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Structure of the short range amplitude for general scattering relations

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Abstract. We consider scattering by short range perturbations of the semi-classical Laplacian. We prove that when a polynomial bound on the resolvent holds the scattering amplitude is a semi-classical Fourier integral operator associated to the scattering relation near a non-trapped ray. Compared to previous work, we allow the scattering relation to have more general structure.

Keywords: short range perturbations, scattering amplitude, scattering relation, semi-classical Fourier integral operators

1. Introduction and statement of results

We study the structure of the scattering amplitude associated to the semi-classical Schrödinger operator with a short range potential on $\mathbb{R}^n$. We prove that, when restricted away from the diagonal on $\mathbb{S}^n \times \mathbb{S}^n$, the natural scattering amplitude quantizes the scattering relation in the sense of semi-classical Fourier integral operators. The scattering relation at energy $\lambda > 0$ here is given roughly by the Hamiltonian flow of the symbol p of the operator between two hypersurfaces "at infinity" inside the energy surface $\{p = \lambda\}$.

1.1. A survey of earlier results

The structure of the scattering matrix has been of significant interest to researchers in mathematical physics. Earlier results have focused primarily on establishing asymptotic expansions of the scattering amplitude. In this section we describe briefly only those asymptotic expansions most relevant to our work and refer to [1] for a more comprehensive survey.

We begin by introducing some notation. Let $P(h) = -\frac{1}{2}h^2\Delta + V$, $0 < h \ll 1$, where $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is such that for some $\rho > 1$ and all $\alpha \in \mathbb{N}^n$

$$\left|\partial_x^\alpha V(x)\right| \leqslant C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathbb{R}^n, \tag{1}$$

where $\langle x\rangle = (1 + \|x\|^2)^{1/2}$.

Let $\lambda > 0$ and for $\omega \in \mathbb{S}^{n-1}$ and $z \in \omega^\perp$ we denote by

$$\gamma_\infty(\cdot; z, \sqrt{2\lambda}\omega) = \{q_\infty(\cdot; z, \sqrt{2\lambda}\omega), p_\infty(\cdot; z, \sqrt{2\lambda}\omega)\}$$

the unique phase trajectory, i.e., the integral curve of the Hamiltonian vector field H_p of $p(x,\xi) = \frac{1}{2}\|\xi\|^2 + V(x)$, such that

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$$\lim_{t\to-\infty}(q_\infty(t;z,\sqrt{2\lambda}\omega)-\sqrt{2\lambda}\omega t-z)=0,\qquad \lim_{t\to-\infty}(p_\infty(t;z,\sqrt{2\lambda}\omega))-\sqrt{2\lambda}\omega)=0$$

in the C^∞ topology for $\omega\in\mathbb{S}^{n-1}$ and $z\in\omega^\perp$.

If $\lim_{t\to\infty}\|q_\infty(t;z,\sqrt{2\lambda}\omega)\|=\infty$, then, setting $\mathbb{S}^{n-1}_{2\lambda}=\{x\in\mathbb{R}^n\colon\ \|x\|=2\lambda\}$, we have that there exist an open set $U\subset T^*\mathbb{S}^{n-1}_{2\lambda}$ with $(\sqrt{2\lambda}\omega,z)\in U$ and functions $\xi_\infty\in C^\infty(T^*\mathbb{S}^{n-1}_{2\lambda}\cap U;\mathbb{S}^{n-1})$ and $x_\infty\in C^\infty(T^*\mathbb{S}^{n-1}_{2\lambda}\cap U;\mathbb{R}^n)$ such that

$$\lim_{t\to\infty}(q_\infty(t;z,\sqrt{2\lambda}\omega)-\sqrt{2\lambda}\xi_\infty(\sqrt{2\lambda}\omega,z)t-x_\infty(\sqrt{2\lambda}\omega,z))=0,$$
$$\lim_{t\to\infty}(p_\infty(t;z,\sqrt{2\lambda}\omega)-\sqrt{2\lambda}\xi_\infty(\sqrt{2\lambda}\omega,z))=0$$

in the C^∞ topology for $\omega\in\mathbb{S}^{n-1}$ and $z\in\omega^\perp$. The trajectory $\gamma_\infty(\cdot;z,\sqrt{2\lambda}\omega)$ is then said to have initial direction ω and final direction $\theta=\xi_\infty(\sqrt{2\lambda}\omega,z)$ and is called an (ω,θ)-trajectory. We also make the following

Definition 1. The outgoing direction $\theta\in\mathbb{S}^{n-1}$ is called *non-degenerate*, or *regular*, for the incoming direction $\omega\in\mathbb{S}^{n-1}$ if $\theta\neq\omega$ and for all $z'\in\omega^\perp$ with $\xi_\infty(\sqrt{2\lambda}\omega,z')=\theta$, the map $\omega^\perp\ni z\mapsto\xi_\infty(\sqrt{2\lambda}\omega,z)\in\mathbb{S}^{n-1}$ is non-degenerate at z'.

Several authors, working under the assumption that a certain final direction θ is non-degenerate for a given initial direction ω, have proved asymptotic expansions of the scattering amplitude A of the form

$$K_{A(\lambda,h)}(\omega,\theta)=\sum_{j=1}^{l}\hat\sigma(z_j,\omega;\lambda)^{-1/2}\exp(\mathrm{i}h^{-1}S_j-\mathrm{i}\mu_j\pi/2)+\mathcal{O}(h),\tag{2}$$

where $(z_j)_{j=1}^l\equiv(\xi_\infty^{-1}(\sqrt{2\lambda}\omega,\cdot))(\theta_0)$, $\hat\sigma(z_j,\omega;\lambda)=\det(\mathrm{J}\xi_\infty(\sqrt{2\lambda}\omega,\cdot))(z_j)$, with J denoting the Jacobian matrix,

$$S_j=\int_{-\infty}^{\infty}\left(\frac{1}{2}\|p_\infty(t;z,\sqrt{2\lambda}\omega)\|^2-V(q_\infty(t;z,\sqrt{2\lambda}\omega))-\lambda\right)\mathrm{d}t-\langle x_\infty(\sqrt{2\lambda}\omega,z),\sqrt{2\lambda}\theta\rangle\tag{3}$$

is a (modified) action along the j-th (ω,θ) trajectory, and μ_j is the path index of that trajectory. Vainberg [11] has studied smooth compactly supported potentials V at energies $\lambda>\sup V$. Guillemin [6] has established a similar asymptotic expansion in the setting of smooth compactly-supported metric perturbations of the Laplacian. Working with trapping potential perturbations of the Laplacian satisfying (1) with $\rho>\max(1,\frac{n-1}{2})$, Yajima [12] has proved such an asymptotic expansion in the L^2 sense. For non-trapping short-range ($\rho>1$) potential perturbations of the Laplacian, Robert and Tamura [10] have established such a pointwise asymptotic expansion. This result has been extended to the case of trapping energies by Michel [9] under an additional assumption on the distribution of the resonances of $P(h)$.

In [1] we have proved, without making the non-degeneracy assumption, that the scattering amplitude for smooth compactly supported potential and metric perturbations of the Euclidean Laplacian at both trapping and non-trapping energies is a semi-classical Fourier integral operator associated to the scattering relation. We have further showed how the expansion (2) follows from the general theory of semi-classical Fourier integral operators developed in [2], once the non-degeneracy assumption on the initial and final directions is made. The asymptotic expansion we have obtained in [1], however, is more general than one given in (2) in that it holds microlocally near (ω, θ) trajectories and not only for fixed initial and final directions.

In this paper we extend our results from [1] to the case of short-range perturbations of the Laplacian when the scattering amplitude is restricted away from the diagonal in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. Our method combines the study of the microlocal structure of the cut-off resolvent and the general results on semi-classical Fourier integral operators established in [2].

