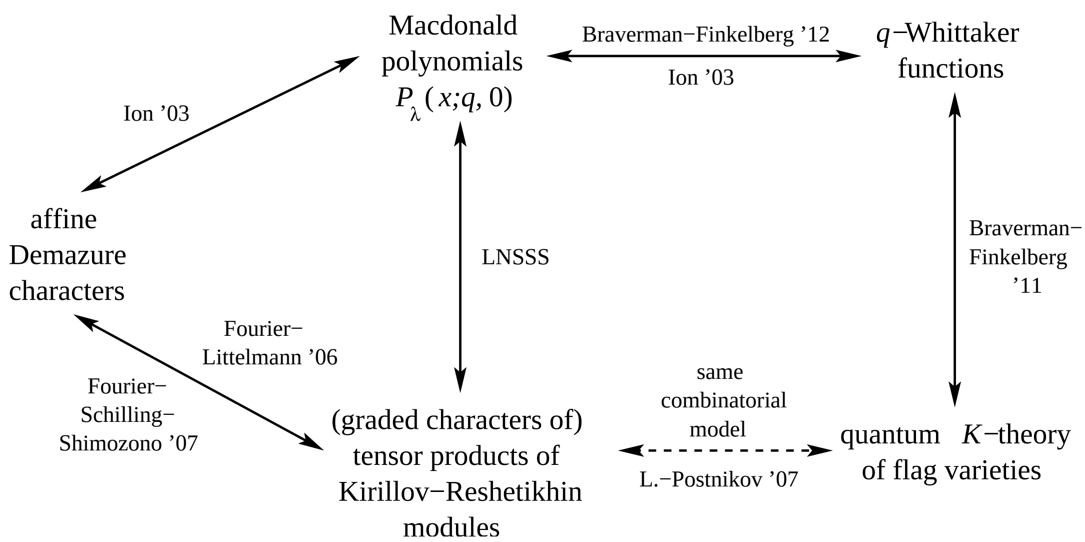

FIGURE 1

quantum alcove model came up earlier, in [32], where we conjectured that it underlies the Chevalley multiplication formula in the quantum K-theory of flag varieties G/B, cf. Section 2.4. Beside experimental evidence, there is recent theoretical evidence for this due to Braverman-Finkelberg [BF11, BF12]. In simply laced types, they relate $P_\mu(x;q,0)$ to the *J-function* in quantum K-theory (via their q-Whittaker functions), while, in principle, one can derive the structure constants in quantum K-theory from the J-function (although this is hard).

Two reviewers of our work pointed out the following. "It is this lack of understanding of the geometry that arguably makes the non-simply laced cases notoriously hard and, for the time being, the study of the structure of KR modules through the combinatorics of their crystal basis seems to be the only option... It is particularly significant that a number of combinatorial structures that appeared separately... are very clearly related and it is reasonable to expect that these connections will generate further research." "This $P = X$ equality in the non simply laced cases is undoubtedly a strong result interesting for a large mathematical audience."

I will now discuss some applications of our results. First, we define a statistic on the quantum alcove model called *height*, and in [9, 19] we show that it efficiently calculates the corresponding *energy function* (the latter comes from solvable lattice models, and it gives a natural grading on a tensor product of KR crystals [ST12]). In [21, 22] we translate the height statistic into a more transparent one on the tableau models in types A and C (in type A we recover the Lascoux-Schützenberger *charge* statistic [LS79]); with my students, C. Briggs and A. Schultze, we have been addressing the type B and D cases. In [7] we give a uniform realization, based on some combinatorial moves called *quantum Yang-Baxter moves*, of the *combinatorial R-matrix* (this is the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals).

Concerning generalizations of the above results, in [8] we show that the specialization at $t = 0$ of a *non-symmetric* Macdonald polynomial equals the graded character of a Demazure-type submodule of the above tensor product of KR modules; we also extend our combinatorial models.

The quantum alcove and quantum LS paths models were implemented in `Sage`[S$^+$12]. Using this, we carried out a computer verification in exceptional types of certain properties of the KR crystals $B^{r,1}$ [9], which were conjectured in [HKO$^+$99], but were only known in classical types.

In [5] we extend the above work to arbitrary KR crystals, and a combinatorial interpretation of the *cluster algebra* relations satisfied by the KR characters is derived.

We conclude with some recent developments inspired by our work. Based on [9], Chari-Ion [CI15] showed that $P_\mu(x;q,0)$ is essentially the character of a *local Weyl module* for a *current algebra*; they also derive a *BGG reciprocity theorem* for current algebras, which is relevant to the *categorification* of Macdonald polynomials in [Kho13]. A combinatorial realization of the crystal bases of *level* 0 *extremal weight modules* for affine algebras, and the corresponding Demazure modules, is given in [INS16] in terms of so-called *semi-infinite LS paths*. A crystal-theoretic interpretation of the relation between local and global Weyl modules for current algebras is given in [NS16], based on the above models. In [NNS17, FKM17] the specialization of a non-symmetric Macdonald polynomial at $t=\infty$ is expressed in terms of quantum LS paths, and a representation theoretic interpretation is given in a special case. An extension of the results in [9] to the twisted type $A_{2n}^{(2)}$ is given in [Nom16]. Finally, so-called *generalized Weyl modules* are defined and studied in our setup in [FM17, Nom17]; they are closely related to our Demazure-type submodules of KR modules in [8].

1.4. **Crystal posets.** In [11] we investigate the way in which well-known properties of the weak Bruhat order on a Weyl group can be lifted (or not) to a corresponding crystal graph, viewed as a partially ordered set (*crystal poset*); the latter projects to the weak order via the so-called *key map* [Lit94, 30]. First, a crystal theoretic analogue of the statement that any two reduced expressions for the same Coxeter group element are related by Coxeter moves is proven for all lower intervals $(\hat{0},v)$ in a crystal graph. Second, the Möbius function of lower intervals is shown to always be 0 or ± 1, and a precise description is given. Moreover, the corresponding order complex is proven always to be homotopy equivalent to a ball or to a sphere of some dimension, despite often not being shellable. The main tools used are the alcove model (see Section 1.1) and the *Quillen fiber lemma*. Several applications are given. All these results are shown to fail for arbitrary intervals in a crystal poset.

