Semi-Classical-Fourier-Integral-Operator-Valued Pseudodifferential Operators and Scattering in a Strong Magnetic Field

Ivana Alexandrova

1400 Washington Avenue, Earth Sciences 110, Department of Mathematics and Statistics, State University of New York, Albany, NY 12222, 518-437-4489, ialexandrova@albany.edu

**Abstract**

We analyze the microlocal structure of the semi-classical scattering amplitude for Schrödinger operators with a strong magnetic and a strong electric fields at non-trapping energies. For this purpose we develop a framework and establish some of the properties of semi-classical-Fourier-integral-operator-valued pseudodifferential operators and prove that the leading term of the scattering amplitude is given by such an operator.

# 1 Introduction

We consider the scattering amplitude for a Schrödinger operator with a strong magnetic and a strong electric fields in the semi-classical limit. To study its microlocal structure we develop a framework of semi-classical-Fourier-integral-operator-valued pseudodifferential operators. We prove that the leading term of the scattering amplitude is given by such a semi-classical-Fourier-integral-operator-valued pseudodifferential operator. We also show how our results formalize an instance of Bohr's quantum-classical correspondence principle by relating, in the semi-classical limit, the scattering amplitude (a quantum object) and the phase functions parameterizing the scattering relation (classical objects).

For $n \geq 3$ and $b > 0$ let $H(b) = H_0(b) + bV$, where the electric potential $V$ satisfies

$$V = V(x, y, z) = V^{\infty}(z) + W(x, y, z), W \in C_c^{\infty}(\mathbb{R}^n), V^{\infty} \in C_c^{\infty}\left(\mathbb{R}^{n-2d}\right), V^{\infty}, W \geq 0, \tag{1}$$

$$H_0(b) = \left|i\nabla_w - bA(w)\right|^2,$$

and the magnetic potential $A \in C^{\infty}(\mathbb{R}^n; \mathbb{R}^n)$ generates the magnetic field $B$, given by the anti-symmetric matrix $B(w) \overset{def}{=} \left(\partial_{w_i} A_j(w) - \partial_{w_j} A_i(w)\right)_{i,j}$, which is independent of $w$. We let $2d = \dim RanB$ and consider the case $0 < 2d < n$. Under this assumption there exist Cartesian coordinates in which $H_0$ and $H$ take the forms

$$H_0(b) = \sum_{j=1}^{d}\left[\left(D_{x_j} - \frac{b\mu_j}{2}y_j\right)^2 + \left(D_{y_j} + \frac{b\mu_j}{2}x_j\right)^2\right] - \Delta_z$$

and

$$H(b) = \sum_{j=1}^{d}\left[\left(D_{x_j} - \frac{b\mu_j}{2}y_j\right)^2 + \left(D_{y_j} + \frac{b\mu_j}{2}x_j\right)^2\right] - \Delta_z + bV(x, y, z) \tag{2}$$

where $(x, y, z) = (x_1, \ldots, x_d, y_1, \ldots, y_d, z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{n-2d}$ and $\mu_j > 0$, $j = 1, \ldots, d$. The parameter $b$ is called the strength of the magnetic field. Throughout this discussion we will use it interchangeably with the semi-classical parameter $h$, which we define as $h = \frac{1}{\sqrt{b}}$. We further let $\mathcal{K} \subset \mathbb{R}^{2d}_{x,y}$ and $K \subset \mathbb{R}^{n-2d}_z$ be compact sets such that $supp\, W \subset \mathcal{K} \times K$ and $supp\, V^{\infty} \subset K$.

It follows from the results of [6] that the wave operators for the pair $(H_0(b), H(b))$ defined by

$$W_{\pm} = \text{s-}\lim_{t \to \pm\infty} U(t)U_0(-t) \text{ in } L^2(\mathbb{R}^n)$$

exist and are complete, where

$$U(t) = e^{-\frac{i}{h}tH(b)},\ U_0(t) = e^{-\frac{i}{h}tH_0(b)},\ t \in \mathbb{R}.$$

We can therefore define the scattering operator $S$ by setting $S = W_+^* W_-$.

To define the scattering amplitude and give its representation which we will use to study its microlocal structure, we need to introduce some terminology and notation. The Schödinger operator with constant magnetic field on $L^2\left(\mathbb{R}^{2d}\right)$ given by

$$\hat{H}_0(b) = \sum_{j=1}^{d} \left[ \left( D_{x_j} - \frac{b\mu_j}{2} y_j \right)^2 + \left( D_{y_j} + \frac{b\mu_j}{2} x_j \right)^2 \right].$$

The spectrum of $\hat{H}_0(b)$ consists of the Landau levels $\mathbb{L} \stackrel{def}{=} \left\{ b\Lambda_Q : Q \in \mathbb{N}_0^d \right\}$, where for $Q = (q_1, \ldots, q_d) \in \mathbb{N}_0^d$ we have set $\Lambda_Q = (2q_1+1)\mu_1 + \cdots + (2q_d+1)\mu_d$. For $\lambda > 0$ we define the set $\mathcal{L}(\lambda) = \{Q \in \mathbb{N}_0^d : \Lambda_Q < \lambda\}$ and its cardinality $N(\lambda) = |\mathcal{L}(\lambda)|$. Let $\left\{Q_1, \ldots, Q_{N(\lambda)}\right\}$ be an enumeration of the elements of $\mathcal{L}(\lambda)$, which we fix for the remainder of the paper. We further let $K(\lambda) = |\{\Lambda_Q : Q \in \mathcal{L}(\lambda)\}|$ and $\{\lambda_1, \ldots, \lambda_{K(\lambda)}\} \equiv \{\Lambda_Q : Q \in \mathcal{L}(\lambda)\}$, where for all $k$ we have set $\lambda_k < \lambda_{k+1}$. For $k = 1, \ldots, K(\lambda)$, we also let $\Pi_k : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ denote the projection onto the eigenspace of $\hat{H}_0(b)$ associated to the eigenvalue $b\lambda_k$.

For $\varphi \in L^2_\alpha(\mathbb{R}^{n-2d})$ (see Section 2), $E > 0$, and $\xi \in \mathbb{S}^{n-2d-1}$ we set

$$\left(\hat{\mathcal{F}}_0(E)\varphi\right)(\xi) = \frac{E^{\frac{n-2d-2}{4}}}{\sqrt{2}\,(2\pi)^{\frac{n-2d}{2}}} \int_{\mathbb{R}^{n-2d}} e^{-i\sqrt{E}\langle z,\xi\rangle} \varphi(z) dz.$$

For $E > b\Lambda_0 = b(\mu_1 + \cdots + \mu_d)$ we also set

$$\begin{aligned} \mathcal{F}_0(E) : \quad L^2\left(\mathbb{R}^{2d}_{x,y}, L^2_\alpha\left(\mathbb{R}^{n-2d}_z\right)\right) &\to L^2\left(\mathbb{R}^{2d}_{x,y}, L^2\left(\mathbb{S}^{n-2d-1}\right)\right) \\ \varphi &\mapsto \textstyle\sum_{k=1}^{K(E)} \left(\Pi_k \otimes \hat{\mathcal{F}}_0\left(E - b\lambda_k\right)\right)\varphi. \end{aligned}$$

To define the scattering amplitude we recall that the absolutely continuous spectrum of $H_0(b)$ is $(b\Lambda_0, \infty)$ and that for the operator

$$\mathcal{F} : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}_+^*, L^2\left(\mathbb{R}^{2d} \times \mathbb{S}^{n-2d-1}\right), dE\right)$$

defined by $(\mathcal{F}\varphi)(E) = \mathcal{F}_0(E)\varphi$, we have that $\mathcal{F}H_0\mathcal{F}^*$ is the multiplication by $E$ on $L^2\left(\mathbb{R}_+^*, L^2\left(\mathbb{R}^{2d} \times \mathbb{S}^{n-2d-1}\right)\right)$ and that for $t > 0$ the operators $\mathcal{F}S(b)\mathcal{F}^*$ and $e^{it\mathcal{F}H_0\mathcal{F}^*}$ commute. Therefore, for $E > b\Lambda_0$, there exists an operator

$$S(E,b) : L^2\left(\mathbb{R}^{2d} \times \mathbb{S}^{n-2d-1}\right) \to L^2\left(\mathbb{R}^{2d} \times \mathbb{S}^{n-2d-1}\right)$$

satisfying

$$S(E,b)\mathcal{F}_0(E) = \mathcal{F}_0(E)S(b).$$

We further have that for $E \in (b\Lambda_0, \infty) \setminus (\mathbb{L} \cup \sigma_{pp}(H(b)))$ the scattering amplitude at energy $E$, i.e., the operator $\mathcal{T}(E,b) \stackrel{def}{=} S(E,b) - I$, is given by

$$\mathcal{T}(E,b) = -2i\pi \mathcal{F}_0(E)[\Delta_z, \chi_1] R(E+i0) [\Delta_z, \chi_2] \mathcal{F}_0(E)^*, \tag{3}$$

where $\chi_j \in C_c^\infty\left(\mathbb{R}^{n-2d}_z\right)$ are such that $1_K \prec \chi_2 \prec \chi_1$, $R(E+i0) = \lim_{\epsilon \downarrow 0} \left(H(b) - (E+i\epsilon)\right)^{-1}$ with the limit taken in the space $\mathcal{B}\left(L^2_\alpha\left(\mathbb{R}^n\right), L^2_{-\alpha}\left(\mathbb{R}^n\right)\right)$ (see Section 2 and [16, Remark 1.1]).

**Remark.** We note that the representation of the scattering amplitude $\mathcal{T}(E,b)$ given in (3) is independent of the choice of functions $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^{n-2d})$ as long as they satisfy the property $1_K \prec \chi_2 \prec \chi_1$. This can be seen through the following argument. Let $\tilde{\chi}_2 \in C^\infty(\mathbb{R}^{n-2d})$ be such that $1_K \prec \tilde{\chi}_2 \prec \chi_1$. Then

$$\begin{aligned} &\mathcal{F}_0(E)\left[\Delta_z, \chi_1\right] R(E+i0) [\Delta_z, \tilde{\chi}_2 - \chi_2] \mathcal{F}_0(E)^* \\ &= \mathcal{F}_0(E)\left[\Delta_z, \chi_1\right] R(E+i0) \left( \left(H(b) - E\right)\left(\tilde{\chi}_2 - \chi_2\right) - \left(\tilde{\chi}_2 - \chi_2\right)\left(\Delta_z + \hat{H}_0(b) - E\right)\right) \mathcal{F}_0(E)^* \\ &= \mathcal{F}_0(E)\left[\Delta_z, \chi_1\right] R(E+i0) \left(H(b) - E\right)\left(\tilde{\chi}_2 - \chi_2\right) \mathcal{F}_0(E)^* \\ &\quad - \mathcal{F}_0(E)\left[\Delta_z, \chi_1\right] R(E+i0) \left(\tilde{\chi}_2 - \chi_2\right)\left(\Delta_z + \hat{H}_0(b) - E\right) \mathcal{F}_0(E)^* = 0, \end{aligned}$$

since for the first term in the last expression above we have

$$[\Delta_z, \chi_1]\, R(E+i0)\,(H(b)-E)\,(\tilde{\chi}_2 - \chi_2) = [\Delta_z, \chi_1]\,(\tilde{\chi}_2 - \chi_2) = 0$$

and for the second term above we have

$$\left(\Delta_z + \hat{H}_0(b) - E\right)\mathcal{F}_0(E)^* = \sum_{k=1}^{K(E)} \left(\Delta_z + \hat{H}_0(b) - E\right)\left(\Pi_k \otimes \hat{\mathcal{F}}_0\left(E - b\lambda_k\right)^*\right) = 0.$$

The independence of the representation of $\mathcal{T}(E,b)$ of the choice of $\chi_1$ with the property above is proven similarly. $\square$

Here we study the structure of the scattering amplitude at energies $b\lambda > b\Lambda_0$ such that $b\lambda \neq b\lambda_k$ for all $k = 1, \ldots, K(\lambda)$, as the strength of the magnetic field $b$ goes to infinity.

The structure of the scattering amplitude associated to a Schrödinger operator with an electric potential has been of considerable interest to researchers in mathematical physics. To review these results we begin by introducing some notation which we will also use later in this paper.

Let $P(h) = -\frac{h^2}{2}\Delta_z + V(z)$, where $V \in C^\infty\left(\mathbb{R}^{n-2d}; \mathbb{R}\right)$, $n \geq 4$, $0 < 2d \leq n-2$, is a short range potential, i.e., there exists $\rho > 1$ such that for every $\alpha \in \mathbb{N}_0^{n-2d}$

$$|\partial_z^\alpha V(z)| \leq C_\alpha \langle z\rangle^{-\rho-|\alpha|} \tag{4}$$

with $\langle z\rangle = (1+\|z\|^2)^{\frac{1}{2}}$. The Hamiltonian vector field of the symbol $p(z,\xi) = \frac{1}{2}\|\xi\|^2 + V(z)$ of $P$ is defined as $H_p = \sum_{j=1}^{n-2d}\left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial z_j} - \frac{\partial p}{\partial z_j}\frac{\partial}{\partial \xi_j}\right)$.