1.2. Statement of main theorem

We consider the semi-classical Schrödinger operator $P(h) = -\frac{1}{2}h^2\Delta + V$, on $\mathbb{R}^n$ for $n \geqslant 2$, $0 < h \leqslant 1$, with the potential $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ satisfying (1). Let $P_0(h) = -\frac{1}{2}h^2\Delta$. Then $P(h)$ and $P_0(h)$ admit unique self-adjoint realizations on $L^2(\mathbb{R}^n)$ with domains $H_h^2(\mathbb{R}^n)$, the semi-classical Sobolev spaces of order 2 (see Appendix A). It is well-known that the wave operators

$$W_\pm = \text{s-}\lim_{t\to\pm\infty} U(t)U_0(-t) \quad \text{in } L^2(\mathbb{R}^n)$$

exist and are complete, where

$$U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{h}tP(h)}, \qquad U_0(t) = \mathrm{e}^{-\frac{\mathrm{i}}{h}tP_0(h)}, \quad t \in \mathbb{R}.$$

We can therefore define the scattering operator

$$S = W_+^* W_- = \overline{\mathcal{F}}_h^{-1} \int_{\hat\sigma(P(h))} \bigoplus S(\lambda, h)\,\mathrm{d}\lambda \overline{\mathcal{F}}_h,$$

where $\overline{\mathcal{F}}_h$ denotes the unitary h-dependent Fourier transform on $L^2(\mathbb{R}^n)$ (see Appendix A) and $\hat\sigma(P(h)) \subset \mathbb{R}^+$ denotes a core of the spectrum of $P(h)$. The operator $S(\lambda, h)$ is called the scattering matrix at energy $\lambda > 0$ and is a unitary operator on $L^2(\mathbb{S}^{n-1})$. The scattering amplitude $A(\lambda, h)$ is defined by $A(\lambda, h) = c(n, \lambda, h)(I - S(\lambda, h))$, where

$$c(n, \lambda, h) = (2\lambda)^{-\frac{n-1}{4}}(2\pi h)^{\frac{n-1}{2}}\,\mathrm{e}^{-\mathrm{i}\frac{(n-5)\pi}{4}}.$$

It is well known that the Schwartz kernel $K_{A(\lambda,h)}$ of $A(\lambda, h)$ satisfies $K_{A(\lambda,h)} \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}\backslash\mathrm{diag}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}))$.

To state our Main Theorem, we let $R(\lambda + \mathrm{i}0, h) = \lim_{\varepsilon\downarrow 0}(P(h) - \lambda - \mathrm{i}\varepsilon)^{-1}$, where the limit is taken in the space $\mathcal{B}(L^2_\alpha(\mathbb{R}^n), L^2_{-\alpha}(\mathbb{R}^n))$, $\alpha > \frac{1}{2}$, with $L^2_\alpha(\mathbb{R}^n) = \{f\colon \langle\cdot\rangle^\alpha f \in L^2(\mathbb{R}^n)\}$. We further refer the reader to Section 3 for the definitions non-trapped trajectories and the scattering relation $SR_{\overline{U}}(\lambda)$. The class of semi-classical Fourier integral operators $\mathcal{I}_h^r$ is defined in Appendix A, where we also review the notion of pseudodifferential operators of principal type.

We are now ready to state our

Main Theorem. *Let $\lambda > 0$ be such that the operator $P(h) - \lambda$ is of principal type. Let also*

$$\|R(\lambda + \mathrm{i}0, h)\|_{\mathcal{B}(L^2_\alpha(\mathbb{R}^n), L^2_{-\alpha}(\mathbb{R}^n))} = \mathcal{O}(h^s), \quad \textit{for some } s \in \mathbb{R} \textit{ and some } \alpha > \frac{1}{2}. \tag{4}$$

Let $(\omega, z) \in T^*\mathbb{S}^{n-1}$ be such that $\gamma_\infty(\cdot; z, \sqrt{2\lambda}\omega)$ is a non-trapped trajectory.

Then there exists an open set $U \subset T^*\mathbb{S}^{n-1}$ with $(\omega, z) \in U$ such that

$$A(\lambda, h) \in \mathcal{I}_h^{\frac{n}{2}}\big(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \backslash \operatorname{diag}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), SR_{\overline{U}}(\lambda)\big).$$

We remark that [2, Theorem 1] gives a characterization of semi-classical Fourier integral distributions as oscillatory integrals. Applied to the scattering amplitude here this characterization says approximately that for every non-degenerate phase function ϕ which locally parameterizes $SR_{\overline{U}}(\lambda)$ we can find a symbol a admitting an asymptotic expansion in h such that $CK_{A(\lambda,h)}$, where C is a microlocal cut-off to $SR_{\overline{U}}(\lambda)$ (see Appendix A), can be represented as an oscillatory integral with phase ϕ and symbol a. From the discussion in [2, Section 4.1] we further know that such a non-degenerate phase function always exists, and therefore we can always express $CK_{A(\lambda,h)}$ as an oscillatory integral admitting an asymptotic expansion in h. In the special case when the non-degeneracy assumption holds, we recover the phases (3) in (2) – see Theorem 1 below. We expect that a finer analysis based on our method would give a precise description of the amplitudes as well. What is different here is the fact that we can handle the cases in which the non-degeneracy assumption fails.

We now introduce some of the notation we shall use below. For a sequentially continuous operator $T: C_c^\infty(\mathbb{R}^m) \to \mathcal{D}'(\mathbb{R}^n)$ we shall denote by K_T its Schwartz kernel. On any smooth manifold M we denote by σ the canonical symplectic form on T^*M and everywhere below we work with the canonical symplectic structure on T^*M. The integral curve of the Hamiltonian vector field H_p with initial conditions $(x_0, \xi_0) \in T^*\mathbb{R}^n$ will be denoted by $\gamma(\cdot; x_0, \xi_0) = (x(\cdot; x_0, \xi_0), \xi(\cdot; x_0, \xi_0))$. If $C \subset T^*M_1 \times T^*M_2$, where M_j, $j = 1, 2$, are smooth manifolds, we will use the notation $C' = \{(x, \xi; y, -\eta)\colon (x, \xi; y, \eta) \in C\}$. For $\gamma \geqslant 0$ we shall also use $\|\cdot\|_{\pm\gamma,\mp\gamma}$ to denote the norm of a linear operator between the spaces $L^2_{\pm\gamma}(\mathbb{R}^n)$ and $L^2_{\mp\gamma}(\mathbb{R}^n)$, while $\|\cdot\|$ will denote the Euclidean norm. Lastly, we set $B(0, r) = \{x \in \mathbb{R}^n\colon \|x\| < r\}$ and $B(0, r, r+1) = \{x \in \mathbb{R}^n\colon r < \|x\| < r+1\}$ and use C_j, $j \in N$, to denote unspecified real constants.

This paper is organized as follows. We review the definition of semi-classical Fourier integral distributions and operators in Appendix A, where we also recall the relevant part of semi-classical analysis. The semi-classical version of Isozaki–Kitada's representation of the short-range scattering amplitude, which we will use in this article, is presented in Section 2.1. Two preliminary lemmas on the structure of the semi-classical amplitude, are given in Section 2.2. The scattering relation is defined in Section 3, where we also prove that it can be parameterized by the modified actions when the non-degeneracy assumption is made. The proof of the Main Theorem is presented in Section 4 and its applications to non-trapping and trapping perturbations are discussed in Section 5.1 and Section 5.2, respectively. Finally, the theorem giving the microlocal representation of the scattering amplitude as an oscillatory integral under the non-degeneracy assumption is proved in Section 5.3.

2. Preliminaries

In this section we introduce some of the preliminary results we shall use throughout the paper.

2.1. *Representation of the scattering amplitude*

Here we present the representation of the short range scattering amplitude which we shall use in the proof of our Main Theorem. The construction is close to the one used by Robert and Tamura [10] and constitutes a semi-classical adaptation of the representation of the short range amplitude originally established by Isozaki and Kitada [8].

We begin with the following

Definition 2. Let $\Omega \subset T^*\mathbb{R}^n$ be an open subset. We denote by $A_m(\Omega)$ the class of symbols a such that $(x,\xi) \mapsto a(x,\xi,h)$ belongs to $C^\infty(\Omega)$ and

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)\right| \leqslant C_{\alpha\beta}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{-L}, \quad \text{for all } (x,\xi) \in \Omega, (\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n,\ L > 0.$$

We denote $A_m(T^*\mathbb{R}^n)$ by A_m.

We also use the notation

$$\Gamma_\pm(R,d,\sigma) = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\colon\ \|x\| > R,\ \frac{1}{d} < \|\xi\| < d, \pm\cos(x,\xi) > \pm\sigma\right\}$$

with $R > 1$, $d > 1$, $\sigma \in (-1,1)$, and $\cos(x,\xi) = \frac{\langle x,\xi\rangle}{\|x\|\,\|\xi\|}$, for the outgoing and incoming subsets of phase space, respectively.