1.5. **Kostka-Foulkes polynomials.** Lusztig [Lus83] defined the *Kostka-Foulkes polynomial* $K_{\lambda\mu}(t)$ as a t-analogue of the multiplicity of a weight μ in the irreducible representation $V(\lambda)$ of highest weight λ of a semisimple Lie algebra. This has many remarkable properties: for instance, it is a special *affine Kazhdan-Lusztig polynomial*. While there is a combinatorial formula for $K_{\lambda\mu}(t)$ in type A, finding such formulas beyond type A was a long-standing problem. In [3] we give the first such formula, for $K_{\lambda,0}(t)$ in type C; the special case $\mu=0$ is, in fact, the most complex one. In [2] we study the so-called atomic decomposition of $K_{\lambda\mu}(t)$ and t-characters, which reveals the combinatorics of the geometric construction of representations via the Satake correspondence [MV07].

1.6. **q-analogues of weight multiplicities for Lie superalgebras.** In [28], we define and study a generalization of Lusztig's q-analogue of weight multiplicities [Lus83] to the general linear Lie superalgebra $\mathfrak{gl}(n,m)$ and the orthosymplectic superalgebra $\mathfrak{spo}(2n,M)$. We prove the positivity property of this q-analogue, and exhibit a positive combinatorial formula in the case of $\mathfrak{gl}(n,m)$ by generalizing the *charge* statistic [LS79].

1.7. **Combinatorial basis constructions for Lie algebra representations.** Crystals only partially encode the structure of a representation (they encode what happens "at zero temperature", as explained in Section 1.1). In order to completely recover the action of a Lie algebra on a basis, we need a more complex structure, known as a representation diagram; this is a graph on the same vertices as the crystal graph but containing more (colored) edges, plus complex numbers as edge labels. The theory of representation diagrams was developed in [Don03, Don97]; it is part of an emerging area known as "constructive representation theory".

For $\mathfrak{sl}_n$, the celebrated *Gelfand-Tsetlin basis* [GC50] is the only known basis for which the Lie algebra action is given by explicit formulas. All the known proofs of the Gelfand-Tsetlin construction use sophisticated algebraic tools. Using the setup of representation diagrams, in the joint paper with Hersh [26], we rederived the construction in a simple way, together with a certain minimality property of the corresponding graph.

1.8. **Whittaker functions and Demazure characters.** *Whittaker functions* on p-adic groups G are a basic tool in the theory of automorphic forms, e.g., in the construction of L-functions. They are associated with an unramified principal series representation of G. The *Casselman-Shalika formula* expresses a spherical Whittaker function as the corresponding irreducible character times a scaling factor. I am interested in the recently discovered, deep connections of Whittaker functions to combinatorial representation theory and Schubert calculus on flag manifolds.

The *Iwahori Whittaker functions* generalize the spherical ones in the same way as the non-symmetric Macdonald polynomials generalize the symmetric ones. Brubaker, Bump, and Licata [BBL15] expressed the Iwahori Whittaker functions in terms of the *Demazure-Lusztig operators*, and observed intriguing connections with singularities of Schubert varieties. In [6] we investigate a generalization of the Casselman-Shalika formula, namely the expansion of an Iwahori Whittaker function in terms of Demazure characters. The related coefficients, which are analogues of the scaling factor in the Casselman-Shalika formula, are computed in some cases; not surprisingly (see above), these cases correspond to smooth Schubert varieties.

A surprising application of the above work involves the K-theory *stable basis* in Schubert calculus, which currently attracts considerable interest; namely, the transition matrix between the stable basis and the Schubert classes is closely related to the matrix considered above [SZZ17].

2. Modern Schubert calculus

The object of study in modern Schubert calculus is the generalized flag variety G/B, where G is a connected, simply connected, semisimple complex Lie group, T a maximal torus, $B \supseteq T$ a Borel subgroup, and B^- the opposite Borel subgroup; more generally, we can consider a partial flag variety G/P, where P is a parabolic subgroup of G. In type A, SL_n/B is the variety Fl_n of complete flags $(0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n)$ in $\mathbb{C}^n$. For each Weyl group element w, the subset $X_w^\circ := B^- wB/B$ is the corresponding Schubert cell, and its closure X_w is the corresponding *Schubert variety*, of complex codimension $\ell(w)$. We denote by σ_w the cohomology class of X_w, which is in $H^{2\ell(w)}(G/B)$. The collection of all these classes forms a basis of $H^*(G/B)$ (cf. [BGG73]).

The central problem in this area is understanding the multiplication of Schubert classes:

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{uv}^w \sigma_w \,, \tag{1}$$

where the (Schubert) structure constants c_{uv}^w are nonnegative integers, since they count points in a suitable triple intersection of Schubert varieties. Finding a combinatorial interpretation for c_{uv}^w (and, in particular, a proof of their nonnegativity which bypasses geometry) is known as the *Schubert problem* for the cohomology of G/B. The importance of this problem stems from the geometric significance of the Schubert structure constants, and from the fact that a combinatorial interpretation for them would facilitate a deeper study of their properties (such as their symmetries, vanishing etc.). The Schubert problem proved to be a very hard problem; for two-step (partial) flag varieties, one solution to it was given by Coskun in [Cos09]. This problem generalizes to K-theory, quantum cohomology, and T-equivariant cohomology/K-theory.

In order to address the Schubert problem, it is useful to have polynomial representatives for Schubert classes. In type A these are the *Schubert polynomials* (for cohomology) and *Grothendieck polynomials* (for K-theory) due to Lascoux and Schützenberger [LS82, Las90]. These polynomials are indexed by permutations in the symmetric group S_n.

2.1. **The cohomology of the classical flag variety.** In [39], I investigate some of the connections not yet understood between several combinatorial structures for the construction of Schubert polynomials. In particular, I introduce a crystal-like structure (cf. Section 1.1) on one particular family of combinatorial objects in this area.

In [45, 46], I extend the work of Fomin and Greene on noncommutative Schur functions [FG98] (which is a useful tool for proving certain positivity results for symmetric functions) by defining noncommutative analogues of Schubert polynomials. Several applications are given.