Let $\lambda > 0$ and for $\omega \in \mathbb{S}^{n-2d-1}$ and $w \in \omega^\perp$ we denote by

$$\gamma_\infty\left(\cdot; w, \sqrt{2\lambda}\omega\right) = \left\{q_\infty\left(\cdot; w, \sqrt{2\lambda}\omega\right), p_\infty\left(\cdot; w, \sqrt{2\lambda}\omega\right)\right\}$$

the unique phase trajectory, i. e. integral curve of $H_p$, such that

$$\lim_{t\to-\infty}\left(q_\infty\left(t; w, \sqrt{2\lambda}\omega\right) - \sqrt{2\lambda}\omega t - w\right) = 0,$$
$$\lim_{t\to-\infty}\left(p_\infty\left(t; w, \sqrt{2\lambda}\omega\right) - \sqrt{2\lambda}\omega\right) = 0$$

in the $C^\infty$ topology for $(\omega, w) \in T^*\mathbb{S}^{n-2d-1}$.

If $\lim_{t\to\infty}\left\|q_\infty\left(t; w, \sqrt{2\lambda}\omega\right)\right\| = \infty$, then there exist an open set $\mathcal{U} \subset T^*\mathbb{S}^{n-2d-1}$ with $(\omega, w) \in \mathcal{U}$ and functions $\xi_\infty \in C^\infty\left(T^*\mathbb{S}^{n-2d-1} \cap \mathcal{U}; \mathbb{S}^{n-2d-1}\right)$ and $x^\infty \in C^\infty\left(T^*\mathbb{S}^{n-2d-1} \cap \mathcal{U}; \mathbb{R}^{n-2d}\right)$ such that in $C^\infty(\mathcal{U})$

$$\begin{aligned} \lim_{t\to\infty}\left(q_\infty\left(t; w, \sqrt{2\lambda}\omega\right) - \sqrt{2\lambda}\xi_\infty\left(\sqrt{2\lambda}\omega, w\right)t - x^\infty\left(\sqrt{2\lambda}\omega, w\right)\right) = 0,\\ \lim_{t\to\infty}\left(p_\infty\left(t; w, \sqrt{2\lambda}\omega\right) - \sqrt{2\lambda}\xi_\infty\left(\sqrt{2\lambda}\omega, w\right)\right) = 0. \end{aligned} \tag{5}$$

The trajectory $\gamma_\infty\left(\cdot; w, \sqrt{2\lambda}\omega\right)$ is then said to have initial direction $\omega$ and final direction $\theta = \xi_\infty\left(\sqrt{2\lambda}\omega, w\right)$ and is called an $(\omega,\theta)-$trajectory.

**Definition 1.1.** *The outgoing direction $\theta \in \mathbb{S}^{n-2d-1}$ is called non-degenerate, or regular, for the incoming direction $\omega \in \mathbb{S}^{n-2d-1}$ at energy $\lambda > 0$ if $\theta \neq \omega$ and for all $z' \in \omega^\perp$ with $\xi_\infty\left(\sqrt{2\lambda}\omega, z'\right) = \theta$, the map $\omega^\perp \ni z \mapsto \xi_\infty\left(\sqrt{2\lambda}\omega, z\right) \in \mathbb{S}^{n-2d-1}$ is non-degenerate at $z'$.*

Several authors, working under the assumption that a certain final direction $\theta$ is non-degenerate for a given initial direction $\omega$, have proved asymptotic expansions of the Schwartz kernel $K_{\mathcal{T}}$ scattering amplitude of the form

$$K_{\mathcal{T}(\lambda,h)}(\omega,\theta)=\sum_{j=1}^{l}\hat{\sigma}\left(z_j;\lambda\right)^{-1/2}\exp\left(\frac{i}{h}S_j-i\mu_j\frac{\pi}{2}\right)+\mathcal{O}\left(h\right), \tag{6}$$

where

$$\left(z_j\right)_{j=1}^{l}=\left(z_j\left(\lambda,\omega,\theta\right)\right)_{j=1}^{l}\equiv\left(\xi_\infty^{-1}\left(\sqrt{2\lambda}\omega,\cdot\right)\right)(\theta), \tag{7}$$

the angular density of the trajectory $\gamma_\infty\left(\cdot;w,\sqrt{2\lambda}\omega\right)$ is given by

$$\hat{\sigma}\left(z_j;\lambda\right)=\det\left(\mathrm{J}\,\xi_\infty\left(\sqrt{2\lambda}\omega,\cdot\right)\right)\left(z_j\right), \tag{8}$$

with J denoting the Jacobian matrix,

$$S_j=\int_{-\infty}^{\infty}\left(\frac{1}{2}\left\|p_\infty\left(t;z_j,\sqrt{2\lambda}\omega\right)\right\|^2-V\left(q_\infty\left(t;z_j,\sqrt{2\lambda}\omega\right)\right)-\lambda\right)dt-\left\langle x^\infty\left(\sqrt{2\lambda}\omega,z_j\right),\sqrt{2\lambda}\theta\right\rangle \tag{9}$$

is a modified action along the $j-$th $(\omega,\theta)-$trajectory, and $\mu_j$ is the path index of that trajectory. Vainberg [24] has studied smooth compactly supported potentials $V$ at energies $\lambda>\sup V$. Guillemin [12] has established a similar asymptotic expansion in the setting of smooth compactly-supported metric perturbations of the Laplacian. Working with non-trapping potential perturbations of the Laplacian satisfying (4) with $\rho>\max\left(1,\frac{n-1}{2}\right)$, Yajima [26] has proved such an asymptotic expansion in the $L^2$ sense. Robert and Tamura [22] have shown that this asymptotic expansion at non-trapping energies further holds in the pointwise sense for all short-range ($\rho>1$) potentials. This result has been extended to the case of trapping energies by Michel [18] under an additional assumption on the distribution of the resonances of $P(h)$. On the other hand, the first to investigate the microlocal structure of the scattering amplitude was Protas [20], who proved that the scattering amplitude for compactly supported potentials at non-trapping energies and with a fixed initial direction is an FIO.

In [2] we have proved, without making the non-degeneracy assumption, that the scattering amplitude for smooth compactly supported potential and metric perturbations of the Euclidean Laplacian at both trapping and non-trapping energies, provided that a polynomial in $h$ bound on the resolvent is known in the former case, is a semi-classical Fourier integral operator associated to the scattering relation. The scattering relation $SR_{\mathcal{U}}(\lambda)$ at energy $\lambda>0$ for $\mathcal{U}$ as in (5) is defined near a non-trapped trajectory as follows:

$$SR_{\mathcal{U}}(\lambda)\overset{def}{=}\left\{\left(\xi_\infty\left(\sqrt{2\lambda}\omega',w'\right),x_\infty\left(\sqrt{2\lambda}\omega',w'\right);\omega',-w'\right):\left(\omega',w'\right)\in\mathcal{U}\right\}, \tag{10}$$

where $x_\infty\left(\sqrt{2\lambda}\omega',w'\right)\in\mathbb{R}^{n-2d-1}$ is the projection of $x^\infty\left(\sqrt{2\lambda}\omega',w'\right)$ onto $\xi_\infty\left(\sqrt{2\lambda}\omega',w'\right)^\perp$. The scattering relation $SR_{\mathcal{U}}(\lambda)$ is a Lagrangian submanifold of $T^*\mathbb{S}^{n-2d-1}\times T^*\mathbb{S}^{n-2d-1}$. In [2] we have further showed how the expansion (6) follows from the general theory of semi-classical Fourier integral operators developed in [1], once the non-degeneracy assumption on the initial and final directions is made. The asymptotic expansion we have obtained in [2], however, is more general than one given in (6) in that it holds microlocally near $(\omega,\theta)$ trajectories and not only for fixed initial and final directions.

For the proof we use the representation of the scattering amplitude at an energy $E>0$

$$\mathcal{T}\left(E,h\right)=c(h)\mathcal{F}_h\left(E\right)\left[h^2\Delta_z,\chi_1\right]R\left(E+i0,h\right)\left[h^2\Delta_z,\chi_2\right]\mathcal{F}_h\left(E\right)^*, \tag{11}$$

where $c(h)$ is a constant, $\chi_1,\chi_2\in C_c^\infty\left(\mathbb{R}^n\right)$ satisfy $V\prec\chi_1\prec\chi_2$, the operator $\mathcal{F}_h\left(E\right):L^2\left(\mathbb{R}^n\right)\to L^2\left(\mathbb{S}^{n-1}\right)$ is the restriction of the semi-classical Fourier transform to the sphere of radius $\sqrt{E}$ :

$$\left(\mathcal{F}_h\left(E\right)\varphi\right)\left(\xi\right)=c_1(h)\int_{\mathbb{R}^n}e^{-\frac{i\sqrt{E}}{h}z\cdot\xi}\varphi(z)\,dz,$$

and the resolvent $R(E+i0;h)=\lim_{\epsilon\downarrow 0}(P-(E+i\epsilon))^{-1}$, where the limit is taken in $\mathcal{B}\left(L^2_\alpha(\mathbb{R}^n);L^2_{-\alpha}(\mathbb{R}^n)\right)$, $\alpha>\frac{1}{2}$, $L^2_{\pm\alpha}(\mathbb{R}^n)=\left\{f:\left(1+|\cdot|^2\right)^{\pm\frac{\alpha}{2}}f\in L^2(\mathbb{R}^n)\right\}$. In (11) we further express the resolvent in the form

$$R(E+i0;h)=\frac{i}{h}\int_0^T e^{-\frac{i}{h}t(P(h)-E)}\,dt+R(E+i0;h)\,e^{-\frac{i}{h}T(P(h)-E)} \tag{12}$$

and show that for $T>0$ large enough the remainder term $R(E+i0;h)\,e^{-\frac{i}{h}T(P(h)-E)}$ makes a negligible contribution of order $\mathcal{O}(h^\infty)$ to (11) near a non-trapped trajectory. To analyze the first term in (27) and therefore the scattering amplitude (11) we show that $e^{-\frac{i}{h}t(P(h)-E)}$ is an h-FIO associated to $(graph\exp(tH_p))'$, $t>0$, near a non-trapped trajectory.

In [3] we have extended our results from [2] to the case of short-range perturbations of the Laplacian when the scattering amplitude is restricted away from the diagonal in $\mathbb{S}^{n-2d-1}\times\mathbb{S}^{n-2d-1}$.

The microlocal structure of the scattering amplitude in the setting of geometric scattering theory has also been analyzed. Hassell and Wunsch [13] have shown that on asymptotically Euclidean spaces the scattering matrix at non-trapping energies is a Legendrian-Lagrangian distribution associated to the total sojourn relation. Vasy [25], on the other hand, has proven that the scattering matrix is an FIO quantizing the geodesic flow between the two asymptotically hyperbolic ends of an asymptotically de Sitter-like spaces.

The structure of the scattering amplitude for the Schrödinger operator with a two-dimensional strong magnetic field ($d=1$) and $\mu_1=1$ has recently been studied by Michel [17]. The classical trajectories in this setting for any $d\in\mathbb{N}$ are the integral curves of

$$H_{p_{xy}}=\sum_{j=1}^{n-2d}\left(\frac{\partial p_{xy}}{\partial\xi_j}\frac{\partial}{\partial z_j}-\frac{\partial p_{xy}}{\partial z_j}\frac{\partial}{\partial\xi_j}\right)$$

with $p_{xy}(z,\xi)=\|\xi\|^2+V(x,y,z)$ for $V$ as in (1) and (2) and with $(x,y)\in\mathbb{R}^{2d}$ treated as parameters. To describe the classical quantities associated to the symbols $p_{xy}$ we shall use the notation introduced earlier but with the variables $x$ and $y$ added as parameters or subscripts.