For $\alpha > \frac{1}{2}$, we denote the bounded operator $F_0(\lambda,h)\colon L^2_\alpha(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ by

$$\big(F_0(\lambda,h)f\big)(\omega) = (2\pi h)^{-\frac{n}{2}}(2\lambda)^{\frac{n-2}{4}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{\mathrm{i}}{h}\sqrt{2\lambda}\langle\omega,x\rangle} f(x)\,\mathrm{d}x, \quad \lambda > 0.$$

The starting point of Robert and Tamura's [10] construction of a representation of the scattering amplitude, as in Isozaki and Kitada [8], is a set of WKB parametrices for the wave operators. For $R_0 \gg 0$, $1 < d_4 < d_3 < d_2 < d_1 < d_0$, and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 < 1$ Robert and Tamura [10] construct phase functions $\Phi_\pm$ and symbols $(a_{\pm j})_{j=0}^\infty$ and $(b_{\pm j})_{j=0}^\infty$ such that:

1. $\Phi_\pm \in C^\infty(T^*\mathbb{R}^n)$ solve the eikonal equation

$$\frac{1}{2}\left\|\nabla_x \Phi_\pm(x,\xi)\right\|^2 + V(x) = \frac{1}{2}\|\xi\|^2$$

in $(x,\xi) \in \Gamma_\pm(R_0,d_0,\pm\sigma_0)$, respectively.
2. $\Phi_\pm(\cdot,\cdot\cdot) - \langle\cdot,\cdot\cdot\rangle \in A_0(\Gamma_\pm(R_0,d_0,\pm\sigma_0))$.
3. For all $(x,\xi) \in T^*\mathbb{R}^n$

$$\left|\frac{\partial^2\Phi_\pm}{\partial_{x_j}\partial_{\xi_k}}(x,\xi) - \delta_{jk}\right| < \varepsilon(R_0),$$

where δ_{jk} is the Kronecker delta and $\varepsilon(R_0) \to 0$ as $R_0 \to \infty$.
4. $(a_{\pm j})_j^\infty$ and $(b_{\pm j})_j^\infty$ are determined inductively as solutions to certain transport equations and satisfy $a_{\pm j} \in A_{-j}(\Gamma_\pm(3R_0,d_1,\mp\sigma_1))$, $\operatorname{supp} a_{\pm j} \subset \Gamma_\pm(3R_0,d_1,\mp\sigma_1)$, $b_{\pm j} \in A_{-j}(\Gamma_\pm(5R_0,d_3,\pm\sigma_4))$, $\operatorname{supp} b_{\pm j} \subset \Gamma_\pm(5R_0,d_3,\pm\sigma_4)$.

Using the Borel process, we now find symbols $a_\pm \in A_0(\Gamma_\pm(3R_0, d_1, \mp\sigma_1))$ and $b_\pm \in A_0(\Gamma_\pm(5R_0, d_3, \pm\sigma_4))$ such that $a_\pm \sim \sum_{j=0}^\infty h^j a_{\pm j}$ and $b_\pm \sim \sum_{j=0}^\infty h^j b_{\pm j}$.

For a symbol c and a phase function ϕ, we denote by $I_h(c, \phi)$ the oscillatory integral

$$I_h(c,\phi) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\frac{\mathrm{i}}{h}(\phi(x,\xi) - \langle y,\xi\rangle)} c(x,\xi)\,\mathrm{d}\xi$$

and let

$$K_{\pm a}(h) = P(h) I_h(a_\pm, \Phi_\pm) - I_h(a_\pm, \Phi_\pm) P_0(h),$$
$$K_{\pm b}(h) = P(h) I_h(b_\pm, \Phi_\pm) - I_h(b_\pm, \Phi_\pm) P_0(h).$$

The scattering amplitude $A(\lambda, h)$ for $\lambda \in (\frac{1}{2d_4^2}, \frac{d_4^2}{2})$ is then given by (see [8, Theorem 3.3])

$$A(\lambda,h) = -2\pi\mathrm{i} c(n,\lambda,h)\big(T_{+1}(\lambda,h) + T_{-1}(\lambda,h) - T_2(\lambda,h)\big),$$

where

$$T_{\pm 1}(\lambda,h) = F_0(\lambda,h) I_h(a_\pm, \Phi_\pm)^* K_{\pm b}(h) F_0^*(\lambda,h)$$

and

$$T_2(\lambda,h) = F_0(\lambda,h) K_{+a}^*(h) R(\lambda + \mathrm{i}0, h)\big(K_{+b}(h) + K_{-b}(h)\big) F_0^*(\lambda,h).$$

2.2. *Two preparatory lemmas*

The following two lemmas will be useful in studying the structure of the scattering amplitude.

Lemma 1. *Let $\|W\|_{\gamma,-\gamma} = \mathcal{O}(h^s)$, $h \to 0$, for some $\gamma \geqslant 0$ and some $s \in \mathbb{R}$. Then $K_W \in \mathcal{D}_h'(\mathbb{R}^{2n})$.*

Proof. By Schwartz Kernel Theorem, for some $h_0 > 0$ and every $h \in (0, h_0]$, there exists $w_h \in \mathcal{D}'(\mathbb{R}^{2n})$ such that $\langle W\varphi, \psi\rangle = \langle w_h, \varphi \otimes \psi\rangle$, $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. Let $c > d > 0$ and set $K(e) = \{x \in \mathbb{R}^n\colon |x_l| < e, l = 1, \ldots, n\}$ for $e > 0$. Let $\chi \in C_c^\infty(K(d) \times K(d))$ and $\rho \in C_c^\infty(K(c))$ be such that $\rho = 1$ on $K(d)$. Then, by the proof of Schwartz Kernel Theorem [5, Theorem 6.1.1], we have that

$$\big\langle w_h, \chi\,\mathrm{e}^{-\frac{\mathrm{i}}{h}(\langle\cdot,\xi\rangle + \langle\cdot\cdot,\eta\rangle)}\big\rangle = \sum_{\mathbb{Z}^n\times\mathbb{Z}^n} \hat{\chi}_{m,k} \big\langle W\rho E\big(\langle m,\cdot\rangle\big), \rho E\big(\langle k,\cdot\cdot\rangle\big)\big\rangle,$$

where $E(t) = \mathrm{e}^{\frac{2\pi\mathrm{i}t}{c}}$, $t \in \mathbb{R}$, and $\hat{\chi}_{m,k} = c^{-2n} \int_{K(c)\times K(c)} \chi(x,y)\,\mathrm{e}^{-\frac{\mathrm{i}}{h}(\langle x,\xi\rangle + \langle y,\eta\rangle)} E(-m\cdot x - k\cdot y)\,\mathrm{d}x\,\mathrm{d}y$. Integration by parts now gives

$$\begin{aligned}&\big(1 + \|m\|\big)^M \big(1 + \|k\|\big)^M \hat{\chi}_{m,k}\\ &\quad\leqslant C_1 h^{-2M} \big\langle(\xi,\eta)\big\rangle^M \sum_{|\alpha|\leqslant M, |\beta|\leqslant M} \big\|\partial_x^\alpha \partial_y^\beta \chi\big\|_{L^\infty(\mathbb{R}^{2n})}, \quad m,k \in \mathbb{Z}^n,\ M \in \mathbb{N}_0. \end{aligned} \tag{5}$$

We also have

$$\left|\langle W\rho E(\langle m,\cdot\rangle),\rho E(\langle k,\cdot\cdot\rangle)\rangle\right|\leqslant C_2h^s. \tag{6}$$

From estimates (5) and (6) we obtain

$$\left|\sum_{\mathbb{Z}^n\times\mathbb{Z}^n}\hat{\chi}_{m,k}\langle W\rho E(\langle m,\cdot\rangle),\rho E(\langle k,\cdot\cdot\rangle)\rangle\right|\leqslant C_3\sum_{|\alpha|\leqslant M,|\beta|\leqslant M}\|\partial_x^\alpha\partial_x^\beta\chi\|_{L^\infty(\mathbb{R}^{2n})}h^{s-2M}\langle(\xi,\eta)\rangle^M, \tag{7}$$

with

$$C_3=C_1C_2\sum_{\mathbb{Z}^n\times\mathbb{Z}^n}(1+\|m\|)^{-M}(1+\|k\|)^{-M}<\infty,$$

if M is taken large enough. Therefore $K_W\in\mathcal{D}_h'(\mathbb{R}^{2n})$. □

Lemma 2. *Let $\nu:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ be given by $\nu(x,y)=(y,x)$.*
Then $\nu^*K_{R(\lambda+\mathrm{i}0,h)}=K_{R(\lambda+\mathrm{i}0,h)}$ for every $\lambda>0$.

Proof. For $u,v\in L^2(\mathbb{R}^n)$ let $\langle u,v\rangle=\int uv$. Let u and v further satisfy $u,v\in C_c^\infty(\mathbb{R}^n)$ and let $z\in\mathbb{C}$ be such that $\Im z>0$. We then have

$$\begin{aligned}\langle R(z,h)u,v\rangle&=\langle R(z,h)u,(P(h)-z)R(z,h)v\rangle\\&=\langle (P(h)-z)R(z,h)u,R(z,h)v\rangle\\&=\langle u,R(z,h)v\rangle.\end{aligned}$$

Using the fact that

$$R(\lambda+\mathrm{i}0,h)=\lim_{\varepsilon\downarrow 0}R(\lambda+\mathrm{i}\varepsilon,h)\quad\text{in }\mathcal{B}(L^2_\alpha(\mathbb{R}^n),L^2_{-\alpha}(\mathbb{R}^n)),\ \alpha>\frac{1}{2},$$

we obtain

$$\langle R(\lambda+\mathrm{i}0,h)u,v\rangle=\langle u,R(\lambda+\mathrm{i}0,h)v\rangle.$$

Since $C_c^\infty(\mathbb{R}^n)\otimes C_c^\infty(\mathbb{R}^n)$ is dense in $C_c^\infty(\mathbb{R}^{2n})$, this completes the proof of the lemma. □

3. Scattering geometry

In this section we describe the scattering relation and prove that it can be parameterized by the modified actions (3) when the non-degeneracy assumption holds. The scattering relation is a Lagrangian submanifold of $T^*\mathbb{S}^{n-1}\times T^*\mathbb{S}^{n-1}$, which relates the incoming and the outgoing data in the way suggested by Fig. 1.