In [40], we generalize skew Schur (symmetric) functions by defining skew Schubert polynomials. We show that our skew Schubert polynomials expand in the basis of Schubert polynomials with nonnegative integer coefficients that are precisely the type A structure constants c_{uv}^{w} in (1). The results in this paper were extended to quantum cohomology by Postnikov in [Pos05b].

The study of the action of the *Steenrod algebra* on the mod p cohomology of spaces has many applications to the topological structure of those spaces. In [47], I present several combinatorial formulas for the action of Steenrod operations on the cohomology of Grassmannians. This paper was cited in [Woo98], which is the most comprehensive survey on the Steenrod algebra.

In [25] I propose a new approach to the Schubert problem, based on generalizing Fomin's *growth diagrams* (which realize Schützenberger's jeu de taquin [Ful97]) from chains in Young's lattice of partitions [Sta99][Appendix 1] to chains in the Bruhat order on S_n. A general conjecture for the Schubert structure constants is given, and it is proved in special cases.

2.2. **The K-theory of the classical flag variety.** In [37], we give several new formulas for Grothendieck polynomials. One of our substitution formulas was used in [KY04] to attack the Schubert problem for Grothendieck polynomials.

In [34], we derive explicit formulas, with no cancellations, for expanding in the basis of Grothendieck polynomials the product of two such polynomials, one of which is indexed by an arbitrary permutation, and the other by a cycle of the form $(k-p+1, k-p+2, ..., k+1)$ or $(k+p, k+p-1, ..., k)$. These are Pieri-type formulas, generalizing my Monk-type formula in [41].

The Pieri-type formulas in [34] also generalize the corresponding formulas in the K-theory of Grassmannians that I previously obtained in [43]. On the other hand, the latter formulas were used in a crucial way in Buch's work [Buc02a, Buc02b]. For instance, in [Buc02b], they are used in the proof of a combinatorial formula for multiplying any two Schubert classes in the K-theory of Grassmannians. Other applications of the results in [43] are given by Lam and Pylyavskyy [LP07].

2.3. **Schubert calculus for generalized flag varieties.** Let us now turn to the K-theory of generalized flag varieties, where G is a complex semisimple Lie group. The Schubert classes, which form a basis of the K-theory of G/B, are the classes of structure sheaves $\mathcal{O}_w = \mathcal{O}_{X_w}$ of Schubert varieties X_w, for w in the Weyl group.

In [32] we gave a very general Chevalley-type multiplication formula [Che94] in the T-equivariant K-theory $K_T(G/B)$, where T is the corresponding torus. This is a formula for multiplying the class of $\mathcal{O}_w$ with the class of the line bundle $\mathcal{L}_\lambda$ associated to an arbitrary weight λ. Our formula is the natural generalization of my previous type A formula in [41], and it is based on the alcove model (cf. Section 1.1). It is more general than the formulas in [PR99, GR04], which only work for dominant/antidominant weights.

Based on the K-Chevalley formula, in [36] we give a model for $K_T(G/B)$ in terms of a certain braided Hopf algebra called the *Nichols-Woronowicz algebra*. This result generalizes my earlier work [38] and belongs to the research initiated by Fomin and Kirillov [FK99] on the realization of the cohomology and K-theory of G/B as commutative subalgebras of certain noncommutative algebras, cf. [Baz06, KM04, KM05a, KM05b].

In joint work with M. Shimozono [20], we extend the Chevalley formula in [32], as well as the formulas in terms of *Lakshmibai-Seshadri (LS) paths* in [PR99, GR04], from the finite case to Kac-Moody flag manifolds; in doing this, we address some gaps in [PR99, GR04].

2.4. **Quantum cohomology and quantum K-theory.** Let us now consider the quantum cohomology $QH^*(Fl_n)$ of the type A flag variety. This is a certain quotient of

$$\mathbb{Z}[q,x] := \mathbb{Z}[q_1,\ldots,q_{n-1}] \otimes \mathbb{Z}[x_1,\ldots,x_n]\,.$$

Fomin, Gelfand, and Postnikov [FGP97] defined polynomial representatives in $\mathbb{Z}[q,x]$, called *quantum Schubert polynomials*, for the quantum Schubert classes in $QH^*(Fl_n)$, for $w \in S_n$. They also gave a quantum Chevalley formula for multiplying an arbitrary quantum Schubert polynomial with one indexed by an adjacent transposition (simple reflection). The formula is in terms of the quantum Bruhat graph, see Section 1.3.

In [35], we extend the ideas in [FGP97] to the quantum K-theory $QK(Fl_n)$, which is a K-theory version of quantum cohomology (cf. [Lee04, GL03]). We define *quantum Grothendieck polynomials* by extending the quantization map approach to the construction of quantum Schubert polynomials in [FGP97]. Our quantization map is substantially more involved than its quantum cohomology counterpart; its construction is based on the presentation of $QK(Fl_n)$ stated in [KM05c], but only recently proved in [KPSZ17]. Using results in [34], we proved a "quantum K-Chevalley formula" for the quantum Grothendieck polynomials, which is the natural generalization of the corresponding formulas (mentioned above) in K-theory [32] and quantum cohomology [FGP97]; this formula is in terms of the quantum alcove model, which was described in Section 1.3. We conjecture that our quantum Grothendieck polynomials represent Schubert classes in $QK(Fl_n)$. This conjecture is supported by extensive tests performed by A. Buch [Buc10], by the result in [KPSZ17] mentioned above, as well as by the work of Braverman-Finkelberg discussed in Section 1.3.

In [32] we stated our "quantum K-Chevalley formula" in the arbitrary root system setup. This formula determines the multiplicative structure of $QK(G/B)$.

3. Combinatorics related to generalized cohomology theories

In this section, I describe older work of mine on the combinatorics of formal group laws, as well as recent work on generalized cohomology Schubert calculus.

A (one-dimensional, commutative) *formal group law* over a commutative ring R is a formal power series $F(x,y)$ in $R[[x,y]]$ with the following properties: (1) $F(x,0) = F(0,x) = x$; (2) $F(x,y) = F(y,x)$; (3) $F(x,F(y,z)) = F(F(x,y),z)$. By considering the pair $(R,F(x,y))$, one can define the category of such objects. The universal ring is called the *Lazard ring*, and this is isomorphic to $\mathbb{Z}[x_1,x_2,\ldots]$ by a celebrated theorem of Lazard [Haz78].