We make the following:

**Definition 1.2.** *The energy $\lambda>\Lambda_0$ is said to be non-trapping if for all $k=1,\dots,K(\lambda)$, and all $(x,y)\in\mathbb{R}^{2d}$ we have*

$$\lim_{|t|\to\infty}\left|\exp\left(tH_{p_{xy}}\right)(z,\xi)\right|=\infty$$

*for all $(z,\xi)\in T^*\mathbb{R}^{n-2d}$ with $\|\xi\|^2+V(x,y,z)=\lambda-\lambda_k$.*

**Definition 1.3.** *The final direction $\theta_0$ is said to be non-degenerate for the initial direction $\omega_0$ at energy $\lambda>0$ if for all $k=1,\dots,K(\lambda)$, and for all $(x,y)\in\mathbb{R}^{2d}$ we have that if $z'\in\omega_0^\perp$ satisfies $\xi_\infty\left(\sqrt{\lambda-\lambda_k}\,\omega_0,z';x,y\right)=\theta_0$ then the map $\omega_0^\perp\ni z\mapsto\xi_\infty\left(\sqrt{\lambda-\lambda_k}\,\omega_0,z;x,y\right)\in\mathbb{S}^{n-2d-1}$ is non-degenerate at $z'$.*

Working under the assumption that the final direction $\theta_0$ is non-degenerate for the initial direction $\omega_0$, Michel [17] asserts the following asymptotic expansion of the scattering amplitude for all $N\in\mathbb{N}_0$ :

$$K_{\mathcal{T}(\lambda b,b)}(\theta_0,\omega_0)=b^{\frac{n-3}{4}}\sum_{k=1}^{K(\lambda)}(\lambda-\lambda_k)^{\frac{n-3}{4}}\,T_k(\theta_0,\omega_0,\lambda b,b)\Pi_k+\mathcal{O}\left(b^{\frac{n-3}{4}-N}\right)$$

in $\mathcal{B}(L^2(\mathbb{R}^2))$, where

$$T_k(\theta_0,\omega_0,\lambda b,b)=\sum_{j=0}^{N}T^w_{k,j}\left(\theta_0,\omega_0,\lambda,b,\frac{y}{2}-\frac{1}{b}D_x,\frac{x}{2}+\frac{1}{b}D_y\right)$$

for some $T_{k,j}(\theta_0,\omega_0,\lambda,h)\in S^{0,1}_{2d}(1)$, $j\in\mathbb{N}_0$. As before, the form of the leading symbol is given explicitly and is asserted to be equal to

$$T_{k,0}(\theta_0,\omega_0,\lambda,b,y,\eta)=\frac{e^{i(n-3)\pi/4}}{2(2\pi)^{(n-3)/2}}\sum_{l=1}^{l_k}\hat{\sigma}_k\left(\eta,y,z_{k,l}(\eta,y)\right)^{-\frac{1}{2}}e^{ib^{1/2}S_{k,l}(y,\eta)-i\mu_{k,l}\pi/2},$$

where $l_k$ is the number of $(\omega_0,\theta_0)-$trajectories at energy $\lambda-\lambda_k$ and $\mu_{k,l}$ is the Maslov index of the $l-$th such trajectory, both proved to be independent of $(y,\eta)$,

$$\{z_{k,l}(\eta,y)\}_{l=1}^{l_k}=\{z_{k,l}\left(\omega_0,\theta_0;\eta,y\right)\}_{l=1}^{l_k}\overset{def}{\equiv}\left(\xi_\infty^{-1}\left(\sqrt{\lambda-\lambda_k}\omega_0,\cdot;\eta,y\right)\right)(\theta_0)$$

with $\xi_\infty(\eta,y)$ defined as in (5) for the vector field $H_{p_{\eta y}}$,

$$\hat{\sigma}_k\left(\eta,y,z_{k,l}(\eta,y)\right)=\det\left(\mathrm{J}\,\xi_\infty\left(\sqrt{\lambda-\lambda_k}\omega_0,\cdot;\eta,y\right)\left(z_{k,l}\left(\eta,y\right)\right)\right),\tag{13}$$

and

$$\begin{aligned}S_{k,l}(\theta_0,\omega_0;y,\eta)=&\int_{-\infty}^{\infty}\frac{1}{2}\left\|p_\infty\left(t;y,\eta,z_{k,l}\left(\omega_0,\theta_0;\eta,y\right),\sqrt{\lambda-\lambda_k}\omega_0\right)\right\|^2\\&-V\left(\eta,y,q_\infty\left(t;y,\eta,z_{k,l}\left(\omega_0,\theta_0;\eta,y\right),\sqrt{\lambda-\lambda_k}\omega_0\right)\right)-\lambda+\lambda_k\,dt\\&-\left\langle x^\infty\left(y,\eta,\sqrt{\lambda-\lambda_k}\omega_0,z_{k,l}\left(\omega_0,\theta_0;\eta,y\right)\right),\sqrt{\lambda-\lambda_k}\theta_0\right\rangle\end{aligned}\tag{14}$$

is the modified action over the $l-$th $(\omega,\theta)-$trajectory of $H_{p_{\eta y}}$ at energy $\lambda-\lambda_k$. As we point out below, however, the proof of this result as proposed in [17] is incomplete.

In this paper we allow the physically more relevant higher dimensional magnetic field, i.e., $d>1$. This higher dimensional problem is further mathematically more complicated and we generalize the result from [17] also in the following ways: we study (the appropriately defined) microlocal structure of the scattering amplitude and we do so without making the non-degeneracy assumption and without restricting to fixed initial and final directions. To achieve these goals it is necessary to find the right framework within which to define and analyze the microlocal structure of the magnetic scattering amplitude. We show that this framework is provided by the new class of h-FIO-valued symbols and pseudodifferential operators, which we introduce in Definition 4.1 and some of the properties of which we develop below (see, for example, Proposition 4.2 and Lemma 4.3). We then show that the underlying reason for Michel's result is exactly the fact that the scattering amplitude has this microlocal structure and we prove that the form of the asymptotic expansion given in [17] follows as a direct corollary of our results once the non-degeneracy assumption is made. Although we do not pursue this here, we remark also that information about the structure of the scattering amplitude can be useful in solving inverse problems, as has been done, for example, in [2] in a different setting and later used in [23]. We further point out that neither the framework nor the methods of proof employed in our earlier works [2, 3, 4] on the structure of the scattering amplitude without magnetic field are applicable to the setting studied here and we have had to create and develop the new framework of h-FIO-valued symbols and pseudodifferential operators within which to analyze the microlocal structure of the present magnetic scattering amplitude. We believe that this new semi-classical framework will also be of independent interest.

This paper is organized as follows. After introducing some notation in the section below, we discuss the mathematical background of our problem and state our results in Section 3. In Section 4 we introduce and develop the framework of h-FIO-valued pseudodifferential operators, within which we will analyze the microlocal structure of the scattering amplitude. The proof of the Main Theorem, which makes up the main body of the paper, is presented in Section 5. The proofs of Theorem 3.1 and Corollary 3.2 are given in Sections 6 and 7, respectively. In Section 8 we discuss the two special cases of having the potential $V$ depend only on the variable $z$ and having the total dimension equal to $n=2d+1$. The necessary background from semi-classical analysis is presented in the Appendix. Some corrections to [17], necessary for our work, are presented throughout this paper.

## 2 Some Notation

We shall use the notation $B^k(0,R)$ to denote the ball $\{x \in \mathbb{R}^k : \|x\| < R\}$ for some $R > 0$ and $k \in \mathbb{N}$. If $T : C_c^\infty(\mathbb{R}^m) \to \mathcal{D}'(\mathbb{R}^n)$ is a sequentially continuous operator, we shall denote its Schwartz kernel by $K_T$. On any smooth manifold $M$ we denote by $\sigma$ the canonical symplectic form on $T^*M$ and everywhere below we work with the canonical symplectic structure on $T^*M$. If $C \subset T^*M_1 \times T^*M_2$, where $M_j$, $j = 1, 2$, are smooth manifolds, we will use the notation $C' = \{(x, \xi; y, -\eta) : (x, \xi; y, \eta) \in C\}$. For $w \in \mathbb{R}^k$ with $k \in \mathbb{N}_0$ and $\alpha > \frac{1}{2}$ we set $L^2_{\pm\alpha}\left(\mathbb{R}^{n-2d}_z \times \mathbb{R}^k_w\right) = \left\{f \in L^2\left(\mathbb{R}^{n-2d+k}\right) : \langle z\rangle^{\pm\alpha} f \in L^2\left(\mathbb{R}^{n-2d+k}\right)\right\}$ and we let $\|\cdot\|_{-\alpha,\alpha} = \|\cdot\|_{\mathcal{B}\left(L^2_{-\alpha}\left(\mathbb{R}^{n-2d}_z \times \mathbb{R}^k_w\right), L^2_{\alpha}\left(\mathbb{R}^{n-2d}_z \times \mathbb{R}^k_w\right)\right)}$ where the dimension $k$ will be clear from the context.

## 3 Preliminaries and Statements of Results

We follow the ideas of [8] and for $j = 1, \ldots, d$, we consider the canonical transformations $\kappa_j$ on $T^*\mathbb{R}^2$ given by

$$\kappa_j\left(r, s, r^*, s^*\right) = \left(-\frac{\sqrt{\mu_j b}}{2}s - \frac{1}{\sqrt{\mu_j b}}r^*, \frac{1}{2}s - \frac{1}{\mu_j b}r^*, \frac{\sqrt{\mu_j b}}{2}r - \frac{1}{\sqrt{\mu_j b}}s^*, \frac{\mu_j b}{2}r + s^*\right)$$

with inverses

$$\kappa_j^{-1}\left(r, s, r^*, s^*\right) = \left(\frac{1}{\sqrt{\mu_j b}}r^* + \frac{1}{\mu_j b}s^*, s - \frac{1}{\sqrt{\mu_j b}}r, -\frac{\sqrt{\mu_j b}}{2}r - \frac{\mu_j b}{2}s, \frac{1}{2}s^* - \frac{\sqrt{\mu_j b}}{2}r^*\right).$$

From [9] we then know that there exists a non-semi-classical metaplectic operator $V_j$ quantizing the canonical transformation $\kappa_j$ such that $V_j$ is unitary on $L^2(\mathbb{R}^2)$ and

$$V_j^* p^w\left(r, s, D_r, D_s\right) V_j = \left(p \circ \kappa_j^{-1}\right)^w\left(r, s, D_r, D_s\right)$$

for all symbols $p \in S_2^{m,\delta}\left(\langle\xi\rangle^k\right)$ with $m \in \mathbb{R}$, $0 \le \delta \le 1$, $k \in \mathbb{N}_0$. Then the operator

$$U \overset{def}{=} V_1 \otimes \cdots \otimes V_d \otimes I_z$$

is unitary on $L^2(\mathbb{R}^n)$ and satisfies

$$UH(b)U^* = -\Delta_z + bN_x\left(\mu_1, \mu_2, \ldots, \mu_d\right) + bV^w\left(\sqrt{b}\right) =: b\tilde{P}(b), \tag{15}$$

where

$$N_x\left(\mu_1, \mu_2, \ldots, \mu_d\right) = \sum_{k=1}^{d} \mu_k\left(-\frac{\partial^2}{\partial x_k^2} + x_k^2\right)$$

is the $d-$dimensional harmonic oscillator and

$$V^w\left(\sqrt{b}\right) = V^w\left(\frac{D_{x_1}}{\sqrt{\mu_1 b}} + \frac{D_{y_1}}{\mu_1 b}, \ldots, \frac{D_{x_d}}{\sqrt{\mu_d b}} + \frac{D_{y_d}}{\mu_d b}, y_1 - \frac{x_1}{\sqrt{\mu_1 b}}, \ldots, y_d - \frac{x_d}{\sqrt{\mu_d b}}, z\right).$$

We also define the unitary operator $\tilde{U} \overset{def}{=} V_1 \otimes \cdots \otimes V_d \otimes \tilde{I}$ on $L^2\left(\mathbb{S}^{n-1}\right)$, where $\tilde{I}$ is the identity operator on $L^2\left(\mathbb{S}^{n-2d-1}\right)$.

We let $\tilde{\Pi}_k$ denote the projection onto the eigenspace of $N_x\left(\mu_1, \mu_2, \ldots, \mu_d\right)$ associated to the eigenvalue $\lambda_k$. Then $\tilde{\Pi}_k \otimes I = U\left(\Pi_k \otimes I\right)U^*$.

Let further $n_k = \dim\ker\left(N_x\left(\mu_1, \ldots, \mu_d\right) - \lambda_k\right)$ and let $\{\phi_{k1}, \ldots, \phi_{kn_k}\} \subset L^2(\mathbb{R}^d)$ be an orthonormal basis for $\ker\left(N_x\left(\mu_1, \ldots, \mu_d\right) - \lambda_k\right)$. For $j \in \mathbb{N}_0$ let $\varphi_j \in L^2(\mathbb{R})$ be such that

$$\left(D_u^2 + u^2 - 2j - 1\right)\varphi_j(u) = 0 \text{ and } \|\varphi_j\|_{L^2(\mathbb{R})} = 1.$$

Then for every $k \in \{1, \ldots, K(\lambda)\}$ and $j \in \{1, \ldots, n_k\}$ there exists a $d-$tuple $Q_{kj} := (q_{kj1}, \ldots, q_{kjd}) \in \mathbb{N}_0^d$ satisfying

$$\Lambda_{Q_{kj}} = \mu_1 (2q_{kj1} + 1) + \cdots + \mu_d (2q_{kjd} + 1) = \lambda_k$$

such that

$$\phi_{kj} = \varphi_{q_{kj1}} \otimes \varphi_{q_{kj2}} \otimes \cdots \otimes \varphi_{q_{kjd}}.$$

We observe that $\tilde{\Pi}_k$ is also the projection onto $Span(\{\phi_{k1}, \ldots, \phi_{kn_k}\})$.