To make this precise, we first give the following

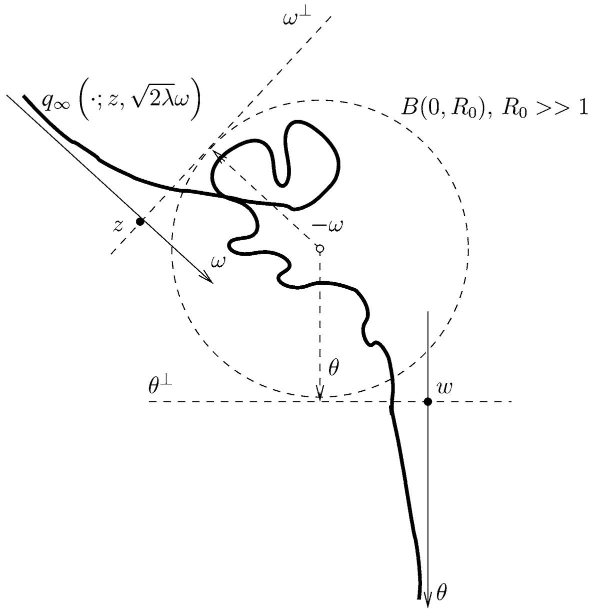

Fig. 1. The scattering relation consists of the points $(\omega, z; \theta, -w)$ related as in this figure.

Definition 3. The trajectory $\gamma(\cdot; x_0, \xi_0)$ is non-trapped if $\lim_{t\to\pm\infty} \|x(t; x_0, \xi_0)\| = \infty$. The energy $\lambda > 0$ is non-trapping if for every $(x_0, \xi_0) \in T^*\mathbb{R}^n$ with $\frac{1}{2}\|\xi_0\|^2 + V(x_0) = \lambda$ the trajectory $\gamma(\cdot; x_0, \xi_0)$ is non-trapped.

Let, now, $\lambda > 0$ be such that the operator $P(h) - \lambda$ is of principal type. Then $\Sigma_\lambda = p^{-1}(\lambda)$ is a smooth $2n-1$-dimensional submanifold of $T^*\mathbb{R}^n$.

Let, further, $(\omega_0, z_0) \in T^*\mathbb{S}^{n-1}$ be such that $\gamma_\infty(\cdot; z_0, \sqrt{2\lambda}\omega_0)$ is a non-trapped trajectory with $\xi_\infty(\sqrt{2\lambda}\omega_0, z_0) \neq \omega_0$. Then there exists $U \subset T^*\mathbb{S}^{n-1}$, open, $(\omega_0, z_0) \in U$, such that for every $(\omega, z) \in U$ the trajectory $\gamma_\infty(\cdot; z, \sqrt{2\lambda}\omega)$ is non-trapped and $\xi_\infty(\sqrt{2\lambda}\omega, z) \neq \omega$. By decreasing U, if necessary, we therefore have that

$$\mathit{SR}_{\overline{U}}(\lambda) = \left\{\left(\omega, z; \xi_\infty\left(\sqrt{2\lambda}\omega, z\right), x_\infty\left(\sqrt{2\lambda}\omega, z\right)\right):\ (\omega, z) \in \overline{U}\right\}' \tag{8}$$

is a closed Lagrangian submanifold of $(T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1}, \pi_1^*\sigma + \pi_2^*\sigma)$, which we call a scattering relation at energy λ (see Fig. 1).

If λ is a non-trapping energy level, we define the scattering relation at energy $\lambda > 0$ as

$$\mathit{SR}(\lambda) = \left\{\left(\omega, z; \xi_\infty\left(\sqrt{2\lambda}\omega, z\right), x_\infty\left(\sqrt{2\lambda}\omega, z\right)\right):\ (\omega, z) \in T^*\mathbb{S}^{n-1}, \omega \neq \xi_\infty\left(\sqrt{2\lambda}\omega, z\right)\right\}'.$$

In this case $\mathit{SR}(\lambda)$ is an open Lagrangian submanifold of $T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1}$.

We now show how, under the assumption that a certain outgoing direction is regular for a given incoming direction, we can find a non-degenerate phase function which parameterizes the scattering relation. We begin with the following

Lemma 3. *Let $\theta_0 \in \mathbb{S}^{n-1}$ be regular for $\omega_0 \in \mathbb{S}^{n-1}$.*

Then there exist $O_j \subset \mathbb{S}^{n-1}$, $j = 1, 2$, open, $\omega_0 \in O_1$, $\theta_0 \in O_2$, and $L \in \mathbb{N}$ such that for every $(\omega, \theta) \in O_1 \times O_2$ the number of (ω, θ) trajectories is at least L.

Proof. By [9, Remark 1.1] and the discussion following it, we have that there exists $L \in \mathbb{N}$ such that the number of (ω_0, θ_0) trajectories is L. Let $(z_l)_{l=1}^L \equiv (\xi_\infty^{-1}(\sqrt{2\lambda}\omega_0, \cdot))(\theta_0)$. By the Implicit Function Theorem, since θ_0 is regular for ω_0, we have that there exist open sets $O_1, O_2 \subset \mathbb{S}^{n-1}$ with $\theta_0 \in O_1$ and $\omega_0 \in O_2$ and functions $z_l \in C^\infty(O_1 \times O_2; \mathbb{R}^{n-1})$, $l = 1, \dots, L$, such that $z_l(\omega_0, \theta_0; \lambda) = z_l$ and $\xi_\infty(\sqrt{2\lambda}\omega, z_l(\omega, \theta; \lambda)) = \theta$, $(\omega, \theta) \in O_1 \times O_2$, which completes the proof. □

As in [1, Lemma 3], we have the following

Lemma 4. *Let $\theta_0 \in \mathbb{S}^{n-1}$ be regular for $\omega_0 \in \mathbb{S}^{n-1}$.*
Then there exist $O_j \subset \mathbb{S}^{n-1}$, $j = 1, 2$, open, $\omega_0 \in O_1$, $\theta_0 \in O_2$, such that the map

$$\theta^\perp \ni w \mapsto \xi_\infty\big(-\sqrt{2\lambda}\theta, w\big) \in \mathbb{S}^{n-1}$$

is non-degenerate at $x_\infty(\sqrt{2\lambda}\omega, z_l(\omega, \theta; \lambda))$, $(\omega, \theta) \in O_1 \times O_2$, $l = 1, \dots, L$.

We now choose O_1 and O_2 in such a way that the conclusions of Lemma 3 and Lemma 4 hold in some open neighborhoods of $\overline{O}_1$ and $\overline{O}_2$ and $\overline{O}_1 \cap \overline{O}_2 = \emptyset$. We set

$$SR_l(\lambda) = \big\{\big(\omega, \theta, z_l(\omega, \theta; \lambda), -x_\infty\big(\sqrt{2\lambda}\omega, z_l(\omega, \theta; \lambda)\big)\colon (\omega, \theta) \in \overline{O}_1 \times \overline{O}_2\big\}, \quad l = 1, \dots, L. \tag{9}$$

The same proof as in [10, Lemma 3.2] now shows that there exist $T_0 \gg 0$ and open sets $U^l_{\omega,\theta} \subset \omega^\perp$ with $z_l(\omega, \theta; \lambda) \in U^l_{\omega,\theta}$ for $l = 1, \dots, L$ and $(\omega, \theta) \in \overline{O}_1 \times \overline{O}_2$, such that

$$\det\left(\frac{\partial x(t; \cdot, \nabla_x \Phi_-(\cdot, \sqrt{2\lambda}\omega)(y))}{\partial y}(y)\right) \neq 0 \tag{10}$$

for $y \in \{q_\infty(s; z, \sqrt{2\lambda}\omega) \cap B(0, \overline{R}, \overline{R} + 1)\colon z \in U^l_{\omega,\theta}, s < 0\}$, $t > T_0$.