Formal group laws have a close connection to algebraic topology. Indeed, any complex oriented cohomology theory has a formal group law naturally associated with it. For instance, ordinary cohomology $H^*(\cdot)$ has the trivial formal group law $x+y$, K-theory has the multiplicative one $x+y-xy$, hyperbolic cohomology (which corresponds to a singular elliptic curve, and gives a stalk version of the elliptic cohomology of Ginzburg-Kapranov-Vasserot [GKV95]) has $(x+y-\mu_1 xy)/(1+\mu_2 xy)$, and complex cobordism $MU^*(\cdot)$ has the universal formal group law.

3.1. **Combinatorics of formal group laws.** In [50, 52, 53], we studied the combinatorics related to the coefficients of the (iterated) universal formal group law. This work was based on the connection between formal group theory and the combinatorics related to *incidence Hopf algebras* [DRS72, Sch94] via *umbral calculus* [RT93].

In [48], I give a shorter proof of Lazard's theorem than the classical one, based on a new approach, which involves the combinatorics of symmetric functions. I also defined and studied new polynomial generators for the Lazard ring.

Other combinatorial results related to formal group laws are contained in my paper [49]. This is devoted to the construction and study of a generalization of the *necklace algebra* defined by Metropolis and Rota [MR83]. The latter was introduced in order to simplify the construction of the universal ring of *Witt vectors* (associated with a commutative ring). My generalized necklace algebra corresponds to an arbitrary formal group law, whereas the one of Metropolis and Rota corresponds to the multiplicative formal group $x+y-xy$. A q-deformation of the classical necklace algebra is defined by considering the formal group law $x+y-qxy$. The constructions in [49] were rephrased in the language of profinite groups and categories in a series of papers by Y.-T. Oh [Oh05, Oh06, Oh07a, Oh07b, Oh07c, Oh08a, Oh08b].

3.2. **Schubert calculus in equivariant hyperbolic cohomology.** Modern Schubert calculus has been focusing on the cohomology and K-theory, as well as their T-equivariant versions, for generalized flag manifolds G/B, where $G \supset B \supset T$ are a connected complex semisimple Lie group, a Borel subgroup, and the corresponding torus. The main results in Schubert calculus for other cohomology theories have only been obtained recently, in [CPZ13, HHH05, HK11] etc. After the above theory has been developed, the next step is to give explicit formulas, thus generalizing well-known results in cohomology and K-theory, which are usually based on combinatorial structures. With this goal in mind, I started a long-term collaboration with Kirill Zainoulline.

In [10] we extend to equivariant hyperbolic cohomology the combinatorial formulas of Billey and Graham-Willems [Bi99, Gr02, Wil04] in equivariant cohomology and K-theory for the localization of Schubert classes at torus fixed points, which are based on the concept of a *root polynomial*.

The main difficulty in Schubert calculus beyond K-theory is the fact that the topologically defined cohomology classes corresponding to a Schubert variety depend on its chosen Bott-Samelson desingularization, and thus on a reduced word for the given Weyl group element (which is not the case in ordinary cohomology and K-theory). In [4, 13] we define Schubert classes independently of a reduced word in equivariant hyperbolic cohomology (for the complete flag variety G/B and the partial one G/P). Our construction is based on the *Kazhdan-Lusztig basis* of a corresponding Hecke algebra. In [13] we formulate two important conjectures about our *Kazhdan-Lusztig (KL) Schubert classes*: (1) a positivity conjecture; (2) the fact that they coincide with the topologically defined Schubert classes in the case of smooth Schubert varieties. We prove the second conjecture, called smoothness conjecture, in several cases. In [4] we also use the new geometric framework to reinterpret many classical results in Kazhdan-Lusztig theory, and also to obtain some new ones. In [1] we give a geometric interpretation of the KL Schubert classes for the type A Grassmannian, as the classes of the small resolutions of Schubert varieties constructed by Zelevinsky [BL00].

References

[1] C. Lenart, K. Zainoulline, and C. Zhong. A geometric interpretation of the Kazhdan-Lusztig Schubert classes for Grassmannians. In preparation, 2019.

[2] C. Lecouvey and C. Lenart. Atomic decomposition of characters and crystals. `arXiv:1809.01262`, 2018.

[3] C. Lenart and C. Lecouvey. Combinatorics of generalized exponents. *Int. Math. Res. Not.*, 2018, `DOI 10.1093/imrn/rny157`.

[4] C. Lenart, K. Zainoulline, and C. Zhong. Parabolic Kazhdan-Lusztig basis, Schubert classes and equivariant oriented cohomology. *J. Inst. Math. Jussieu*, 2019, `DOI 10.1017/s1474748018000592`.

[5] C. Lenart and T. Scrimshaw. On higher level Kirillov–Reshetikhin crystals, Demazure crystals, and related uniform models. *J. Algebra*, 539:285–304, 2019.

[6] K.-H. Lee, C. Lenart, and D. Liu. Whittaker functions and Demazure characters (Appendix by D. Muthiah and A. Puskás). *J. Inst. Math. Jussieu*, 18:759–781, 2019.

[7] C. Lenart and A. Lubovsky. A uniform realization of the combinatorial R-matrix. *Adv. Math.*, 334:151–183, 2018.

[8] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals III: Nonsymmetric Macdonald polynomials at $t = 0$ and Demazure characters. *Transform. Groups*, 22:1041–1079, 2017.

[9] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals II: Alcove model, path model, and $P = X$. *Int. Math. Res. Not.*, no. 14, 4259–4319, 2017.

[10] C. Lenart and K. Zainoulline. Towards generalized cohomology Schubert calculus via formal root polynomials. *Math. Res. Lett.*, 24:839–877, 2017.

[11] P. Hersh and C. Lenart. From the weak Bruhat order to crystal posets. *Math. Z.*, 286:1435–1464, 2017.

[12] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. Affine crystals, Macdonald polynomials, and combinatorial models. Extended abstract. *Rev. Roumaine Math. Pures Appl.*, 62:113–135, 2017.

[13] C. Lenart and K. Zainoulline. A Schubert basis in equivariant elliptic cohomology. *New York J. Math.*, 23:711–737, 2017.