Lastly, for $k \in \{1, \ldots, K(\lambda)\}$ and $i, j \in \{1, \ldots, n_k\}$, we define the operator

$$\tilde{\Pi}_{kij}: \ L^2\left(\mathbb{R}^d\right) \ni f \mapsto \langle f, \phi_{ki} \rangle_{L^2(\mathbb{R}^d_x)} \phi_{kj} \in L^2\left(\mathbb{R}^d\right)$$

and the operator $\Pi_{kij} \in \mathcal{B}\left(L^2\left(\mathbb{R}^{2d}\right)\right)$ by setting and $\Pi_{kij} \otimes I = U^* \left(\tilde{\Pi}_{kij} \otimes I\right) U.$

To simplify the notation below we further let

$$\frac{y}{2} - \frac{D_x}{\mu b} \text{ stand for the operator-valued vector } \left(\frac{y_1}{2} - \frac{D_{x_1}}{\mu_1 b}, \ldots, \frac{y_d}{2} - \frac{D_{x_d}}{\mu_d b}\right)$$

and

$$\frac{x}{2} + \frac{D_y}{\mu b} \text{ denote the operator-vector } \left(\frac{x_1}{2} + \frac{D_{y_1}}{\mu_1 b}, \ldots, \frac{x_d}{2} + \frac{D_{y_d}}{\mu_d b}\right).$$

If $\lambda > \Lambda_0$ is a non-trapping energy, $k \in \{1, \ldots, K(\lambda)\}$, and $(x, y) \in \mathbb{R}^{2d}$, we also define the scattering relation at energy $\lambda - \lambda_k$ for the symbol $p_{xy}$ by setting

$$SR_{xy}(\lambda - \lambda_k) \stackrel{def}{=} \Big\{ \left(\xi_\infty\left(\sqrt{2(\lambda - \lambda_k)}\omega', w'; x, y\right), x_\infty\left(\sqrt{2(\lambda - \lambda_k)}\omega', w'; x, y\right); \omega', -w'\right) : (\omega', w') \in T^*\mathbb{S}^{n-2d-1}\Big\}.$$

With the notation introduced above and in Definitions 9.2, 9.5, and 4.1 we are now ready to state our **Main Theorem.** *Let $n \geq 2d + 3$ and let $\lambda > \Lambda_0$ be non-trapping.*

*Then, with $c_0(h) = 2^{-1} h^{2d-n-1} (2\pi)^{2d-n+1}$, we have that*

$$\mathcal{T}(\lambda b, b) = c_0(h) \sum_{k=1}^{K(\lambda)} \sum_{i,j=1}^{n_k} g_k^w \left(\frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b}\right) \Pi_{kij} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1} \times \mathbb{R}^{2d}))}(h),$$

*for some $g_k \in S^{2d-n,1}_{2d}\left(1; \cup_{(y,\eta) \in \mathbb{R}^{2d}} \mathcal{I}_h^{2d-n-1}\left(\mathbb{S}^{n-2d-1} \times \mathbb{S}^{n-2d-1}, SR_{y\eta}(\lambda - \lambda_k)\right)\right)$ for all $k \in \{1, \ldots, K(\lambda)\}$.*

We further have

**Theorem 3.1.** *Let $n \geq 2d + 3$, let $\lambda > \Lambda_0$ be non-trapping, and let $k \in \{1, \ldots, K(\lambda)\}$.*

*Then there exist $q_k, \sigma_k \in \mathbb{N}$, $\left\{\{m_{kq\sigma}\}_{\sigma=1}^{\sigma_q}\right\}_{q=1}^{q_k} \subset \mathbb{N}_0$, open sets $V_{kq\sigma} \subset \mathbb{R}^{m_{kq\sigma}}$ and $\mathcal{Y}_{kq} \subset \mathbb{R}^{2d}$ with $\mathcal{K} \subset \cup_{q=1}^{q_k} \mathcal{Y}_{kq}$, and functions $\varphi_{kq\sigma} \in C^\infty(\mathbb{S}^{n-2d-1} \times \mathbb{S}^{n-2d-1} \times V_{kq\sigma} \times \mathcal{Y}_{kq})$ such that for every $(\eta, y) \in \mathcal{Y}_{kq}$, $\left\{\{\varphi_{kq\sigma}(\cdot; y, \eta)\}_{q=1}^{q_k}\right\}_{\sigma=1}^{\sigma_k}$ are non-degenerate phase functions locally parameterizing the scattering relation $SR_{\eta y}(\lambda - \lambda_k)$ and satisfying $SR_{\eta y}(\lambda - \lambda_k) \subset \cup_{q=1}^{q_k} \cup_{\sigma=1}^{\sigma_k} \Lambda_{\varphi_{kq\sigma}(\cdot; y, \eta)}$. There also exist $N_k \in \mathbb{N}$, $\{m_{kN}\}_{N=1}^{N_k} \subset \mathbb{N}_0$, open sets $V_{kN} \subset \mathbb{R}^{m_{kN}}$, and non-degenerate phase functions $\psi_{kN} \in C^\infty\left(\mathbb{S}^{n-2d-1} \times \mathbb{S}^{n-2d-1} \times V_{kN}\right)$ for $N \in \{1, \ldots, N_k\}$, locally parameterizing $SR_{\eta y}(\lambda - \lambda_k)$ and such that $SR_{\eta y}(\lambda - \lambda_k) \subset \cup_{N=1}^{N_k} \Lambda_{\psi_{kN}}$ for $(\eta, y) \in \mathcal{K}^c$. For any such families of phase functions there further exist symbols $a_{kq\sigma} \in S^{d - \frac{n - m_{kq\sigma} + 3}{2}, 0}_{2n-2d-2+m_{kq\sigma}}(1)$, for $q \in \{1, \ldots, q_k\}$, $\sigma \in \{1, \ldots, \sigma_k\}$, $a_{kN} \in S^{d - \frac{n - m_{kN} + 3}{2}, 0}_{2n-4d-2+m_{kN}}(1)$, for $N \in \{1, \ldots, N_k\}$, and $\{\chi_k\}_{k=1}^{K(\lambda)} \subset S^{0,0}_{2d}(1)$*

*such that*

$$\mathcal{T}(\lambda b, b) = c_0(h) \sum_{k=1}^{K(\lambda)} \sum_{q=1}^{q_k} \sum_{\sigma=1}^{\sigma_k} \sum_{i,j=1}^{n_k} \left( \int_{\mathbb{R}^{m_{kq\sigma}}} e^{i\sqrt{b}\varphi_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot;)} a_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot;) \, d\tau \right)^w \left( \frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b} \right) \Pi_{kij}$$

$$+ c_0(h) \sum_{k=1}^{K(\lambda)} \sum_{i,j=1}^{n_k} \sum_{N=1}^{N_k} \chi_k^w \left( \frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b} \right) \Pi_{kij} \int_{\mathbb{R}^{m_N}} e^{\frac{i}{h}\psi_{kN}(\theta,\omega,\tau)} a_{kN}(\theta,\omega,\tau) \, d\tau + \mathcal{O}(h)$$

*in* $\mathcal{B}\left(L^2\left(\mathbb{S}^{n-2d-1} \times \mathbb{R}^{2d}\right)\right)$, *where* $\theta, \omega \in \mathbb{S}^{n-2d-1}$ *and* $c_0(h) = 2^{-1} h^{2d-n-1} (2\pi)^{2d-n+1}$.

**Remark:** It is clear from the proof below that for every $i, j, k, q$, and $\sigma$, the operators $\Pi_{kij}$ and $\left( \int_{\mathbb{R}^{m_{kq\sigma}}} e^{i\sqrt{b}\varphi_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)} a_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot) \, d\tau \right)^w \left( \frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b} \right)$ commute.

We now state a corollary to these theorems under the assumption that the final direction $\theta_0$ is non-degenerate for the initial direction $\omega_0$ (see Definition 1.3). For that we need to make some preliminary remarks and introduce some notation.

We recall from [17] that under this assumption for every $k \in \{1, \ldots, K(\lambda)\}$ there exists $l_k \in \mathbb{N}$ such that the number of $(\omega_0, \theta_0)$ trajectories for all $(x, y)$ is $l_k$.

Let $k \in \{1, \ldots, K(\lambda)\}$. Let $(\eta', y') \in \mathcal{K}$ and define

$$\omega_0^{\perp} \equiv \mathbb{R}^{n-2d-1} \supset \left\{ z^0_{kl,\eta' y'} \right\}_{l=1}^{l_k} \stackrel{def}{\equiv} \left( \xi_\infty \left( \sqrt{\lambda - \lambda_k}\, \omega_0, \cdot; \eta', y' \right) \right)^{-1} (\theta_0).$$

Since the map $\omega_0^{\perp} \ni z \mapsto \xi_\infty \left( \sqrt{\lambda - \lambda_k}\, \omega_0, z; \eta', y' \right) \in \mathbb{S}^{n-2d-1}$ is non-degenerate at $z^0_{kl,\eta' y'}$ for every $l \in \{1, 2, \ldots, l_k\}$, it follows, by the Implicit Function Theorem, that for every $l \in \{1, 2, \ldots, l_k\}$, there exist open sets

$$O^-_{kl,\eta' y'} \times O^+_{kl,\eta' y'} \times \mathcal{Y}_{kl,\eta' y'} \subset \mathbb{S}^{n-2d-1} \times \mathbb{S}^{n-2d-1} \times \mathbb{R}^{2d}$$

containing $(\omega_0, \theta_0, \eta', y')$ and unique functions

$$z_{kl,\eta' y'} \in C^\infty \left( O^-_{kl,\eta' y'} \times O^+_{kl,\eta' y'} \times \mathcal{Y}_{kl,\eta' y'}; \mathbb{R}^{n-2d-1} \right)$$

such that

$$\xi_\infty \left( \sqrt{\lambda - \lambda_k}\, \omega, z_{kl,\eta' y'}(\omega, \theta, \eta, y); \eta, y \right) = \theta$$

for all $(\omega, \theta, \eta, y) \in O^-_{kl,\eta' y'} \times O^+_{kl,\eta' y'} \times \mathcal{Y}_{kl,\eta' y'}$. We let

$$\mathcal{Y}_{k,\eta' y'} = \bigcap_{l=1}^{l_k} \mathcal{Y}_{kl,\eta' y'}.$$

Since $\mathcal{K}$ is compact, there exists $s_k \in \mathbb{N}$ and points $\{(\eta_s, y_s)\}_{s=1}^{s_k} \subset \mathcal{K}$ such that the sets $\mathcal{Y}_{ks} \stackrel{def}{=} \mathcal{Y}_{k,\eta_s y_s}$, $s \in \{1, \ldots, s_k\}$, satisfy

$$\mathcal{K} \subset \bigcup_{s=1}^{s_k} \mathcal{Y}_{ks}.$$

Let further

$$O_\pm = \bigcap_{k=1}^{K(\lambda)} \bigcap_{l=1}^{l_k} \bigcap_{s=1}^{s_k} O^\pm_{kl,\eta_s y_s}.$$

For $s \in \{1, \ldots, s_k\}$ and $l \in \{1, \ldots, l_k\}$ we can then define the functions

$$z_{ksl} \in C^\infty \left( O_- \times O_+ \times \mathcal{Y}_{ks}; \mathbb{R}^{n-2d-1} \right)$$

by setting

$$z_{ksl} \stackrel{def}{=} z_{kl,\eta_s y_s}|_{O_- \times O_+ \times \mathcal{Y}_{ks}}.$$

We remark that the sets $\{\{\mathcal{Y}_{ks}\}_{s=1}^{s_k}\}_{k=1}^{K(\lambda)}$ can clearly be chosen to be connected and bounded.

Now, since $V$ is constant in $(\eta', y')$ for $(\eta', y') \in \mathcal{K}^c$, as above it follows that, perhaps after decreasing the sets $O_-$ and $O_+$ around $\omega_0$ and $\theta_0$, respectively, there further exist unique functions $\{z_{k0l}\}_{l=1}^{l_k} \subset C^\infty\left(O_- \times O_+ \times \mathcal{K}^c; \mathbb{R}^{n-2d-1}\right)$ such that

$$\xi_\infty\left(\sqrt{\lambda - \lambda_k}\,\omega, z_{k0l}\left(\omega, \theta, \eta, y\right); \eta, y\right) = \theta$$

for all

$$(\omega, \theta, \eta, y) \in O_- \times O_+ \times \mathcal{Y}_{k0},$$

where we have set $\mathcal{Y}_{k0} = \mathcal{K}^c$.

For $s \in \{0, \ldots, s_k - 1\}$ we renumber the functions $z_{ks1}, \ldots, z_{ksl_k}$ in such a way that for some $y' \in \mathcal{Y}_{ks} \cap \mathcal{Y}_{ks+1}$ and all $l \in \{1, \ldots, l_k\}$, $z_{ksl}\left(\omega_0, \theta_0, y'\right) = z_{ks+1l}\left(\omega_0, \theta_0, y'\right)$. By the uniqueness guaranteed by the Implicit Function Theorem, we then have that for all $s, s' \in \{0, \ldots, s_k\}$ and for all $y \in \mathcal{Y}_{ks} \cap \mathcal{Y}_{ks'}$, $z_{ksl}\left(\cdot, \cdot\cdot, y\right) = z_{ks'l}\left(\cdot, \cdot\cdot, y\right)$. Using a partition of unity subordinate to the cover $\{\mathcal{Y}_{ks}\}_{s=0}^{s_k}$ we can then define functions $\{z_{kl}\}_{l=1}^{l_k} \subset C^\infty\left(O_- \times O_+ \times \mathbb{R}^{2d}\right)$ such that for all $s \in \{1, \ldots, s_k\}$ we have $z_{kl}|_{O_- \times O_+ \times \mathcal{Y}_{ks}} = z_{ksl}$.