Let, now, $t_0 > T_0$ be fixed. From (10) it follows that for $(\omega, \theta) \in \overline{O}_1 \times \overline{O}_2$ we can define the (modified) action along the segment of the l-th (ω, θ)-trajectory $\gamma_l(\omega, \theta, \lambda) = (x_l(\omega, \theta, \lambda), \xi_l(\omega, \theta, \lambda)) \stackrel{\text{def}}{=} \gamma_\infty(\cdot; z_l(\omega, \theta; \lambda), \sqrt{2\lambda}\omega)$, $l = 1, \dots, L$, between the points

$$y_l(s; \omega, \theta, \lambda) \stackrel{\text{def}}{=} q_\infty\big(s; z_l(\omega, \theta; \lambda), \sqrt{2\lambda}\omega\big) \cap B(0, \overline{R}, \overline{R} + 1)$$

for some $s < 0$ and $x_l(t_0; s, \omega, \theta, \lambda) \stackrel{\text{def}}{=} q_\infty(s + t_0; z_l(\omega, \theta; \lambda), \sqrt{2\lambda}\omega)$ and we set

$$S_l(\omega, \theta) = \Phi_-(y_l(s; \omega, \theta, \lambda), \sqrt{2\lambda}\omega) + \int_0^{t_0} L(x, \dot{x})\,\mathrm{d}t - \Phi_+\big(x_l(t_0; s, \omega, \theta, \lambda), \sqrt{2\lambda}\theta\big) - \lambda t_0, \tag{11}$$

where $L(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|_g^2 - V(x)$ is the Lagrangian, and the integral is taken over the segment of the bicharacteristic curve $x_l(\omega, \theta, \lambda)$ connecting $y_l(s; \omega, \theta, \lambda)$ and $x_l(t_0; s, \omega, \theta, \lambda)$.

From the representations [10, (4.5)]

$$\Phi_-(x,\sqrt{2\lambda}\omega) = 2\tau\lambda + \int_{-\infty}^{\tau}\left(\frac{1}{2}\|p_\infty(t;z,\sqrt{2\lambda}\omega)\|^2 - V\big(q_\infty(t;z,\sqrt{2\lambda}\omega)\big) - \lambda\right)\mathrm{d}t \tag{12}$$

for $x = q_\infty(\tau;z,\sqrt{2\lambda}\omega) \in B(0,\overline{R},\overline{R}+1)$, $\tau < 0$, and [10, (4.4)]

$$\begin{aligned}\Phi_+(x,\xi) &= 2\lambda\tau + \langle x_\infty(\sqrt{2\lambda}\omega,z),\xi\rangle \\ &\quad - \int_{\tau}^{\infty}\left(\frac{1}{2}\|p_\infty(t;z,\sqrt{2\lambda}\omega)\|^2 - V\big(q_\infty(t;z,\sqrt{2\lambda}\omega)\big) - \lambda\right)\mathrm{d}t \end{aligned}\tag{13}$$

for $(x,\xi) \in \Gamma_+(R_0,d_0,-\sigma_0)$ with $x = q_\infty(\tau;z,\sqrt{2\lambda}\omega)$, $\xi = \lim_{t\to\infty} p_\infty(t;z,\sqrt{2\lambda}\omega)$, we see that $S_l(\omega,\theta)$ is independent of the choice of s with the specified properties.

We now have the following

Lemma 5. *Let $\theta_0 \in \mathbb{S}^{n-1}$ be regular for $\omega_0 \in \mathbb{S}^{n-1}$.*
Then $SR_l(\lambda) = \Lambda_{S_l}$, where $\Lambda_{S_l} = \{(\omega,\theta,d_\omega S_l,d_\theta S_l)\colon (\omega,\theta) \in \overline{O}_1 \times \overline{O}_2\}$, $l = 1,\dots,L$.

Proof. We consider

$$\begin{aligned} d_\theta S_l(\omega,\theta) &= d_\theta\left(\Phi_-\big(y_l(s;\omega,\theta,\lambda),\sqrt{2\lambda}\omega\big) + \int_0^{t_0} L(x,\dot{x})\,\mathrm{d}t\right) - d_\theta\Phi_+\big(x_l(t_0;s,\omega,\cdot,\lambda),\sqrt{2\lambda}\,\cdot\big)(\theta) \\ &= \big\langle \xi\big(t_0;y_l(s;\omega,\theta,\lambda),\nabla_x\Phi_-\big(y_l(s,\omega,\theta,\lambda),\sqrt{2\lambda}\omega\big)\big),d_\theta x_l(t_0;s,\omega,\cdot,\lambda)(\theta)\big\rangle \\ &\quad - \big\langle \nabla_x\Phi_+\big(x_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\theta\big),d_\theta x_l(t_0;s,\omega,\cdot,\lambda)(\theta)\big\rangle \\ &\quad - d_\theta\big\langle\nabla_\xi\Phi_+\big(x_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\theta\big),\sqrt{2\lambda}\,\cdot\big\rangle(\theta) \\ &= -d_\theta\big\langle\nabla_\xi\Phi_+\big(x_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\theta\big),\sqrt{2\lambda}\,\cdot\big\rangle(\theta), \end{aligned}\tag{14}$$

where (10) has allowed us to use [3, Theorem 46.C] to obtain the second equality. Lastly, we recall from [10, Lemma 4.1] and the proof of [10, Lemma 4.6] that

$$\nabla_\xi\Phi_+\big(x_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\theta\big) = x_\infty\big(\sqrt{2\lambda}\omega,z_l(\omega,\theta;\lambda)\big) + \sqrt{2\lambda}\theta(t_0+s). \tag{15}$$

To compute $d_\omega S_l$ we first reparameterize the phase trajectories in the reverse direction, which is equivalent to exchanging the roles of the initial and final conditions and reversing the directions. Using (12) and (13) we further re-write $S_l(\omega,\theta)$ in the following way

$$S_l(\omega,\theta) = -\Phi_+\big(x_l(s;\omega,\theta,\lambda),\sqrt{2\lambda}\theta\big) + \int_0^{t_0} L(x_l,\dot{x}_l)\,\mathrm{d}t + \Phi_-\big(y_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\omega\big) - \lambda t_0,$$

where $x_l(s;\omega,\theta,\lambda) \overset{\text{def}}{=} q_\infty(s;z_l(\omega,\theta;\lambda),\sqrt{2\lambda}\omega) \cap B(0,\overline{R},\overline{R}+1)$ for some $s > 0$,

$$y_l(t_0;s,\omega,\theta,\lambda) \overset{\text{def}}{=} q_\infty\big(s-t_0;z_l(\omega,\theta;\lambda),\sqrt{2\lambda}\theta\big)$$

and the integral is taken over the segment of the bicharacteristic curve $x_l(\omega,\theta,\lambda)$ connecting $x_l(s;\omega,\theta,\lambda)$ and $y_l(t_0;s,\omega,\theta,\lambda)$. We observe that this bicharacteristic curve is uniquely defined by Lemma 4 and (10).

Lemma 4 and (10) further allow us to proceed as in (14) and we obtain

$$\begin{aligned} d_\omega S_l(\omega,\theta) &= d_\omega\Big(-\Phi_+\big(x_l(s;\omega,\theta,\lambda),\sqrt{2\lambda}\theta\big)+\int_0^{t_0}L(x,\dot x)\,\mathrm{d}t\Big)\\ &\qquad + d_\omega\Phi_-\big(y_l(t_0;s,\cdot,\theta,\lambda),\sqrt{2\lambda}\,\cdot\,\big)(\omega)\\ &= d_\omega\big\langle\nabla_\xi\Phi_-\big(y_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\omega\big),\sqrt{2\lambda}\,\cdot\,\big\rangle(\omega). \end{aligned} \tag{16}$$

As above, we have that

$$\nabla_\xi\Phi_-\big(y_l(t_0;s,\omega,\theta,\lambda),\sqrt{2\lambda}\omega\big) = z_l(\omega,\theta;\lambda)+\sqrt{2\lambda}\omega(s-t_0). \tag{17}$$

From (14), (15), (16), and (17) we therefore have that S_l is a non-degenerate phase function such that $SR_l(\lambda)=\Lambda_{S_l}$. □

We remark that (12) and (13) allow us to rewrite $S_l(\omega,\theta)$ in the following way

$$\begin{aligned} S_l(\omega,\theta) &= \int_{-\infty}^{\infty}\Big(\frac{1}{2}\big\|p_\infty\big(t;z_l(\omega,\theta;\lambda),\sqrt{2\lambda}\omega\big)\big\|^2 - V\big(q_\infty\big(t;z_l(\omega,\theta;\lambda),\sqrt{2\lambda}\omega\big)\big)-\lambda\Big)\,\mathrm{d}t\\ &\qquad - \big\langle x_\infty\big(\sqrt{2\lambda}\omega,z_l(\omega,\theta;\lambda)\big),\sqrt{2\lambda}\theta\big\rangle, \end{aligned} \tag{18}$$

which is the same as the modified actions given by (3).

4. Proof of Main Theorem

We now turn to the proof of the Main Theorem.

Proof. Since $S(\lambda,h)$ is a unitary operator on $L^2(\mathbb{S}^{n-1})$, we have, by Lemma 1, that $K_{S(\lambda,h)}\in\mathcal{D}'_h(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ and therefore $K_{T(\lambda,h)}\in\mathcal{D}'_h(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$.