[14] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. Quantum Lakshmibai-Seshadri paths and root operators. In *Proceedings of the 5th Mathematical Society of Japan Seasonal Institute. Schubert Calculus (Osaka, Japan, 2012)*, volume 71 of *Advanced Studies in Pure Mathematics*, pages 267–294, 2016.

[15] C. Lenart and K. Ramer. On combinatorial formulas for non-symmetric Macdonald polynomials. `www.albany.edu/~lenart`, 2016.

[16] C. Lenart. Combinatorial representation theory of Lie algebras. Richard Stanley's work and the way it was continued. In *The Mathematical Legacy of Richard Stanley*, edited by P. Hersh, T. Lam, P. Pylyavskyy, and V. Reiner, 263–277, Amer. Math. Soc., Providence, RI, 2016.

[17] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals I: Lifting the parabolic quantum Bruhat graph. *Int. Math. Res. Not.*, no. 7, 1848–1901, 2015.

[18] C. Lenart and A. Lubovsky. A generalization of the alcove model and its applications. *J. Algebraic Combin.*, 41:751–783, 2015.

[19] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. Explicit description of the degree function in terms of quantum Lakshmibai-Seshadri paths. In *Proceedings of Tsukuba Workshop on Infinite Dimensional Lie Theory and Related Topics – History and Development, 2014, University of Tsukuba, Japan*, volume 37 of *Toyama Math. J.*, pages 107–130, 2015.

[20] C. Lenart and M. Shimozono, Equivariant K-Chevalley rules for Kac-Moody flag manifolds. *Amer. J. Math.*, 136:1175–1213, 2014.

[21] C. Lenart and A. Schilling. Crystal energy via the charge in types A and C. *Math. Z.*, 273:401–426, 2013.

[22] C. Lenart. From Macdonald polynomials to a charge statistic beyond type A. *J. Combin. Theory Ser. A*, 119:683–712, 2012.

[23] C. Lenart. Hall-Littlewood polynomials, alcove walks, and fillings of Young diagrams (Appendix with A. Lubovsky). *Discrete Math.*, 311:258–275, 2011.

[24] C. Lenart. Haglund-Haiman-Loehr type formulas for Hall-Littlewood polynomials of type B and C. *Algebra Number Theory*, 4:887–917, 2010.

[25] C. Lenart. Growth diagrams for the Schubert multiplication. *J. Combin. Theory Ser. A*, 117:842–856, 2010.

[26] P. Hersh and C. Lenart. Combinatorial constructions of weight bases. The Gelfand-Tsetlin basis. *Electron. J. Combin.*, 17, #R33, 2010.

[27] C. Lenart. Combinatorial formulas for Macdonald and Hall-Littlewood polynomials. 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), 549–560, *Discrete Math. Theor. Comput. Sci. Proc.*, AK, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.

[28] C. Lecouvey and C. Lenart. On q-analogs of weight multiplicities for the Lie superalgebras $\mathfrak{gl}(n, m)$ and $\mathfrak{spo}(2n, M)$. *J. Algebraic Combin.*, 30:141–163, 2009.

[29] C. Lenart. On combinatorial formulas for Macdonald polynomials. *Adv. Math.*, 220:324–340, 2009.

[30] C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.

[31] C. Lenart. On the combinatorics of crystal graphs. II. The crystal commutor. *Proc. Amer. Math. Soc.*, 136:825–837, 2008.

[32] C. Lenart and A. Postnikov. Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not.*, pages 1–65, 2007. Art. ID rnm038.

[33] C. Lenart. On the combinatorics of crystal graphs, I. Lusztig's involution. *Adv. Math.*, 211:204–243, 2007.

[34] C. Lenart and F. Sottile. A Pieri-type formula for the K-theory of a flag manifold. *Trans. Amer. Math. Soc.*, 359:2317–2342, 2007.