We now define the actions over the $(\omega_0, \theta_0)-$trajectories by setting

$$\begin{aligned} S_{kl}(\theta, \omega; y, \eta) = \int_{-\infty}^{\infty} \frac{1}{2}\left\|p_\infty\left(t; y, \eta, z_{kl}\left(\omega, \theta; \eta, y\right), \sqrt{\lambda - \lambda_k}\omega\right)\right\|^2 \\ - V\left(\eta, y, q_\infty\left(t; y, \eta, z_{kl}\left(\omega, \theta; \eta, y\right), \sqrt{\lambda - \lambda_k}\omega\right)\right) - \lambda + \lambda_k\, dt \\ - \left\langle x^\infty\left(y, \eta, \sqrt{\lambda - \lambda_k}\omega, z_{kl}\left(\omega, \theta; \eta, y\right)\right), \sqrt{\lambda - \lambda_k}\theta\right\rangle \end{aligned} \tag{16}$$

for $(\omega, \theta, \eta, y) \in O_- \times O_+ \times \mathbb{R}^{2d}$, $l = 1, \ldots, l_k$, $k = 1, \ldots, K(\lambda)$.

We can now state the following

**Corollary 3.2.** *Let $n \geq 2d + 3$ and let $\lambda > \Lambda_0$ be non-trapping. Let $\theta_0$ be non-degenerate for $\omega_0$.*

*Then, with the notation as above, there exist symbols $\left\{\{a_{kl}\}_{l=1}^{l_k}\right\}_{k=1}^{K(\lambda)} \subset S_{2n-4d-2}^{d-\frac{n+3}{2},0}(1)$ satisfying supp $a_{kl} \Subset O_- \times O_+ \times \mathbb{R}^{2d}$ such that*

$$K_{\mathcal{T}(b\lambda, b)}(\theta_0, \omega_0) = c_0(h)\sum_{k=1}^{K(\lambda)}\sum_{l=1}^{l_k}\sum_{i,j=1}^{n_k}\left(e^{i\sqrt{b}S_{kl}(\theta_0, \omega_0; \cdot, \cdot\cdot)}a_{kl}\left(\theta_0, \omega_0; \cdot, \cdot\cdot\right)\right)^w\left(\frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b}\right)\Pi_{kij} + \mathcal{O}(h)$$

*in $\mathcal{B}\left(L^2\left(\mathbb{R}^{2d}\right)\right)$, where $c_0(h) = 2^{-1}h^{2d-n-1}(2\pi)^{2d-n+1}$.*

**Remark:** The existence of the full asymptotic expansion asserted in [17, Theorem 2] depends heavily on the diagonalization [17, Proposition 3.4], which asserts essentially that the effective Hamiltonian defined by (22) can be diagonalized up to any order through conjugation by a unitary operator. The proof of [17, Proposition 3.4], however, relies on knowing that the terms of the asymptotic expansion of the effective Hamiltonian $E_{-+}(\lambda)$, the existence of which is established in [17, Proposition 2.4], are symmetric matrices. This fact has not been proven in [17] and a proof is unavailable. Thus [17] has only established the form of the leading term of the scattering amplitude. If [17, Proposition 3.4] can be proven as stated, then it is clear that the same proof as given below establishes our Main Theorem, Theorem 3.1, and Corollary 3.2 up to an remainder term of order $\mathcal{O}\left(h^\infty\right)$ and for dimensions $n \geq 2d + 2$. We remark that our restricting the dimension to $n \geq 2d + 3$ in the results above is due only to this lack of completeness which we have discovered in the proof of [17, Proposition 3.4]. We do not attempt to prove [17, Proposition 3.4] as stated because we believe its proof belongs in [17] and because whether or not [17, Proposition 3.4] holds does not affect our work here.

# 4 Semi-Classical FIO-Valued Symbols

In this section we develop the microlocal framework, which we have found to be the correct one within which to study the structure of the scattering amplitude in the present setting. We refer to the Appendix for the necessary background from semi-classical analysis and define semi-classical-Fourier-integral-operator-valued ($h-$FIO-valued) symbols and pseudodifferential operators as follows:

**Definition 4.1.** *If $X = \mathbb{R}^k$ or $X = \mathbb{S}^k$ for some $k \in \mathbb{N}$ and $g : \mathbb{R}^{2d} \to [0, \infty)$ is an order function for some $d \in \mathbb{N}$, an operator-valued symbol $a \in S^{m,\delta}_{2d}\left(g; \mathcal{B}\left(L^2(X)\right)\right)$ is said to be a semi-classical-FIO-valued symbol, or an element of $S^{m,\delta}_{2d}\left(g; \cup_{w\in\mathbb{R}^{2d}} \mathcal{I}^r_{h^\delta}\left(X \times X, \Lambda_w\right)\right)$ for some order $r \in \mathbb{R}$ and a family of Lagrangian submanifolds $\Lambda_w \subset T^*\mathbb{R}^{2k}$, if for every $w \in \mathbb{R}^{2d}$, $a(w) \in \mathcal{I}^r_{h^\delta}\left(X \times X, \Lambda_w\right)$ and for every $\alpha \in \mathbb{N}^{2d}_0$ and every $w \in \mathbb{R}^{2d}$, $\partial^\alpha_w a(w) \in \mathcal{I}^{r+\delta|\alpha|}_{h^\delta}\left(X \times X, \Lambda_w\right)$.*

*An h-FIO-valued (semi-classical-)pseudodifferential-operator in $t-$quantization is an operator of the form*

$$Op^t_h(a) = a^t\left(x, hD_x\right) = \frac{1}{(2\pi h)^{2d}} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left((1-t)x + ty, \xi\right) d\xi$$

*for some $a \in S^{m,\delta}_{2d}\left(g; \cup_{w\in\mathbb{R}^{2d}} \mathcal{I}^r_{h^\delta}\left(X \times X, \Lambda_w\right)\right)$. We set $Op^w_h(a) = a^w\left(x, hD_x\right) := Op^{\frac{1}{2}}_h(a)$ and denote the space of all such operators by $\Psi^{m,\delta}_h\left(g, \mathbb{R}^d; \cup_{w\in\mathbb{R}^{2d}} \mathcal{I}^r_{h^\delta}\left(X \times X, \Lambda_w\right)\right)$.*

*If $b \in S^{m,\delta}_{3d}\left(g; \cup_{w\in\mathbb{R}^{2d}} \mathcal{I}^r_{h^\delta}\left(X \times X, \Lambda_w\right)\right)$ we further define*

$$Op_h(b) = \frac{1}{(2\pi h)^{2d}} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} b\left(x, y, \xi\right) d\xi.$$

The following proposition shows that semi-classical-FIO-valued pseudodifferential operators have the desired composition behavior. We note that the case of having the (semi-classical-FIO-valued) symbols of class $\delta = 1/2$ is non-trivial. We use this proposition in the proof of the Main Theorem below in Section 5.

**Proposition 4.2.** *Let $k, l_1, l_2, l_3 \in \mathbb{N}$, $r_1, r_2 \in \mathbb{R}$, and $0 < \delta \leq \frac{1}{2}$. Let $\{\Lambda_j\left(x, \xi\right)\}_{(x,\xi)\in\mathbb{R}^{2k}}$, $j = 1, 2$, be smooth families of Lagrangian submanifolds of $T^*\mathbb{R}^{l_j} \times T^*\mathbb{R}^{l_{j+1}}$, such that for every $(x, \xi) \in \mathbb{R}^{2k}$ the intersection of the manifolds $\Lambda_1(x,\xi)' \times \Lambda_2(x,\xi)'$ and $T^*\mathbb{R}^{l_1} \times diag\left(T^*\mathbb{R}^{l_2} \times T^*\mathbb{R}^{l_2}\right) \times T^*\mathbb{R}^{l_3}$ at every point is clean with excess e, proper, and connected.*

*Let $a_j \in S^{0,\delta}_{2k}\left(1; \cup_{(x,\xi)\in\mathbb{R}^{2k}} \mathcal{I}^{r_j}_{h^\delta}\left(\mathbb{R}^{l_j} \times \mathbb{R}^{l_{j+1}}, \Lambda_j\left(x, \xi\right)\right)\right)$, $j = 1, 2$, be such that for some $J_j \in \mathbb{N}$, $\{m_{j\nu}\}^{J_j}_{\nu=1} \subset \mathbb{N}_0$, symbols $a_{j\nu} \in S^{0,0}_{2k+l_j+l_{j+1}+m_{j\nu}}(1)$ and functions $\varphi_{j\nu} \in C^\infty\left(\mathbb{R}^{2k+l_j+l_{j+1}+m_{j\nu}}\right)$, $\nu \in \{1, \cdots, J_j\}$, with the property that for every $(x, \xi) \in \mathbb{R}^{2k}$ and every $j, \nu$, $\varphi_{j\nu}\left(x, \xi, \cdot, \cdot, \cdot\right)$ is a non-degenerate phase function with $\Lambda_{\varphi_{j\nu}(x,\xi)} = \Lambda_j\left(x, \xi\right)$ locally and for all $s, q \in \{1, \ldots, k\}$, $\partial_{x_s}\partial_{x_q}\varphi_{j\nu}$ is bounded, we have that for $(x, \xi) \in \mathbb{R}^{2k}$*

$$a_j\left(x, \xi\right) = \sum^{J_j}_{\nu=1} h^{\delta\left(r_j - \frac{m_{j\nu}}{2} - \frac{l_j+l_{j+1}}{4}\right)} \int_{\mathbb{R}^{m_{j\nu}}} e^{\frac{i}{h^\delta}\varphi_{jk}(x,\xi,z_j,z_{j+1},\theta_{j\nu})} a_{j\nu}\left(x, \xi, z_j, z_{j+1}, \theta_{j\nu}\right) d\theta_{j\nu}.$$

*Then, if for $0 \leq t \leq 1$, the symbol $c_t \in S^{0,\delta}_{2k}\left(1; \mathcal{B}\left(L^2\left(\mathbb{R}^{l_3}\right); L^2\left(\mathbb{R}^{l_1}\right)\right)\right)$ is such that*

$$Op^t_h\left(c_t\right) = Op^t_h\left(a_1\right) Op^t_h\left(a_2\right),$$

*it satisfies*

$$c_t \in S^{0,\delta}_{2k}\left(1; \cup_{(x,\xi)\in\mathbb{R}^{2k}} \mathcal{I}^{r_1+r_2+\frac{e}{2}}_{h^\delta}\left(\mathbb{R}^{l_1} \times \mathbb{R}^{l_3}, \left(\Lambda_1\left(x, \xi\right)' \circ \Lambda_2\left(x, \xi\right)'\right)'\right)\right).$$

*Proof.* From [15, Theorem 2.7.4] we have that for every $t \in [0, 1]$ there exists a non-degenerate quadratic form $\mathcal{M}_t$ on $\mathbb{R}^{4k}$ such that

$$c_t(x, \xi) = \left(e^{ih\mathcal{M}_t(D_x,D_\xi,D_y,D_\eta)}\left(a_1(x, \xi)a_2(y, \eta)\right)\right)|_{y=x,\eta=\xi}.$$

Let $d_t(x,y,\xi,\eta) = e^{ih\mathcal{M}_t(D_x,D_\xi,D_y,D_\eta)}(a_1(x,\xi)\otimes a_2(y,\eta))$. From [9, Theorem 7.6] we have that

$$d_t \in S^{0,\delta}_{4k}\left(1 : \mathcal{B}\left(L^2\left(\mathbb{R}^{l_2+l_3}\right); L^2\left(\mathbb{R}^{l_1+l_2}\right)\right)\right). \tag{17}$$

First we prove that for every $(x,y,\xi,\eta)\in\mathbb{R}^{4k}$,

$$d_t(x,y,\xi,\eta) \in \mathcal{I}^{r_1+r_2}_{h^\delta}\left(\mathbb{R}^{l_1+l_2}\times\mathbb{R}^{l_2+l_3}, \Lambda_1(x,\xi)\times\Lambda_2(y,\eta)\right). \tag{18}$$