Since we are working away from the diagonal in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ we can use integration by parts, as in [10] and [9], and obtain

$$K_{T_{\pm1}} = \mathcal{O}_{L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\setminus\mathrm{diag}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}))}(h^\infty).$$

Therefore, by (7), we obtain

$$\mathit{WF}^f_h(K_{T_{\pm1}})=\emptyset. \tag{19}$$

The same proof as that of [10, Lemma 2.1] now gives the following estimates for $\gamma>\frac{n}{2}$ close to $\frac{n}{2}$

$$\begin{aligned} &\big\|K^*_{+a}(h)R(\lambda+\mathrm{i}0,h)K_{+b}(h)\big\|_{-\gamma,\gamma}=\mathcal{O}(h^\infty),\\ &\big\|K^*_{+a}(h)R(\lambda+\mathrm{i}0,h)(1-\chi_b)K_{-b}(h)\big\|_{-\gamma,\gamma}=\mathcal{O}(h^\infty),\\ &\big\|\big((1-\chi_a)K_{+a}\big)^*(h)R(\lambda+\mathrm{i}0,h)K_{-b}(h)\big\|_{-\gamma,\gamma}=\mathcal{O}(h^\infty), \end{aligned} \tag{20}$$

where $\chi_a \in C_c^\infty(B(0,20R_0+1))$, $\chi_a(x)=1$, $|x|<20R_0$ and $\chi_b \in C_c^\infty(B(0,10R_0+1))$, $\chi_b(y)=1$, $|y|<10R_0$.

From (19), (20), and (7) we then conclude, as in [10, Corollary], that for every $\chi \in C^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\backslash \mathrm{diag}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}))$

$$WF_h^f\big(\chi\big(K_{A(\lambda,h)}-c_1(n,\lambda,h)G_0\big)\big)=\emptyset, \tag{21}$$

where

$$G_0(\theta,\omega;\lambda,h)=\big\langle \mathrm{e}^{-\frac{\mathrm{i}}{h}\Phi_+(\cdot,\sqrt{2\lambda}\theta)}g_{+a}(\cdot,\theta;h)\otimes \mathrm{e}^{\frac{\mathrm{i}}{h}\Phi_-(\cdot\cdot,\sqrt{2\lambda}\omega)}g_{-b}(\cdot\cdot,\omega;h),K_{R(\lambda+\mathrm{i}0,h)}\big\rangle,$$
$$g_{+a}(x,\theta;h)=\mathrm{e}^{-\frac{\mathrm{i}}{h}\Phi_+(x,\sqrt{2\lambda}\theta)}\big[\chi_a,P_0(h)\big]a_+\big(x,\sqrt{2\lambda}\theta;h\big)\,\mathrm{e}^{\frac{\mathrm{i}}{h}\Phi_+(x,\sqrt{2\lambda}\theta)},$$
$$g_{-b}(y,\omega;h)=\mathrm{e}^{-\frac{\mathrm{i}}{h}\Phi_-(y,\sqrt{2\lambda}\omega)}\big[\chi_b,P_0(h)\big]b_-\big(y,\sqrt{2\lambda}\omega;h\big)\,\mathrm{e}^{\frac{\mathrm{i}}{h}\Phi_-(y,\sqrt{2\lambda}\omega)},$$

and

$$c_1(n,\lambda,h)=2\pi(2\lambda)^{\frac{n-3}{4}}(2\pi h)^{-\frac{n+1}{2}}\,\mathrm{e}^{-\frac{\mathrm{i}(n-3)\pi}{4}}.$$

Let, now, $\bar{p}\in SR_U(\lambda)$ be such that $\tilde{\pi}_1(\bar{p})=(\omega,z)$, where $\tilde{\pi}_1: T^*\mathbb{S}^{n-1}\times T^*\mathbb{S}^{n-1}\to T^*\mathbb{S}^{n-1}$ is the canonical projection onto the first factor. Let $A_j\in \Psi_h^0(1,\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\backslash\mathrm{diag}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}))$, $j=0,\dots,N$, have compact wavefront sets near $\bar{p}$ and satisfy $\sigma_0(A_j)|_{SR_{\overline{U}}(\lambda)}=0$, $j<N$. We also set $\varphi_+(x,\theta)=\Phi_+(x,\sqrt{2\lambda}\theta)$, $(x,\sqrt{2\lambda}\theta)\in\Gamma_+(R_0,d_0,\sigma_0)$, and $\varphi_-(y,\omega)=\Phi_-(y,\sqrt{2\lambda}\omega)$, $(y,\sqrt{2\lambda}\omega)\in\Gamma_-(R_0,d_0,-\sigma_0)$. First, we shall prove that the generalization of Egorov's Theorem to manifolds of unequal dimensions [2, Theorem 2] can be applied to the semi-classical Fourier integral operator given by the Schwartz kernel

$$\mathrm{e}^{-\frac{\mathrm{i}}{h}\varphi_+}g_{+a}\otimes\mathrm{e}^{\frac{\mathrm{i}}{h}\varphi_-}g_{-b},$$

associates to the Lagrangian submanifold of $T^*\mathbb{R}^n\times T^*\mathbb{R}^n$

$$\begin{aligned}\Lambda(\lambda)=\big\{&\big(x,y,-\nabla_x\varphi_+(x,\theta),\nabla_y\varphi_-(y,\omega);\theta,\omega,-\nabla_\theta\varphi_+(x,\theta),\nabla_\omega\varphi_+(y,\omega)\colon\\ &\big(x,\sqrt{2\lambda}\theta\big)\in\Gamma_+(R_0,d_0,\sigma_0)\cap\big(T^*(\mathrm{supp}\,\nabla\chi_a)\times T^*\mathbb{S}^{n-1}_{2\lambda}\big),\\ &\big(y,\sqrt{2\lambda}\omega\big)\in\Gamma_-(R_0,d_0,-\sigma_0)\cap\big(T^*(\mathrm{supp}\,\nabla\chi_b)\times T^*\mathbb{S}^{n-1}_{2\lambda}\big)\big)\big\}.\end{aligned}$$

For every $(x,\xi)\in\Gamma_\pm(R_0,d_0,\pm\sigma_0)$ there exist unique phase trajectories $(q_\pm(\cdot;x,\xi),p_\pm(\cdot;x,\xi))$ such that $q_\pm(0;x,\xi)=x$ and $\lim_{t\to\pm\infty}p_\pm(t;x,\xi)=\xi$, respectively (see [10, Subsection 4.1] as well as the discussion following [7, Definition 1.10]). Furthermore, by the construction of $\Phi_\pm$,

$$\nabla_x\Phi_\pm\big(q_\pm(t;x,\xi),\xi\big)=p_\pm(t;x,\xi).$$

By [10, Lemma 4.1], we also have that

$$\lim_{t\to\pm\infty}\big\|q_\pm(t;x,\xi)-\xi t-\nabla_\xi\Phi_\pm(x,\xi)\big\|=0.$$

These considerations imply that

$$\pi_1|_{\Lambda(\lambda)} \text{ is an immersion,} \tag{22}$$

where

$$\pi_1 : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1} \to T^*\mathbb{R}^n \times T^*\mathbb{R}^n$$

is the canonical projection. With (22) the hypotheses of the generalization of Egorov's Theorem to manifolds of unequal dimensions [2, Theorem 2] are satisfied and applying [2, Theorem 2] we obtain that there exist $B_j \in \Psi_h^0(1, \mathbb{R}^n \times \mathbb{R}^n)$, $j = 0, \ldots, N$, satisfying the following conditions

1. B_j, $j = 0, \ldots, N$, have compact wavefront sets near a point $\bar{q} \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ such that $\hat{\pi}_1(\bar{q}) \in \gamma_\infty(\cdot; z, \sqrt{2\lambda}\theta)$, where $\hat{\pi}_1 : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is the canonical projection onto the first factor.
2. $\sigma_0(B_j)|_{\Lambda_R(\lambda)} = 0, j < N$, where $\Lambda_R(\lambda) = \bigcup_{t>0}(\operatorname{graph}\exp(tH_p)|_{\Sigma_\lambda})'$.
3. Near $(\bar{p}, \bar{q})$,

$$\left(\prod_{j=0}^{N} A_j\right)\left(\mathrm{e}^{-\frac{\mathrm{i}}{h}\varphi_+} g_{+a} \otimes \mathrm{e}^{\frac{\mathrm{i}}{h}\varphi_-} g_{-b}\right) \equiv \left(\mathrm{e}^{-\frac{\mathrm{i}}{h}\varphi_+} g_{+a} \otimes \mathrm{e}^{\frac{\mathrm{i}}{h}\varphi_-} g_{-b}\right)\left(\prod_{j=0}^{N} B_j\right). \tag{23}$$

Assumption (4) and Lemma 1, now, imply that $K_{R(\lambda+\mathrm{i}0,h)} \in \mathcal{D}_h'(\mathbb{R}^{2n})$. From (23) we therefore obtain

$$\left(\prod_{j=0}^{N} A_j\right) K_{A(\lambda,h)} \equiv c_1(n,\lambda,h)\left(\mathrm{e}^{-\frac{\mathrm{i}}{h}\varphi_+} g_{+a} \otimes \mathrm{e}^{\frac{\mathrm{i}}{h}\varphi_-} g_{-b}\right)\left(\prod_{j=0}^{N} B_j\right)(\chi_2 \otimes \chi_1) K_{R(\lambda+\mathrm{i}0,h)}, \tag{24}$$

near $(\bar{p}, \bar{q})$, where $\chi_j \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$, $j = 1, 2$, are such that $\chi_2 = 1$ on $\operatorname{supp} g_{+a}$, $\chi_1 = 1$ on $\operatorname{supp} g_{-b}$, and $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$.