[35] C. Lenart and T. Maeno. Quantum Grothendieck polynomials. `arXiv:math.CO/0608232`, 2006.
[36] C. Lenart and T. Maeno. Alcove path and Nichols-Woronowicz model of the equivariant K-theory of generalized flag varieties. *Int. Math. Res. Not.*, pages Art. ID 78356, 14, 2006.
[37] C. Lenart, S. Robinson, and F. Sottile. Grothendieck polynomials via permutation patterns and chains in the Bruhat order. *Amer. J. Math.*, 128:805–848, 2006.
[38] C. Lenart. The K-theory of the flag variety and the Fomin-Kirillov quadratic algebra. *J. Algebra*, 285:120–135, 2005.
[39] C. Lenart. A unified approach to combinatorial formulas for Schubert polynomials. *J. Algebraic Combin.*, 20:263–299, 2004.
[40] C. Lenart and F. Sottile. Skew Schubert polynomials. *Proc. Amer. Math. Soc.*, 131:3319–3328, 2003.
[41] C. Lenart. A K-theory version of Monk's formula and some related multiplication formulas. *J. Pure Appl. Algebra*, 179:137–158, 2003.
[42] C. Lenart. The many faces of modern combinatorics. `www.albany.edu/~lenart`, 2003. Preprint.
[43] C. Lenart. Combinatorial aspects of the K-theory of Grassmannians. *Ann. Combin.*, 4:67–82, 2000.
[44] C. Lenart. Lagrange inversion and Schur functions. *J. Algebraic Combin.*, 11:69–78, 2000.
[45] C. Lenart. A Robinson-Schensted-Knuth type correspondence in Schubert calculus and its applications. In *Proceedings of the 11th International Conference on Formal Power Series and Algebraic Combinatorics*, Marc Noy and Oriol Serra, editors, Universitat Politècnica de Catalunya, Barcelona, 1999, 287–298.
[46] C. Lenart. Noncommutative Schubert calculus and Grothendieck polynomials. *Adv. Math.*, 143:159–183, 1999.
[47] C. Lenart. The combinatorics of Steenrod operations on the cohomology of Grassmannians. *Adv. Math.*, 136:251–283, 1998.
[48] C. Lenart. Symmetric functions, formal group laws, and Lazard's theorem. *Adv. Math.*, 134:219–239, 1998.
[49] C. Lenart. Necklace algebras and Witt vectors associated with formal group laws. *J. Algebra*, 199:703–732, 1998.
[50] C. Lenart and N. Ray. Hopf algebras of set systems. *Discrete Math.*, 180:255–280, 1998.
[51] C. Lenart. A generalized distance in graphs and centered partitions. *SIAM J. Discrete Math.*, 11:293–304, 1998.
[52] C. Lenart and N. Ray. Chromatic polynomials of partition systems. *Discrete Math.*, 167/168:419–444, 1997. Proceedings of the 15th British Combinatorial Conference (Stirling, England, 1995).
[53] C. Lenart. *Combinatorial Models for Certain Structures in Formal Group Theory and Algebraic Topology.* PhD thesis, University of Manchester, 1996. 185 pp.
[54] C. Lenart. Defining separability of two fuzzy clusters by a fuzzy decision hyperplane. *Pattern Recognition*, 26:1351–1356, 1993.
[55] C. Lenart. *Classification and Learning in Pattern Recognition.* PhD thesis, University of Cluj-Napoca, Romania, 1992. 194 pp.
[56] I. Haidu, I. Lazăr, C. Lenart, and A. Imbroane. Modelling of natural hydroenergy organization of the small basins. In *Energy and the Environment into the 1990s: Proceedings of the 1st World Renewable Energy Congress*, Reading, UK, Pergamon Press, 1990, 3159–3167.
[57] C. Lenart, Method for improving the results of certain clustering procedures, *Studia Univ. "Babeş-Bolyai"*, Mathematica 35 (1990), 55–63.
[58] C. Lenart, Classification with fuzzy relations II, *Studia Univ. "Babeş-Bolyai"*, Mathematica 34 (1989), 63–67.
[59] C. Lenart, Classification with fuzzy relations I, *Studia Univ. "Babeş-Bolyai"*, Mathematica 33 (1988), 52–55.
[60] D. Dumitrescu and C. Lenart, Hierarchical classification for linear clusters, *Studia Univ. "Babeş-Bolyai"*, Mathematica 33 (1988), 48–51.
[61] C. Lenart and D. Dumitrescu, Convex decomposition of fuzzy partitions, *Univ. of Cluj-Napoca Research Seminars*, Preprint no. 5 (1987), 46–54.
[62] C. Lenart. Two-hour tutorial on the alcove model. *Whittaker Functions, Schubert Calculus and Crystals*, Institute for Computational and Experimental Research in Mathematics (ICERM), Brown University, `http://icerm.brown.edu/video_archive`, February-March 2013.
[63] C. Lenart. Maple package for Hall-Littlewood polynomials of type A and C, 2008, `www.albany.edu/~lenart`.
[64] C. Lecouvey and C. Lenart, Maple package for Lusztig's q-analog of weight multiplicities and its generalization to Lie superalgebras, 2007, `www.albany.edu/~lenart`.
[65] C. Lenart. Alcove_path, Maple package for the alcove model, 2006, `www.albany.edu/~lenart`.
[66] C. Lenart. Package for clustering with fuzzy sets, 1989–1992, 1997, available upon request.

[Ada11] W. Adamczak. On combinatorial models for representations of classical Lie algebras. PhD thesis, University at Albany, 2011. 117 pp.

[Baz06] Y. Bazlov. Nichols-Woronowicz algebra model for Schubert calculus on Coxeter groups. *J. Algebra*, 297:372–399, 2006.

[BGHT99] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler. Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions. *Methods Appl. Anal.*, 6:363–420, 1999.

[BGG73] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Schubert cells and cohomology of the spaces G/P. *Russian Math. Surveys*, 28:1–26, 1973.

[Bi99] S. Billey. Kostant polynomials and the cohomology ring for G/B. *Duke J. Math.*, 96:205–224, 1999.

[BL00] S. Billey and V. Lakshmibai. *Singular Loci of Schubert Varieties*, volume 182 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2000.

[BF11] A. Braverman and M. Finkelberg. Semi-infinite Schubert varieties and quantum K-theory of flag manifolds. *J. Amer. Math. Soc.*, 27:1147–1168, 2014.

[BF12] A. Braverman and M. Finkelberg. Weyl modules and q-Whittaker functions. *Math. Ann.*, 359:45–59, 2014.

[BBL15] B. Brubaker, D. Bump, and A. Licata. Iwahori Whittaker functions and Demazure operators. Rallis memorial volume, *J. Number Theory*, 146:41–68, 2015.

[Buc02a] A. Buch. Grothendieck classes of quiver varieties. *Duke Math. J.*, 115:75–103, 2002.

[Buc02b] A. Buch. A Littlewood-Richardson rule for the K-theory of Grassmannians. *Acta Math.*, 189:37–78, 2002.

[Buc10] A. Buch. Personal communication, June 2010.

[BCMP16] A. Buch, P.-E. Chaput, L. Mihalcea, and N. Perrin. A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties. *Algebr. Geom.*, 5:568–595, 2018.

[BM11] A. Buch and L. Mihalcea. Quantum K-theory of Grassmannians. *Duke Math. J.*, 156:501–538, 2011.

[CPZ13] B. Calmès, V. Petrov, and K. Zainoulline. Invariants, torsion indices and oriented cohomology of complete flags. *Ann. Sci. École Norm. Sup. (4)*, 46:405–448, 2013.

[CI15] V. Chari and B. Ion. BGG reciprocity for current algebras. *Compos. Math.*, 151:1265–1287, 2015.

[CSS14] V. Chari, L. Schneider, P. Shereen, and J. Wand. Modules with Demazure flags and character formulae. *SIGMA*, 10, 032, 2014, 16 pp.

[Che94] C. Chevalley. Sur les décompositions cellulaires des espaces G/B. In *Algebraic Groups and Generalizations: Classical Methods*, volume 56 Part 1 of *Proceedings and Symposia in Pure Mathematics*, pages 1–23. Amer. Math. Soc., 1994.

[Cos09] I. Coskun. A Littlewood-Richardson rule for two-step flag varieties. *Invent. Math.* 176:325–395, 2009.

[Don97] R. Donnelly. *Explicit Constructions of Representations of Semisimple Lie Algebras*. PhD thesis, University of North Carolina at Chapel Hill, 1997.