As for every $(x,y,\xi,\eta)\in\mathbb{R}^{4k}$ we have that

$$a_1(x,\xi)\otimes a_2(y,\eta) \in \mathcal{I}^{r_1+r_2}_{h^\delta}\left(\mathbb{R}^{l_1+l_2}\times\mathbb{R}^{l_2+l_3}, \Lambda_1(x,\xi)\times\Lambda_2(y,\eta)\right)$$

it is clear that (18) will follow from

**Lemma 4.3.** *Let $\varphi\in C^\infty\left(\mathbb{R}^{q+l+m}\right)$ be such that for every $x\in\mathbb{R}^q$, $\varphi(x,\cdot;\cdot\cdot)$ is a non-degenerate phase function, $a\in S^{0,0}_{q+l+m}(1)$, $\mathcal{M}$ — a non-degenerate quadratic form on $\mathbb{R}^q$, and $\delta\in(0,\frac{1}{2}]$. Suppose that for all $j,k\in\{1,\ldots,q\}$, $\partial_{x_j}\partial_{x_k}\varphi$ is bounded.*

*Then there exists $\tilde{a}_\delta\in S^{0,0}_{q+l+m}(1)$ such that*

$$e^{ih\mathcal{M}(D_x)}\left(\int_{\mathbb{R}^m} e^{\frac{i}{h^\delta}\varphi(\cdot,z;\theta)}a(\cdot,z,\theta)\,d\theta\right)(x) = \int_{\mathbb{R}^m} e^{\frac{i}{h^\delta}\varphi(x,z;\theta)}\tilde{a}_\delta(x,z,\theta)\,d\theta.$$

*If $m=0$, then if $\delta\in\left(\frac{1}{2},1\right]$ there exists $\tilde{a}_\delta\in S^{0,0}_{q+l}(1)$ such that*

$$e^{ih\mathcal{M}(D_x)}\left(e^{\frac{i}{h^\delta}\varphi(\cdot,z)}a(\cdot,z)\right)(x) = e^{\frac{i}{h^\delta}\varphi(x,z)-\frac{i}{h^{2\delta-1}}\mathcal{M}\left(\varphi'_x(x,z)\right)}\tilde{a}_\delta(x,z).$$

*Proof of Lemma 4.3:* We have that

$$e^{ih\mathcal{M}(D_x)}\left(\int_{\mathbb{R}^m} e^{\frac{i}{h^\delta}\varphi(\cdot,z;\theta)}a(\cdot,z,\theta)\,d\theta\right)(x) = \int_{\mathbb{R}^m} e^{ih\mathcal{M}(D_x)}\left(e^{\frac{i}{h^\delta}\varphi(\cdot,z;\theta)}a(\cdot,z,\theta)\right)(x)d\theta$$
$$= \frac{1}{(2\pi h)^q}\int\int\int e^{\frac{i}{h}\left(\langle x,\xi\rangle+\mathcal{M}(\xi)-\langle y,\xi\rangle+h^{1-\delta}\varphi(y,z;\theta)\right)}a(y,z,\theta)\,d\xi dy d\theta.$$

We set $\mu=\mu(h)=h^{1-\delta}$ and obtain that

$$e^{ih\mathcal{M}(D_x)}\left(\int_{\mathbb{R}^m} e^{\frac{i}{h^\delta}\varphi(\cdot,z;\theta)}a(\cdot,z,\theta)\,d\theta\right)(x) \quad = \frac{1}{(2\pi h)^q}\int\int\int e^{\frac{i}{h}(\langle x,\xi\rangle+\mathcal{M}(\xi)-\langle y,\xi\rangle+\mu\varphi(y,z;\theta))}a(y,z,\theta)\,d\xi dy d\theta.$$

Let $\Phi(y,\xi;x,z,\theta,\mu) = \langle x,\xi\rangle+\mathcal{M}(\xi)-\langle y,\xi\rangle+\mu\varphi(y,z;\theta)$. The non-degenerate critical point of $\Phi(\cdot,\cdot\cdot;x,z,\theta,\mu)$ is determined by setting

$$\begin{cases} 0=\partial_y\Phi(y,\xi;x,z,\theta,\mu) = -\xi+\mu\varphi'_y(y,z;\theta) \\ 0=\partial_\xi\Phi(y,\xi;x,z,\theta,\mu) = x-y+\mathcal{M}'_\xi(\xi) \end{cases}$$

and calculating

$$\begin{cases} \xi=\mu\varphi'_y(y,z;\theta) \\ x=y-\mathcal{M}'_\xi(\xi)=y-\mu\mathcal{M}'\left(\varphi'_y(y,z;\theta)\right) \end{cases}.$$

The assumption on the boundedness of the derivatives of $\varphi$ and the Implicit Function Theorem now imply that for $h$ small enough there exists a function $g:\mathbb{R}^{q+l+m}\to\mathbb{R}^q$ such that for all $(x,z,\theta)\in\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^m$

$$x = g(x,z,\theta)-\mu\mathcal{M}'\left(\varphi'_y(g(x,z,\theta),z;\theta)\right).$$

Thus for every $(x,z,\theta)\in\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^m$ the function $\Phi(\cdot,\cdot\cdot;x,z,\theta)$ has a unique critical point. Using again the assumption on the boundedness of the derivatives of $\varphi$ we easily verify that for $h$ sufficiently small this critical point if non-degenerate for every $(x,z,\theta)$. From the method of stationary phase we, therefore, obtain that

$$\begin{aligned}
&\int\int\int e^{\frac{i}{h}\left(\langle x,\xi\rangle+\mathcal{M}(\xi)-\langle y,\xi\rangle+\mu\varphi(y,z;\theta)\right)}a\left(y,z,\theta\right)d\xi dyd\theta\\
&=\int e^{\frac{i}{h}\left(-\left\langle \mathcal{M}'\left(\mu\varphi_y'(y,z;\theta)\right),\mu\varphi_y'(y,z;\theta)\right\rangle+\mathcal{M}\left(\mu\varphi_y'(y,z;\theta)\right)+\mu\varphi(y,z;\theta)\right)}\\
&\qquad\qquad\sum_{k=0}^{\infty}h^k\left(A_{2k}\left(D_y\right)a\left(\cdot,z,\theta\right)\right)\left(x+\mu\mathcal{M}'\left(\varphi'\left(g\left(x,z,\theta\right),z,\theta\right)\right)\right)d\theta\\
&=\int e^{\frac{i}{h}\left(-\mathcal{M}\left(\mu\varphi_y'(y,z;\theta)\right)+\mu\varphi(y,z;\theta)\right)}\sum_{k=0}^{\infty}h^k\left(A_{2k}\left(D_y\right)a\left(\cdot,z,\theta\right)\right)\left(x+\mu\mathcal{M}'\left(\varphi'\left(g\left(x,z,\theta\right),z,\theta\right)\right)\right)d\theta\\
&=\int e^{\frac{i}{h^\delta}\left(-h^{1-\delta}\mathcal{M}\left(\varphi_y'(y,z;\theta)\right)+\varphi(y,z;\theta)\right)}\sum_{k=0}^{\infty}h^k\left(A_{2k}\left(D_y\right)a\left(\cdot,z,\theta\right)\right)\left(x+\mu\mathcal{M}'\left(\varphi'\left(g\left(x,z,\theta\right),z,\theta\right)\right)\right)d\theta,
\end{aligned}$$

where for $k\in\mathbb{N}_0$, $A_{2k}(D)$ is a differential operator of order at most $2k$.

This completes the proof of the lemma with

$$\tilde{a}(\delta)=\begin{cases} e^{-ih^{1-2\delta}\mathcal{M}\left(\varphi_y'(y,z;\theta)\right)}\sum_{k=0}^{\infty}h^k\left(A_{2k}\left(D_y\right)a\left(\cdot,z,\theta\right)\right)\left(x+\mu\mathcal{M}'\left(\varphi'\left(g\left(x,z,\theta\right),z,\theta\right)\right)\right) & \text{for } \delta\in\left(0,\frac{1}{2}\right]\\ \sum_{k=0}^{\infty}h^k\left(A_{2k}\left(D_y\right)a\left(\cdot,z\right)\right)\left(x+\mu\mathcal{M}'\left(\varphi'\left(g\left(x,z\right),z\right)\right)\right) & \text{for } \delta\in\left(\frac{1}{2},1\right] \text{ and } m=0\end{cases}$$

and

$$\tilde{\varphi}(\delta)=\begin{cases}\varphi & \text{for } \delta\in\left(0,\frac{1}{2}\right]\\ \varphi-h^{1-\delta}\mathcal{M}\circ\varphi_x' & \text{for } \delta\in\left(\frac{1}{2},1\right] \text{ and } m=0\end{cases}.$$

□

Lemma 4.3 and (17) now imply that there exist $\left\{\{w_{klt}\}_{k=1}^{J_1}\right\}_{l=1}^{J_2}\subset S^{0,0}_{4k+l_1+2l_2+l_3+m_{1k}+m_{2l}}(1)$

$$d_t=\sum_{k=1}^{J_1}\sum_{l=1}^{J_2}h^{\delta\left(r_1+r_2-\frac{m_{1k}+m_{2l}}{2}-\frac{l_1+2l_2+l_3}{4}\right)}\int\int e^{\frac{i}{h^\delta}\left(\varphi_{1k}\left(x,\xi,z,z'',\theta_{1k}\right)+\varphi_{2l}\left(y,\eta,z''',z',\theta_{2l}\right)\right)}$$
$$w_{klt}\left(x,y,\xi,\eta,z,z'',z''',z',\theta_{1k},\theta_{2l}\right)d\theta_{1k}d\theta_{2l}.$$

Therefore

$$\begin{aligned}
K_{c_t(x,\xi)}\left(z,z'\right)&=\int K_{d_t(x,x,\xi,\xi)}\left(z,z'',z'',z'\right)dz''\\
&=\sum_{k=1}^{J_1}\sum_{l=1}^{J_2}h^{\delta\left(r_1+r_2-\frac{m_{1k}+m_{2l}}{2}-\frac{l_1+2l_2+l_3}{4}\right)}\int\int e^{\frac{i}{h^\delta}\left(\varphi_{1k}\left(x,\xi,z,z'',\theta_{1k}\right)+\varphi_{2l}\left(x,\xi,z'',z',\theta_{2l}\right)\right)}\\
&\qquad w_{klt}\left(x,x,\xi,\xi,z,z'',z'',z',\theta_{1k},\theta_{2l}\right)d\theta_{1k}d\theta_{2l}dz''.
\end{aligned}\tag{19}$$

Since at every point the manifolds $\Lambda_1(x_0,\xi_0)\times\Lambda_2(x_0,\xi_0)$ and $T^*\mathbb{R}^l\times diag\left(T^*\mathbb{R}^l\times T^*\mathbb{R}^l\right)\times T^*\mathbb{R}^l$ intersect cleanly with excess $e$, from [10] it follows that (19) defines an h-FIO of order $\delta\left(r_1+r_2+\frac{e}{2}\right)$ associated to $\Lambda_1(x_0,\xi_0)\circ\Lambda_1(x_0,\xi_0)$. □

**Remark:** Proposition 4.2 and Lemma 4.3 for symbols of classes $S^{0,\delta}_{2k}\left(1;\cup_{(x,\xi)\in\mathbb{R}^{2k}}\mathcal{I}^r_{h^\delta}\left(\mathbb{R}^{l_1}\times\mathbb{R}^{l_2},\Lambda\left(x,\xi\right)\right)\right)$ with $0<\delta<\frac{1}{2}$ follow also from the well known asymptotic expansion available in this case (see, for example, [9, Propositions 7.6 and 7.7]). This asymptotic expansion, however, does not hold for $\delta=\frac{1}{2}$ and in that case in general there is no closed form expression for the composition symbol. Through Proposition 4.2 and Lemma 4.3 we are able to provide such a closed form expression for the composition symbol in the special case when the symbols are given by oscillatory integrals, demonstrating that the oscillatory structure is again preserved.