Estimate (4), Lemma 2, and the same proof as in [1, Theorem 2] further give that there exists an open set $Z \subset \Lambda_R(\lambda)$, $\bar{q} \in Z$, such that $(\chi_2 \otimes \chi_1) K_{R(\lambda+\mathrm{i}0,h)} \in I_h^1(\mathbb{R}^{2n}, \Lambda_R(\lambda) \cap \overline{Z})$. Therefore

$$\left(\prod_{j=0}^{N} B_j\right)(\chi_2 \otimes \chi_1) K_{R(\lambda+\mathrm{i}0,h)} = \mathcal{O}_{L^2(\mathbb{R}^{2n})}\left(h^{N-1-\frac{n}{2}}\right), \quad h \to 0. \tag{25}$$

Since $g_{+b}, g_{-a} \in S_{2n-1}^{-1}(1) \cap C_c^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$, we easily find that

$$\left\|\left(\mathrm{e}^{-\frac{\mathrm{i}}{h}\varphi_+} g_{+a} \otimes \mathrm{e}^{\frac{\mathrm{i}}{h}\varphi_-} g_{-b}\right)\right\|_{\mathcal{B}(L^2(\mathbb{R}^n \times \mathbb{R}^n), L^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}))} = \mathcal{O}\left(h^2\right). \tag{26}$$

Estimates (25) and (26) together with (21) and (24) now imply that

$$\left(\prod_{j=0}^{N} A_j\right) K_{A(\lambda,h)} = \mathcal{O}_{L^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})}\left(h^{N-n+\frac{1}{2}}\right),$$

and therefore

$$A(\lambda,h) \in \mathcal{I}_h^{\frac{n}{2}}\left(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \backslash \operatorname{diag}\left(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}\right), SR_{\overline{U}}(\lambda)\right). \quad \square$$

5. Applications

In this section we discuss two applications of our Main Theorem to trapping and non-trapping energies, respectively.

5.1. Non-trapping energies

Corollary 1. *Let $\lambda > 0$ be a non-trapping energy level for P such that $P(h) - \lambda$ is of principal type. Then $A(\lambda, h) \in \mathcal{I}_h^{\frac{n}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \backslash \mathrm{diag}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), SR(\lambda))$.*

Proof. From [10, Lemma 2.2] we have that $\|R(\lambda + \mathrm{i}0, h)\|_{\alpha,-\alpha} = \mathcal{O}(\frac{1}{h}), \alpha > \frac{1}{2}$. The result now follows from the Main Theorem. □

5.2. Trapping energies

Corollary 2. *Let $\lambda > 0$ be a trapping energy level for P such that $P(h) - \lambda$ is of principal type. Let also*

(i) *there exist $\theta_0 \in [0, \pi)$, $R > 0$ such that the potential V extends holomorphically to the domain $D_{R,\theta_0} = \{z \in \mathbb{C}^n\colon \|z\| > R, |\Im z| \leqslant \tan\theta_0 |\Re z|\}$ and $|V(x)| \leqslant C|x|^{-\beta}$ for all $x \in D_{R,\theta_0}$ and some $\beta > 0$, $C > 0$, and*

(ii) *$\mathrm{Res}(P(h)) \cap ([\lambda - \varepsilon, \lambda + \varepsilon] + \mathrm{i}[0, Ch^M]) = \emptyset$ for some $\varepsilon > 0$, $C > 0$, and $M > 0$.*

Lastly, let there exist $(\theta, z) \in T^*\mathbb{S}^{n-1}$ such that $\gamma_\infty(\cdot; z, \sqrt{2\lambda}\theta)$ is a non-trapped trajectory. Then there exists an open set $U \subset T^*\mathbb{S}^{n-1}$ such that

$$A(\lambda, h) \in \mathcal{I}_h^{\frac{n}{2}}\bigl(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \backslash \mathrm{diag}\bigl(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}\bigr), SR_{\overline{U}}(\lambda)\bigr).$$

Proof. We choose U as in (8). From [9, Proposition 4.1], we have that there exists $m \in \mathbb{N}$ such that

$$\|R(\lambda + \mathrm{i}0, h)\|_{\alpha,-\alpha} = \mathcal{O}\left(\frac{1}{h^m}\right), \quad \alpha > \frac{1}{2}.$$

The assertion of the corollary now follows from the Main Theorem. □

5.3. Microlocal representation of the scattering amplitude

Here we show how under the non-degeneracy assumption the expansion (2) follows from the results we have proved in this article and the characterization of semi-classical Fourier integral distributions as oscillatory integrals, which we have developed in [2, Theorem 1]. More precisely, we have the following

Theorem 1. *Let $\theta_0 \in \mathbb{S}^{n-1}$ be regular for $\omega_0 \in \mathbb{S}^{n-1}$ and $L \in \mathbb{N}$ be the number of (ω_0, θ_0) phase trajectories. Let $\lambda > 0$ be such that $P - \lambda$ is of principle type and $\|R(\lambda + \mathrm{i}0, h)\|_{\alpha,-\alpha} = \mathcal{O}(h^m)$, $m \in \mathbb{R}$, $\alpha > \frac{1}{2}$.*

Then there exist $a_l \in S^{n-\frac{1}{2}}_{2n-2}(1)$, $l = 1, \ldots, L$, *such that*

$$K_{A(\lambda,h)} = \mathrm{e}^{\frac{\mathrm{i}}{h}S_l} a_l, \quad l = 1, \ldots, L,$$

microlocally near $SR_l(\lambda)$, *respectively, where* S_l, $l = 1, \ldots, L$, *are as given by* (11).

Proof. By our Main Theorem, $A(\lambda, h) \in \mathcal{I}_h^{\frac{n}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \bigcup_{l=1}^{L} SR_l(\lambda))$. With this and Lemma 5 the hypotheses of [2, Theorem 1] are satisfied and we obtain that there exist $a_l \in S^{n-\frac{1}{2}}_{2n-2}(1)$, $l = 1, \ldots, L$, such that $K_{A(\lambda,h)} = \mathrm{e}^{\frac{\mathrm{i}}{h}S_l} a_l$ microlocally near $SR_l(\lambda)$, $l = 1, \ldots, L$. □

We remark that the conclusion of this theorem holds whenever we have a polynomial bound on the resolvent. We also remark that this theorem recovers the phases (3) in (2), due to (18).

Appendix A. Elements of semi-classical analysis

In this section we recall some of the elements of semi-classical analysis which we use in this paper. First we review the definitions of the following two classes of symbols

$$S^m_{2n}(1) = \left\{a \in C^\infty\left(\mathbb{R}^{2n} \times (0, h_0]\right) \colon \forall \alpha, \beta \in \mathbb{N}^n, \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)\right| \leqslant C_{\alpha,\beta} h^{-m}\right\}$$

and

$$S^{m,k}\left(T^*\mathbb{R}^n\right) = \Big\{a \in C^\infty\left(T^*\mathbb{R}^n \times (0, h_0]\right) \colon \forall K \Subset \mathbb{R}^n, \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x \in K}\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)\right|$$
$$\leqslant C_{K,\alpha,\beta} h^{-m} \langle\xi\rangle^{k-|\beta|},$$

where $h_0 \in (0, 1]$ and $m, k \in \mathbb{R}$. For $a \in S^m_{2n}(1)$ or $a \in S^{m,k}(T^*\mathbb{R}^n)$ we define the corresponding semi-classical pseudodifferential operator of class $\Psi^m_h(1, \mathbb{R}^n)$ or $\Psi^{m,k}_h(\mathbb{R}^n)$, respectively, by setting

$$Op_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint \mathrm{e}^{\frac{\mathrm{i}\langle x-y,\xi\rangle}{h}} a(x, \xi; h) u(y) \, \mathrm{d}y \, \mathrm{d}\xi, \quad u \in \mathcal{S}\left(\mathbb{R}^n\right),$$