[Don03] R. Donnelly. Extremal properties of bases for representations of semisimple Lie algebras. *J. Algebraic Combin.*, 17:255–282, 2003.

[DRS72] P. Doubilet, G.-C. Rota, and R. Stanley. On the foundations of combinatorial theory VI: the idea of generating function. In *Sixth Berkeley Symposium on Mathematical Statistics and Probability*, volume 2, pages 267–318. University of California Press, 1972.

[FKM17] E. Feigin, S. Kato, and I. Makedonskyi. Representation theoretic realization of non-symmetric Macdonald polynomials at infinity. J. Reine Angew. Math., 2019, `DOI 10.1515/crelle-2019-0011`.

[FM17] E. Feigin and I. Makedonskyi. Generalized Weyl modules, alcove paths and Macdonald polynomials. *Sel. Math. New Ser.*, 23:2863–2897, 2017.

[FG98] S. Fomin and C. Greene. Noncommutative Schur functions and their applications. *Discrete Math.*, 193:179–200, 1998.

[FGP97] S. Fomin, S. Gelfand, and A. Postnikov. Quantum Schubert polynomials. *J. Amer. Math. Soc.*, 10:565–596, 1997.

[FK99] S. Fomin and A. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. In *Advances in Geometry*, pages 147–182. Birkhäuser Boston, Boston, MA, 1999.

[FL06] G. Fourier and P. Littelmann. Tensor product structure of affine Demazure modules and limit constructions. *Nagoya Math. J.*, 182:171–198, 2006.

[Ful97] W. Fulton. *Young Tableaux*, volume 35 of *London Math. Soc. Student Texts*. Cambridge Univ. Press, Cambridge and New York, 1997.

[FW04] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13:641–661, 2004.

[GL11] S. Gaussent and P. Littelmann. One-skeleton galleries, the path model, and a generalization of Macdonald's formula for Hall-Littlewood polynomials. *Int. Math. Res. Notices*, (12):2649–2707, 2012.

[GC50] I. Gelfand and M. Cetlin. Finite-dimensional representations of the group of unimodular matrices. *Doklady Akad. Nauk SSSR (N.S.)*, 71:825–828, 1950.

[GKV95] V. Ginzburg, M. Kapranov, and E. Vasserot. Elliptic algebras and equivariant elliptic cohomology. `arXiv:math.q-alg/9505012`, 1995.

[GL03] A. Givental and Y.-P. Lee. Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. *Invent. Math.*, 151:193–219, 2003.

[Gr02] W. Graham. Equivariant K-theory and Schubert varieties. Preprint, 2002.

[GR04] S. Griffeth and A. Ram. Affine Hecke algebras and the Schubert calculus. *European J. Combin.*, 25:1263–1283, 2004.

[HHL05] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18:735–761, 2005.

[HHH05] M. Harada, A. Henriques, and T. Holm. Computation of generalized equivariant cohomologies of Kac-Moody flag varieties. *Adv. Math.*, 197:198–221, 2005.

[HKO$^+$99] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[Haz78] M. Hazewinkel. *Formal Groups and Applications.* Academic Press, New York, 1978.

[HK06] A. Henriques and J. Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132:191–216, 2006.

[HK11] J. Hornbostel and V. Kiritchenko. Schubert calculus for algebraic cobordism. *J. Reine Angew. Math.*, 656:59–85, 2011.

[Ion03] B. Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116(2):299–318, 2003.

[INS16] M. Ishii, S. Naito, and D. Sagaki. Semi-infinite Lakshmibai-Seshadri path model for level-zero extremal weight modules over quantum affine algebras. *Adv. Math.*, 290:967–1009, 2016.

[Jon08] B. Jones. ClassicalCrystalOfAlcovePaths. `sagemath.org`, 2008. Implementation in *Sage* of the alcove model.

[JS10] B. Jones and A. Schilling. Affine structures and a tableau model for E_6 crystals. *J. Algebra*, 324:2512–2542, 2010.

[Kas91] M. Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Math. J.*, 63:465–516, 1991.

[KN94] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *J. Algebra*, 165:295–345, 1994.

[Kho13] A. Khoroshkin. Highest weight categories and Macdonald polynomials. `arXiv:1312.7053`, 2013.

[KM04] A. Kirillov and T. Maeno. Noncommutative algebras related with Schubert calculus on Coxeter groups. *European J. Combin.*, 25:1301–1325, 2004.

[KM05a] A. Kirillov and T. Maeno. A note on quantization operators on Nichols algebra model for Schubert calculus on Weyl groups. *Lett. Math. Phys.*, 72:233–241, 2005.

[KM05b] A. Kirillov and T. Maeno. On some noncommutative algebras related to K-theory of flag varieties. I. *Int. Math. Res. Not.*, (60):3753–3789, 2005.

[KM05c] A. Kirillov and T. Maeno. A note on quantum K-theory of flag varieties. Preprint, 2005.

[KR90] A. N. Kirillov and N. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *J. Soviet Math.*, 52:3156–3164, 1990.

[Klo11] I. Klostermann. Generalization of the Macdonald formula for Hall-Littlewood polynomials. *Adv. Math.* 248:13661403, 2013.

[KPSZ17] P. Koroteev, P. Pushkar, A. Smirnov, and A. Zeitlin. Quantum K-theory of quiver varieties and many-body systems. `arXiv:1705.10419`, 2017.

[KY04] A. Knutson and A. Yong. A formula for K-theory truncation Schubert calculus. *Int. Math. Res. Not.*, 70:3741–3756, 2004.

[LP07] T. Lam and P. Pylyavskyy. Combinatorial Hopf algebras and K-homology of Grassmanians. *Int. Math. Res. Not.*, article ID rnm125, 48 pp., 2007.

[Las90] A. Lascoux. Anneau de Grothendieck de la variété de drapeaux. In *The Grothendieck Festschrift*, volume III of *Progr. Math.*, pages 1–34, Boston, 1990. Birkhäuser.

[LS79] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de H. O. Foulkes. *C. R. Acad. Sci. Paris Sér. I Math.*, 288:95–98, 1979.

[LS82] A. Lascoux and M.-P. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294:447–450, 1982.

[Lee04] Y.-P. Lee. Quantum K-theory I: Foundations. *Duke Math. J.*, 121:389–424, 2004.

[Lit94] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116:329–346, 1994.