# 5 Proof of Main Theorem

As in [17, Corollary 3.2] we have that for $b>0$ large enough $\lambda b\notin\sigma_{pp}(H(b))$. From (3) we therefore have that for $n\geq 2d+1$

$$\mathcal{T}(\lambda b,b)=-2i\pi\mathcal{F}_0(\lambda b)[\Delta_z,\chi_1]R(\lambda b+i0)[\Delta_z,\chi_2]\mathcal{F}_0(\lambda b)^*. \tag{20}$$

We define the operators

$$\begin{array}{rll} R_-: & L^2\left(\mathbb{R}^{n-d}_{y,z}\right)^{N(\lambda)} & \to L^2(\mathbb{R}^n_{x,y,z}) \\ & \left(g_{11},\ldots,g_{1n_1},\ldots,g_{K(\lambda)1},\ldots,g_{K(\lambda)n_{K(\lambda)}}\right) & \mapsto \sum_{k=1}^{K(\lambda)}\sum_{j=1}^{n_k}\phi_{kj}\otimes g_{kj}\end{array}$$

and

$$\begin{array}{rll} R_+: & L^2\left(\mathbb{R}^n_{x,y,z}\right) & \to L^2(\mathbb{R}^{n-d}_{y,z})^{N(\lambda)} \\ & f & \mapsto \left(\langle f,\phi_{11}\rangle_{L^2(\mathbb{R}^d_x)},\ldots,\langle f,\phi_{1n_1}\rangle_{L^2(\mathbb{R}^d_x)},\ldots,\left\langle f,\phi_{K(\lambda)n_{K(\lambda)}}\right\rangle_{L^2(\mathbb{R}^d_x)}\right)\end{array}$$

and observe that

$$R_+R_-=I \text{ and } R_-R_+=\sum_{k=1}^{K(\lambda)}\tilde{\Pi}_k. \tag{21}$$

Similarly to Michel [17] we now consider the Grushin problem of finding the inverse of the operator

$$\mathcal{P}(s)=\begin{pmatrix}\tilde{P}(h)-s & R_-\\ R_+ & 0\end{pmatrix}$$

for $s=\lambda+i\epsilon$ for $\epsilon\geq 0$. The same analysis as in [17, Proposition 2.1] shows that the inverse of the operator $\mathcal{P}(\lambda+i\epsilon)$ exists as a bounded operator from $L^2(\mathbb{R}^n)\times\left(L^2\left(\mathbb{R}^d_y;H^2\left(\mathbb{R}^{n-2d}_z\right)\right)\right)^{N(\lambda)}$ into $D\left(N_x\left(\mu_1,\mu_2,\ldots,\mu_d\right)\right)\otimes L^2\left(\mathbb{R}^d_y\right)\otimes H^2\left(\mathbb{R}^{n-2d}_z\right)\times L^2\left(\mathbb{R}^{n-d}_{y,z}\right)^{N(\lambda)}$, where $D\left(N_x\left(\mu_1,\mu_2,\ldots,\mu_d\right)\right)$ is the domain of $N_x\left(\mu_1,\mu_2,\ldots,\mu_d\right)$ and we set

$$\mathcal{E}(\lambda+i\epsilon)=\mathcal{P}(\lambda+i\epsilon)^{-1}=\begin{pmatrix}E(\lambda+i\epsilon) & E_+(\lambda+i\epsilon)\\ E_-(\lambda+i\epsilon) & -E_{-+}(\lambda+i\epsilon)\end{pmatrix}.$$

where the effective Hamiltonian $E_{-+}(\lambda+i\epsilon)=R_+\left(\tilde{P}(h)-(\lambda+i\epsilon)\right)E_+(\lambda+i\epsilon)$ satisfies

$$E_{-+}(\lambda+i\epsilon)^{-1}=R_+\left(\tilde{P}(h)-(\lambda+i\epsilon)\right)^{-1}R_-. \tag{22}$$

From (20), (15), (21), and (22) we then obtain, as in [17, Section 3.2], that

$$\mathcal{T}(\lambda b,b)=-2\pi i h^2\tilde{U}^*\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)R_-[\Delta_z,\chi_1]E_{-+}(\lambda)^{-1}[\Delta_z,\chi_2]R_+\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)^*\tilde{U}, \tag{23}$$

where $\tilde{\mathcal{F}}_0(E)=\sum_{k=1}^{K(E)}\tilde{\Pi}_k\otimes\hat{\mathcal{F}}_0(E-b\lambda_k)$ for $E>b$ and $E_{-+}(\lambda)^{-1}=\lim_{\epsilon\to0^+}E_{-+}(\lambda+i\epsilon)^{-1}$ with the limit taken in $\mathcal{B}\left(\left(L^2_\alpha\left(\mathbb{R}^{n-d}_{y,z}\right)\right)^{N(\lambda)},\left(L^2_{-\alpha}\left(\mathbb{R}^{n-d}_{y,z}\right)\right)^{N(\lambda)}\right)$. We recall from [17, Proposition 2.4] that

$$E_{-+}(\lambda)=E_0(\lambda)+\mathcal{O}_{\mathcal{B}\left(L^2(\mathbb{R}^{n-d})^{N(\lambda)}\right)}(h), \tag{24}$$

where

$$E_0(\lambda)=E_0^w\left(y,h^2D_y,z,hD_z,\lambda\right)=diag\left(\mathcal{A}(h)+\Lambda_{Q_{kj}}-\lambda,\ k=1,\ldots,K(\lambda),j=1,\ldots,n_k\right)$$

for $\mathcal{A}(h)=a^w\left(y,z,h^2D_y,hD_z\right)$ with $a(y,z,\eta,\xi)=\|\xi\|^2+V(\eta,y,z)$.

Now, the same proof as that of [17, Proposition 3.3] gives the following

**Proposition 5.1.** *Suppose that $\lambda > 0$ is non-trapping.*
*Then for all $\alpha > \frac{1}{2}$ we have*

$$\left\|E_{-+}(\lambda)^{-1}\right\|_{\mathcal{B}\left(L^2_\alpha(\mathbb{R}^{n-d})^{N(\lambda)},\, L^2_{-\alpha}(\mathbb{R}^{n-d})^{N(\lambda)}\right)} = \mathcal{O}\left(\frac{1}{h}\right)$$

*and*

$$\left\|E_0(\lambda)^{-1}\right\|_{\mathcal{B}\left(L^2_\alpha(\mathbb{R}^{n-d})^{N(\lambda)},\, L^2_{-\alpha}(\mathbb{R}^{n-d})^{N(\lambda)}\right)} = \mathcal{O}\left(\frac{1}{h}\right).$$

We also recall the following estimate from [7, Proposition 2.1]

$$\|\mathcal{E}_{Q_\kappa}\tilde{\chi}_1\|_{\mathcal{B}(L^2(\mathbb{R}^{n-2d});L^2(\mathbb{S}^{n-2d-1}))} = \mathcal{O}\left(h^{\frac{n-2d-1}{2}}\right). \tag{25}$$

From (23), (24), Proposition 5.1, the resolvent identity, and (25) we therefore obtain that for $n \geq 2d+3$

$$\mathcal{T}(\lambda b, b) = -2\pi i h^2 \tilde{U}^* \tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right) R_- \left[\Delta_z, \chi_1\right] E_0\left(\lambda\right)^{-1} \left[\Delta_z, \chi_2\right] R_+ \tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)^* \tilde{U} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}^{2d}\times\mathbb{S}^{n-2d-1}))}(h). \tag{26}$$

We now recall that for any $T > 0$

$$E_0(\lambda)^{-1} = \frac{i}{h}\int_0^T e^{-\frac{i}{h}tE_0(\lambda)}dt + E_0(\lambda)^{-1}e^{-\frac{i}{h}TE_0(\lambda)}. \tag{27}$$

To analyze the contribution to the scattering amplitude of the last term in (27), we need the following two propositions. To state the first one we define the incoming and outgoing regions of phase space as follows. For $R > 0$, $d > 1$, and $\sigma \in (-1, 1)$ let

$$\Gamma_\pm(R, d, \sigma) = \left\{(y, z, \eta, \xi) \in T^*\mathbb{R}^{n-d} : \|z\| \geq R, \frac{1}{d} < \|\xi\| < d, \pm\cos(z, \xi) > \pm\sigma\right\},$$

where $\cos(z, \xi) = \frac{\langle z,\xi\rangle}{\|z\|\|\xi\|}$.

Proceeding as in [17, Lemma 4.1] we then have the following

**Proposition 5.2.** *Let $\omega_\pm \in S^{0,0}_{2n-2d}(1)$ be supported in $\Gamma_\pm(R, d, \sigma_\pm)$ with $R > 0$ sufficiently large.*
*Then*

1. *for all $\delta > 1$ and $\alpha > \frac{1}{2}$ we have that*

$$\left\|E_0(\lambda)^{-1}\omega_+^w(y, z, h^2D_y, hD_z)\right\|_{-\alpha+\delta,-\alpha} = \mathcal{O}\left(\frac{1}{h}\right).$$

2. *for all $\alpha \gg 1$, if $\sigma_+ > \sigma_-$ we have that*

$$\left\|\omega_-^w(y, z, h^2D_y, hD_z)E_0(\lambda)^{-1}\omega_+^w(y, z, h^2D_y, hD_z)\right\|_{-\alpha,\alpha} = \mathcal{O}\left(h^\infty\right).$$

3. *for all $\alpha \gg 1$, if $\omega \in S^{0,0}_{2n-2d}(1)$ satisfies $supp_z\omega \subset B^{n-2d}\left(0, \frac{1}{2}\right)$ we have that*

$$\left\|\omega^w(y, z, h^2D_y, hD_z)E_0(\lambda)^{-1}\omega_+^w(y, z, h^2D_y, hD_z)\right\|_{-\alpha,\alpha} = \mathcal{O}\left(h^\infty\right).$$

The following proposition represents the corrected statement and proof of [17, Lemma 4.2]. The most important corrections in the statement concern the definition of the map $\psi_t$ below and the condition which the supports of $\omega_1$ and $\omega_2$ are required to satisfy. The proof contains additional corrections of the proof given in [17]. We also remark that [17, Lemma 4.2] is stated as below but with the operator $E_0(\lambda)$ replaced by the operator $\mathcal{P}_{N_0}(\lambda)$, as defined through [17, Proposition 3.4]; as we remarked in Section 3, however, the proof of [17, Proposition 3.4] is incomplete.

For $t>0$ let $\psi_t : T^*\mathbb{R}^{n-d} \to T^*\mathbb{R}^{n-d}$ be defined by

$$\psi_t\left(y,z,\eta,\xi\right) = \left(y,\eta,\exp\left(tH_{p_{\eta y}}\right)(z,\xi)\right).$$

Then we have the following

**Proposition 5.3.** *Let $\omega_1$, $\omega_2 \in S^{0,0}_{2n-2d}(1)$ be such that supp $\omega_2 \cap \psi_t($supp $\omega_1) = \emptyset$ for some $t>0$ and $\omega_1$ is compactly supported.*

*Then*

$$\left\| \omega_2^w(y,z,h^2D_y,hD_z)e^{-\frac{i}{h}tE_0(\lambda)}\omega_1^w(y,z,h^2D_y,hD_z)\right\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty)$$

*for all $\alpha > \frac{1}{2}$.*

*Proof.* As

$$e^{-\frac{i}{h}tE_0(\lambda)} = diag\left(e^{-\frac{i}{h}t\left(\mathcal{A}(h)-\lambda+\Lambda_{Q_{kj}}\right)},\ k=1,\dots,K(\lambda), j=1,\dots,n_k\right),$$

where $\mathcal{A}(h) = a^w\left(y,z,h^2D_y,hD_z\right)$ with $a\left(y,z,\eta,\xi\right) = \|\xi\|^2 + V\left(\eta,y,z\right)$, it suffices to prove that

$$\left\| \omega_2^w(y,z,h^2D_y,hD_z)e^{-\frac{i}{h}t\mathcal{A}(h)}\omega_1^w(y,z,h^2D_y,hD_z)\right\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty).$$

For this we construct a parametrix for $e^{-\frac{i}{h}t\mathcal{A}(h)}\omega_1^w(y,z,h^2D_y,hD_z)e^{\frac{i}{h}t\mathcal{A}(h)}$ of the form

$$F(t) = f^w(t,y,z,h^2D_y,hD_z)$$

with $f(t) \sim \sum_{j=0}^\infty h^j f_j(t)$, $(f_j(t))_{j=0}^\infty \subset S^{0,0}_{2n-2d}(1)$, depending continuously on $t$, such that

$$hD_tF(t) = [\mathcal{A}(h), F(t)] \text{ and } f(0) = \omega_1. \tag{28}$$

Denoting $\tilde{f}(t,z,\xi) = f^w(t,y,z,h^2D_y,\xi)$ and $\tilde{a}(z,\xi) = a^w(y,z,h^2D_y,\xi)$, we obtain from (28)

$$hD_t\tilde{f}(t) = \tilde{a}\#\tilde{f}(t) - \tilde{f}(t)\#\tilde{a} \text{ and } f(0) = \omega_1,$$

where # denotes the composition of symbols in the class $S^{0,0}_{2n-4d}\left(1;\mathcal{L}\left(L^2\left(\mathbb{R}^d_y\right)\right)\right)$, i.e.,

$$\tilde{a}\#\tilde{f}(t) = \sum_{j\geq 0} h^j \tilde{a}\#_j\tilde{f}(t)$$

with

$$\tilde{a}\#_j\tilde{f}(t) = \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\alpha|}}{(2i)^j\alpha!\beta!}\left(\partial_z^\alpha\partial_\xi^\beta\tilde{a}\right)\left(\partial_\xi^\alpha\partial_z^\beta\tilde{f}(t)\right).$$

Therefore, from (28) we obtain

$$hD_t\tilde{f}(t) = \sum_{j\geq 0} h^j\left(\tilde{a}\#_j\tilde{f}(t) - \tilde{f}(t)\#_j\tilde{a}\right),$$

and using the symbol calculus in $S^{0,0}_{2d}(1)$ with respect to the $h^2$ Weyl quantization in $(y,\eta)$ we obtain

$$hD_tf(t) = \sum_{j,l\geq 0} h^{j+2l}\left(a\#_j\Box_l f(t) - f(t)\#_j\Box_l a\right),$$

where

$$a\#_j\Box_l f(t) = \sum_{|\alpha+\beta|=j}\sum_{|\gamma+\delta|=l} \frac{(-1)^{|\alpha|+|\gamma|}}{(2i)^{j+l}\alpha!\beta!\gamma!\delta!}\left(\partial_y^\gamma\partial_\eta^\delta\partial_z^\alpha\partial_\xi^\beta a\right)\left(\partial_\eta^\gamma\partial_y^\delta\partial_\xi^\alpha\partial_z^\beta f(t)\right).$$

Using the asymptotic expansion of $f(t)$, we therefore have to solve

$$D_t = \sum_{j+m+2l=n+1}\left(a\#_j\Box_l f_m(t) - f_m(t)\#_j\Box_l a\right), \quad n \in \mathbb{N}.$$

For $n = 0$ we then have the equation

$$D_t f_0(t) = a\#_1\Box_0 f_0(t) - f_0(t)\#_j\Box_0 a = iH_{p_{\eta y}} f_0(t).$$

Together with the initial condition $f_0(0) = \omega_1$ this gives $f_0(t) = \omega_1 \circ \psi_t$.