and extending the definition to $\mathcal{S}'(\mathbb{R}^n)$ by duality (see [4]). Here we work only with symbols which admit asymptotic expansions in h and with pseudodifferential operators which are quantizations of such symbols. For $A \in \Psi^m_h(1, \mathbb{R}^n)$ or $A \in \Psi^{m,k}_h(\mathbb{R}^n)$ we shall use $\sigma_0(A)$ to denote its principal symbol. A semi-classical pseudodifferential operator is said to be of principal type if its principal symbol a_0 satisfies

$$a_0 = 0 \Leftrightarrow da_0 \neq 0.$$

For $a \in S^{m,k}(T^*\mathbb{R}^n)$ or $a \in S^m_{2n}(1)$ we define

$$\begin{aligned}\operatorname*{ess\text{-}supp}_h a = &\left\{(x,\xi)\in T^*\mathbb{R}^n \mid \exists\varepsilon>0,\ \partial_x^\alpha\partial_\xi^\beta a(x',\xi') = \mathcal{O}_{C(B((x,\xi),\varepsilon))}\big(h^\infty\big),\ \forall\alpha,\beta\in\mathbb{N}^n\right\}^c\\ &\cup\Bigg(\bigg\{(x,\xi)\in T^*\mathbb{R}^n\backslash\{0\} \mid \exists\varepsilon>0,\ \partial_x^\alpha\partial_\xi^\beta a(x',\xi') = \mathcal{O}\big(h^\infty\langle\xi\rangle^{-\infty}\big),\\ &\quad\text{uniformly in } (x',\xi') \text{ such that } \|x-x'\| + \frac{1}{\|\xi'\|} + \left\|\frac{\xi}{\|\xi\|} - \frac{\xi'}{\|\xi'\|}\right\| < \varepsilon\bigg\}\Big/\mathbb{R}_+\Bigg)^c\\ &\subset T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n,\end{aligned}$$

where we define $S^*\mathbb{R}^n = (T^*\mathbb{R}^n\backslash\{0\})/\mathbb{R}_+$ and denote by $\bullet^c$ the complement of the set $\bullet$. For $A\in \Psi_h^{m,k}(\mathbb{R}^n)$ or $A\in\Psi_h^m(1,\mathbb{R}^n)$ we then define

$$WF_h(A) = \operatorname*{ess\text{-}supp}_h a,\quad A = Op_h(a).$$

We also define the class of semi-classical distributions $\mathcal{D}_h'(\mathbb{R}^n)$ with which we will work here

$$\begin{aligned}\mathcal{D}_h'(\mathbb{R}^n) = \big\{&u\in C_h^\infty\big((0,1];\mathcal{D}'\big(\mathbb{R}^n\big)\big)\colon \forall\chi\in C_c^\infty\big(\mathbb{R}^n\big)\ \exists N\in\mathbb{N} \text{ and } C_N>0\colon\\ &\big|\mathcal{F}_h(\chi u)(\xi)\big| \leqslant C_N h^{-N}\langle\xi\rangle^N\big\},\end{aligned}$$

where

$$\mathcal{F}_h(\chi u)(\xi) = \big\langle \mathrm{e}^{-\frac{\mathrm{i}}{h}\langle\cdot,\xi\rangle}, \chi u\big\rangle,$$

and $\langle\cdot,\cdot\rangle$ denotes the distribution pairing. We also extend this definition in the obvious way to $\mathcal{E}_h'(\mathbb{R}^n)$.

The L^2-based semi-classical Sobolev spaces $H_h^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, which consist of the distributions $u\in\mathcal{E}_h'(\mathbb{R}^n)$ such that $\|u\|^2_{H_h^s(\mathbb{R}^n)} \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}(1+\|\xi\|^2)^s|\mathcal{F}_h(u)(\xi)|^2\,\mathrm{d}\xi<\infty$.

For $u\in\mathcal{D}_h'(\mathbb{R}^n)$ we also define its finite semi-classical wavefront set $WF_h^f(u)$ as follows.

Definition 4. Let $u\in\mathcal{D}_h'(\mathbb{R}^n)$. The point $(x_0,\xi_0)\in T^*(\mathbb{R}^n)$ does not belong to $WF_h^f(u)$ if there exist $\chi\in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0)\neq 0$ and an open neighborhood U of ξ_0, such that $\forall N\in\mathbb{N}$, $\forall\xi\in U$, $|\mathcal{F}_h(\chi u)(\xi)|\leqslant C_N h^N$.

We say that $u = v$ *microlocally* (or $u\equiv v$) near an open set $U\subset T^*\mathbb{R}^n$, if $P(u-v)=\mathcal{O}(h^\infty)$ in $C_c^\infty(\mathbb{R}^n)$ for every $P\in\Psi_h^0(1,\mathbb{R}^n)$ such that

$$WF_h(P)\subset\widetilde{U},\quad \overline{U}\Subset\widetilde{U}\Subset T^*\mathbb{R}^n,\ \widetilde{U}\text{ open}.$$

We also say that u satisfies a property $\mathcal{P}$ *microlocally* near an open set $U\subset T^*\mathbb{R}^n$ if there exists $v\in\mathcal{D}_h'(\mathbb{R}^n)$ such that $u=v$ microlocally near U and v satisfies property $\mathcal{P}$.

We extend these notions to compact manifolds through the following definition of semi-classical pseudodifferential operators on compact manifolds. Let M be a smooth compact manifold and $\kappa_j: M_j\to X_j$, $j=1,\dots,N$, a set of local charts. A linear continuous operator $A: C^\infty(M)\to\mathcal{D}_h'(M)$ belongs to $\Psi_h^m(1,M)$ or $\Psi_h^{m,k}(T^*M)$ if for all $j\in\{1,\dots,N\}$ and $u\in C_c^\infty(M_j)$ we have $Au\circ\kappa_j^{-1} =$

$A_j(u \circ \kappa_j^{-1})$ with $A_j \in \Psi_h^m(1, X_j)$ or $A_j \in \Psi_h^{m,k}(X_j)$, respectively, and $\chi_1 A \chi_2 : \mathcal{D}'_h(M) \to h^\infty C^\infty(M)$ if $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$.

We now define global semi-classical Fourier integral operators.

Definition 5. Let M be a smooth k-dimensional manifold and let $\Lambda \subset T^*M$ be a smooth closed Lagrangian submanifold with respect to the canonical symplectic structure on T^*M. Let $r \in \mathbb{R}$. Then the space $I_h^r(M, \Lambda)$ of semi-classical Fourier integral distributions of order r associated to Λ is defined as the set of all $u \in \mathcal{E}'_h(M)$ such that

$$\left(\prod_{j=0}^{N} A_j\right)(u) = \mathcal{O}_{L^2(M)}\left(h^{N-r-\frac{k}{4}}\right), \quad h \to 0,$$

for all $N \in \mathbb{N}_0$ and for all $A_j \in \Psi_h^0(1, M)$, $j = 0, \ldots, N-1$, with compact wavefront sets and principal symbols vanishing on Λ, and any $A_N \in \Psi_h^0(1, M)$ with compact wavefront set.

A continuous linear operator $T : C_c^\infty(M_1) \to \mathcal{D}'_h(M_2)$, where M_1, M_2 are smooth manifolds, with $K_T \in I_h^r(M_1 \times M_2, \Lambda)$ for some Lagrangian submanifold $\Lambda \subset T^*M_1 \times T^*M_2$ and some $r \in \mathbb{R}$ will be called a global semi-classical Fourier integral operator of order r associated to Λ. We denote the space of these operators by $\mathcal{I}_h^r(M_1 \times M_2, \Lambda)$.

If $X \subset M_1 \times M_2$ is an open set, we shall also write $T \in \mathcal{I}_h^r(X, \Lambda)$ to indicate that $K_T|_X \in I_h^r(X, \Lambda)$, where $\Lambda \subset T^*X$ is a Lagrangian submanifold.

Lastly, we define the microlocal equivalence of two semi-classical Fourier integral operators.

Definition 6. Let M_j, $j = 1, 2$, be smooth manifolds, $\Lambda \subset T^*M_1 \times T^*M_2$ – a Lagrangian submanifold, and $T, T' \in \mathcal{I}_h^r(M_1 \times M_2, \Lambda)$ for some $r \in \mathbb{R}$. For open or closed sets $U \subset T^*M_1$ and $V \subset T^*M_2$ the operators T and T' are said to be *microlocally equivalent* near $U \times V$ if there exist open sets $\widetilde{U} \Subset T^*M_1$ and $\widetilde{V} \Subset T^*M_2$ with $\overline{U} \Subset \widetilde{U}$ and $\overline{V} \Subset \widetilde{V}$ such that for any $A \in \Psi_h^0(1, M_1)$ and $B \in \Psi_h^0(1, M_2)$ with $WF_h(A) \subset \widetilde{U}$ and $WF_h(B) \subset \widetilde{V}$ we have that

$$B(T - T')A = \mathcal{O}(h^\infty) : \mathcal{D}'_h(M_1) \to C^\infty(M_2).$$

We shall also write $T \equiv T'$ near $V \times U$.

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