[Lit95] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math.* (2), 142:499–525, 1995.

[LW08] N. Loehr and G. Warrington. Nested quantum Dyck paths and $\nabla(s_\lambda)$. *Int. Math. Res. Not.*, article ID rnm 157, 2008.

[Lub13a] A. Lubovsky. *Alcove Models for Hall-Littlewood Polynomials and Affine Crystals*. PhD thesis, State University of New York at Albany, 2013.

[Lub13b] A. Lubovsky. CrystalOfAlcovePaths. `sagemath.org`, 2013. Implementation in *Sage* of the quantum alcove model.

[Lus83] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 208–229. Soc. Math. France, Paris, 1983.

[Lus93] G. Lusztig. *Introduction to Quantum Groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993.

[Mac71] I. Macdonald. *Spherical functions on a group of p-adic type*. Ramanujan Institute, Centre for Advanced Study in Mathematics,University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. Oxford University Press, Oxford, second edition, 1995.

[Mac03] I. G. Macdonald. *Affine Hecke Algebras and Orthogonal Polynomials*. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2003.

[MR83] N. Metropolis and G.-C. Rota. Witt vectors and the algebra of necklaces. *Adv. Math*, 50:95–125, 1983.

[Mil16] E. Milićević. Maximal Newton points and the quantum Bruhat graph. `arXiv:1606.07478`, 2016.

[MR16] E. Milićević and A. Ram. Alcove walks and the Peterson isomorphism. talk at Banff International Research Station workshop *Whittaker Functions: Number Theory, Geometry, and Physics*, video available at `www.birs.ca/events/2016/5-day-workshops/16w5039/videos`, 2016.

[MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166:95–143, 2007.

[NNS17] S. Naito, F. Nomoto, and D. Sagaki. Specialization of nonsymmetric Macdonald polynomials at $t = \infty$ and Demazure submodules of level-zero extremal weight modules. *Trans. Amer. Math. Soc.*, pages 1–45, 2017. `DOI: https://doi.org/10.1090/tran/7114`.

[NS16] S. Naito and D. Sagaki. Demazure submodules of level-zero extremal weight modules and specializations of Macdonald polynomials. *Math. Z.*, 283:937–978, 2016.

[Nom16] F. Nomoto. Quantum Lakshmibai-Seshadri paths and the specialization of Macdonald polynomials at $t = 0$ in type $A_{2n}^{(2)}$. `arXiv:1606.01067`, 2016.

[Nom17] F. Nomoto. Generalized Weyl modules and Demazure submodules of level-zero extremal weight modules. `arXiv:1701.08377`, 2017.

[Oh05] Y.-T. Oh. Generalized Burnside-Grothendieck ring functor and aperiodic ring functor associated with profinite groups. *J. Algebra*, 291:607–648, 2005.

[Oh06] Y.-T. Oh. Necklace rings and logarithmic functions. *Adv. Math.*, 205:434–486, 2006.

[Oh07a] Y.-T. Oh. Classification of the ring of Witt vectors and the necklace ring associated with the formal group law $X + Y - qXY$. *J. Algebra*, 310:325–350, 2007.

[Oh07b] Y.-T. Oh. Nested Witt vectors and their q-deformation. *J. Algebra*, 309:683–710, 2007.

[Oh07c] Y.-T. Oh. q-deformation of Witt-Burnside rings. *Math. Z.*, 257:151–191, 2007.

[Oh08a] Y.-T. Oh. q-Analog of the Möbius function and the cyclotomic identity associated to a profinite group. *Adv. Math.*, 219:852–893, 2008.

[Oh08b] Y.-T. Oh. q-deformed necklace rings and q-Möbius function. *J. Algebra*, 320:1599–1625, 2008.

[OS17] D. Orr and M. Shimozono. Specializations of nonsymmetric Macdonald-Koornwinder polynomials. *J. Algebraic Combin.*, pages 1–37, 2017. `DOI 10.1007/s10801-017-0770-6`.

[PR99] H. Pittie and A. Ram. A Pieri-Chevalley formula in the K-theory of a G/B-bundle. *Electron. Res. Announc. Amer. Math. Soc.*, 5:102–107, 1999.

[Pos05b] A. Postnikov. Quantum Bruhat graph and Schubert polynomials. *Proc. Amer. Math. Soc.*, 133:699–709, 2005.

[Ram06] A. Ram. Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux. *Pure Appl. Math. Q.*, 2:963–1013, 2006.

[RY11] A. Ram and M. Yip. A combinatorial formula for Macdonald polynomials. *Adv. Math.*, 226:309–331, 2011.

[RT93] G.-C. Rota and B. Taylor. An introduction to the umbral calculus. In *Analysis, geometry and groups: a Riemann legacy volume*, Hadronic Press Collect. Orig. Artic., pages 513–525. Hadronic Press, Palm Harbor, FL, 1993.

[S$^+$12] W. A. Stein et al. Sage mathematics software (Version 5.4). `http://www.sagemath.org`, 2012. The Sage Development Team.

[ST12] A. Schilling and P. Tingley. Demazure crystals, Kirillov-Reshetikhin crystals, and the energy function. *Electron. J. Combin.*, 19:P2, 2012.

[Sch94] W. R. Schmitt. Incidence Hopf algebras. *J. Pure Appl. Algebra*, 96:299–330, 1994.

[Sch06] C. Schwer. Galleries, Hall-Littlewood polynomials, and structure constants of the spherical Hecke algebra. *Int. Math. Res. Not.*, Art. ID 75395, 31, 2006.

[Sta99] R. P. Stanley. *Enumerative Combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[SZZ17] C. Su, G. Zhao, and C. Zhong. On the K-theory stable bases of the Springer resolution. `arXiv:1708.08013`, 2017. To appear in *Ann. Sci. Éc. Norm. Supér.*, 2019.

[Wil04] M. Willems. Cohomologie et K-théorie équivariantes des variétés de Bott-Samelson et des variétés de drapeaux. *Bull. Soc. Math. France*, 132:569–589, 2004.

[Woo98] R. M. W. Wood. Problems in the Steenrod algebra. *Bull. London Math. Soc.*, 30:449–517, 1998.

[Yip11] M. Yip. Personal communication, 2011.