For $n \geq 1$ we have the equation

$$D_t f_n(t) = iH_{p_{\eta y}} f_n(t) + r_n(t),$$

with

$$r_n(t) = \sum_{j+m+2l=n+1, m<n}\left(a\#_j\Box_l f_m(t) - f_m(t)\#_j\Box_l a\right). \tag{29}$$

Together with the initial condition $f_n(0) = 0$, this equation is solved by $f_n(t) = \frac{1}{i}\int_0^t r_n(s)\circ\psi_{t-s}ds$.

Using the fact that $V$ has compact support, one can easily show that pullback by $\psi_r$, $r \in \mathbb{R}$, preserves the symbol class $S^{0,0}_{2n-2d}(1)$ and therefore $f_j \in S^{0,0}_{2n-2d}(1)$ for every $j \in \mathbb{N}_0$.

Since for $m < n$ the symbol $f_m(t)$ is supported in $\psi_{-t}(supp\ \omega_1)$, it follows from (29) that $r_n(t)$ is supported in $\psi_{-t}(supp\ \omega_1)$. Therefore $f_n(t)$ is supported in $\psi_{-t}(supp\ \omega_1)$. Consequently, we obtain that

$$F(t) = e^{-\frac{i}{h}t\mathcal{A}(h)}\omega_1^w(y,z,h^2D_y,hD_z)e^{\frac{i}{h}t\mathcal{A}(h)} + \mathcal{O}_{\mathcal{B}\left(L^2(\mathbb{R}^{n-d})^{N(\lambda)}\right)}(h^\infty)$$

uniformly with respect to $t$ in any compact set. To prove the result in $L^2_\alpha$, it suffices to work as in [19, Appendix B].

□

Then, from (26), (27), and Propositions 5.1, 5.2, and 5.3, we obtain, as in [22], that for $T_0 > 0$ large enough

$$\mathcal{T}(\lambda b, b) = \frac{2\pi}{h^3}\tilde{U}^*\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)R_-\left[h^2\Delta_z,\chi_1\right]\int_0^{T_0} e^{-\frac{i}{h}tE_0(\lambda)}dt\left[h^2\Delta_z,\chi_2\right]R_+\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)^*\tilde{U} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{2d}))}(h). \tag{30}$$

From the definitions of $\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)$ and $E_0(\lambda)$, we then have that

$$\mathcal{T}(\lambda b, b) = c_0(h)\sum_{\delta,\gamma,\kappa=1}^{N(\lambda)}\tilde{U}^*\tilde{\Pi}_{\gamma\kappa}\otimes F_{\gamma\delta\kappa}(\lambda,h)\,\tilde{U} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{2d}))}(h), \tag{31}$$

where

$$\tilde{\Pi}_{\gamma\kappa}g = \left\langle g, \phi_{Q_\gamma}\right\rangle\phi_{Q_\kappa} \text{ for } g \in L^2\left(\mathbb{R}^d\right),$$

$$F_{\gamma\delta\kappa}(\lambda,h) = \mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right]\int_0^{T_0} e^{-\frac{i}{h}t\left(\mathcal{A}(h)-\lambda+\Lambda_{Q_\delta}\right)}dt\left[h^2\Delta_z,\chi_2\right]\mathcal{E}^*_{Q_\gamma} \tag{32}$$

with $\mathcal{E}_Q = \mathcal{E}_Q(\lambda,h) : L^2\left(\mathbb{R}^{n-2d}\right) \to L^2\left(\mathbb{S}^{n-2d-1}\right)$ for $Q \in \mathcal{L}(\lambda)$ — the operator with Schwartz kernel

$$K_{\mathcal{E}_Q(\lambda,h)}(\omega,z) = (\lambda-\Lambda_Q)^{\frac{n-2d-2}{4}}\, e^{-\frac{i}{h}\sqrt{\lambda-\Lambda_Q}\langle z,\omega\rangle}.$$

We now turn to analyzing the operator $\int_0^{T_0} e^{-\frac{i}{h}t\left(\mathcal{A}(h)-\lambda+\Lambda_{Q_\delta}\right)}dt$, which is the most significant part of the proof of the Main Theorem. We have the following

**Proposition 5.4.** *For $Q \in \mathcal{L}(\lambda)$ there exists*

$$c_Q \in S^{1,1}_{2d}\left(1; \cup_{(y,\eta)\in\mathbb{R}^{2d}} \mathcal{I}_h^{\frac{1}{2}}\left(\mathbb{R}^{n-2d}\times\mathbb{R}^{n-2d}, \Lambda_{\eta y}\left(\lambda-\Lambda_Q\right)\right)\right),$$

*where*

$$\Lambda_{\eta y}\left(\lambda-\Lambda_Q\right) = \left(\cup_{t\in[0,T_0]}\, graph \exp\left(tH_{p_{\eta y}}\right)|_{p^{-1}_{\eta y}(\lambda-\Lambda_Q)}\right)',$$

*such that*

$$\left\|\frac{i}{h}\int_0^{T_0} e^{-\frac{i}{h}t(\mathcal{A}(h)-\lambda+\Lambda_Q)}dt - c_Q^w\left(y,h^2D_y\right)\right\|_{\mathcal{B}\left(L^2\left(\mathbb{R}^{n-d}_{y,z}\right)\right)} = \mathcal{O}(h^\infty).$$

*Proof.* First we approximate the operator $U(t) = e^{-\frac{i}{h}t\mathcal{A}(h)}$ for $t$ sufficiently small modulo an error of $\mathcal{O}(h^\infty)$ in $\mathcal{B}\left(L^2\left(\mathbb{R}^{n-d}\right)\right)$ by an operator of the form $\tau^w\left(t,y,h^2D_y\right)$, where $\tau \in S^{0,1}_{2d}\left(1;\mathcal{B}\left(L^2\left(\mathbb{R}^{n-2d}\right)\right)\right)$ and

$$K_{\tau(t,y,\eta)}(z,z') = \frac{1}{(2\pi h)^{n-2d}}\int_{\mathbb{R}^{n-2d}} e^{\frac{i}{h}\left(\phi(t,y,z,\eta;\theta)-z'\cdot\theta\right)}\upsilon(t,y,z,\eta;\theta)d\theta \tag{33}$$

for suitable smooth families of functions $\mathbb{R}^{2d}\ni(y,\eta)\mapsto\phi(t,y,\eta)\in C^\infty\left(\mathbb{R}^{2n-4d+1}\right)$ and symbols $\mathbb{R}^{2d}\ni(y,\eta)\mapsto\upsilon(t,y,\eta)\sim\sum_{j\geq 0}h^j\upsilon_j(t,y,\eta)$. For this we construct $\tau$ by solving

$$\begin{cases} \left(hD_t+\mathcal{A}(h)\right)\tau^w\left(t,y,h^2D_y\right) = \mathcal{O}(h^\infty) \text{ in } \mathcal{B}\left(L^2\left(\mathbb{R}^{n-d}\right)\right) \\ \tau^w\left(0,y,h^2D_y\right) = I \text{ on } L^2\left(\mathbb{R}^{n-d}\right). \end{cases} \tag{34}$$

Similarly to the proof in [17] we observe that

$$\mathcal{A}(h)\tau^w\left(t,y,h^2D_y\right) = \eta^w(t,y,h^2D_y),$$

where $\eta(t,\cdot)\in S^{0,1}_{2d}\left(1;\mathcal{B}\left(L^2\left(\mathbb{R}^{n-2d}\right)\right)\right)$ satisfies

$$\eta\left(t,y,\eta\right)\sim\sum_{\alpha,\beta\in\mathbb{N}_0^d}\frac{h^{2|\alpha+\beta|}(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\partial_y^\alpha\partial_\eta^\beta a^w\left(y,z,\eta,hD_z\right)\partial_y^\beta\partial_\eta^\alpha\tau\left(t,y,\eta\right).$$

For $N\in\mathbb{N}_0$ we now set

$$L_N\left(y,z,\eta,y^*,z^*,\eta^*\right) = \sum_{|\alpha+\beta|\leq N}\frac{h^{2|\alpha+\beta|}(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\partial_y^\alpha\partial_\eta^\beta a\left(y,z,\eta,z^*\right)\left(\eta^*\right)^\alpha\left(y^*\right)^\beta$$

and

$$\tau_N\left(t,y,\eta\right) = \frac{1}{(2\pi h)^{n-2d}}\sum_{j=0}^N h^j\int_{\mathbb{R}^{n-2d}} e^{\frac{i}{h}\left(\phi(t,y,z,\eta;\theta)-z'\cdot\theta\right)}\upsilon_j\left(t,y,z,\eta;\theta\right)d\theta,$$

and we look for $\tau$ of the form (33) such that for every $N\in\mathbb{N}_0$

$$\begin{cases} \left(hD_t+L_N^w\left(y,z,\eta,hD_y,hD_z,hD_\eta\right)\right)\tau_N(t,y,\eta) = \mathcal{O}\left(h^{N+1}\right) \text{ in } S^{0,1}_{2d}\left(1;\mathcal{B}\left(L^2\left(\mathbb{R}^{n-2d}\right)\right)\right) \\ \tau\left(0,y,\eta\right) = I \text{ in } \mathcal{B}\left(L^2\left(\mathbb{R}^{n-2d}\right)\right). \end{cases} \tag{35}$$

For the function $\phi$ we then have the eikonal equation

$$\begin{cases} \partial_t\phi\left(t,y,z,\eta;\theta\right)+L_0\left(y,z,\eta,\partial_y\phi\left(t,y,z,\eta;\theta\right),\partial_z\phi\left(t,y,z,\eta;\theta\right),\partial_\eta\phi\left(t,y,z,\eta;\theta\right)\right)=0 \\ \phi\left(0,y,z,\eta;\theta\right) = z\cdot\theta \text{ for all } (\eta,y)\in\mathbb{R}^{2d} \end{cases}. \tag{36}$$

Now, since

$$L_0(y,z,\eta,y^*,z^*,\eta^*) = \left\|z^*\right\|^2+V(\eta,y,z)$$

we see that the solutions of the corresponding Hamiltonian system

$$\begin{array}{ll} \dot{Z} = 2Z^*, & \dot{Z}^* = -\nabla_z V(\Theta, Y, Z) \\ \dot{Y} = 0, & \dot{Y}^* = -\nabla_y V(\Theta, Y, Z) \\ \dot{\Theta} = 0, & \dot{\Theta}^* = -\nabla_x V(\Theta, Y, Z) \end{array}$$

further satisfy

$$(Z, Z^*)(t, \eta, y, z, z^*) = \exp\left(tH_{p_{\eta y}}\right)(z, z^*).$$

Proceeding as in the proof of [21, Proposition IV-14], we therefore obtain that there exists $T > 0$ and a unique solution $\phi \in C^\infty\left([-T, T] \times \mathbb{R}^{2n-2d}\right)$ to (36). In addition, the solution $\phi$ has the following property

$$(z, \partial_z \phi\,(t, y, \eta, z; \theta)) = \exp\left(tH_{p_{\eta y}}\left(\partial_\theta \phi\,(t, y, \eta, z; \theta)\,, \theta\right)\right)$$

for $T \in [-t, t]$ and all $(t, y, \eta, z, \theta) \in [-T, T] \times \mathbb{R}^{2n-2d}$. From [21, Proposition IV-14 (ii)] and the fact that the potential $V$ is constant in $(\eta, y)$ outside of a compact set it now follows that if $t \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{N}_0^d$, and $\gamma, \delta \in \mathbb{N}_0^{n-2d}$, then $\partial_t^k \partial_y^\alpha \partial_\eta^\beta \partial_z^\gamma \partial_\theta^\delta \phi\,(t, y, \eta, z; \theta) = \mathcal{O}\left(\langle (z, \theta) \rangle^r\right)$ uniformly in $(t, y, \eta) \in [-T, T] \times \mathbb{R}^{2d}$, where $r = 2$ if $k \in \{0, 1\}$, $\alpha = \beta = 0$, and $\gamma = \delta = 0$, $r = 1$ if $k = 0$, $|\alpha| + |\beta| = 1$, and $\gamma = \delta = 0$, and $r = 0$ if $|\alpha| + |\beta| \geq 2$, $|\gamma| + |\delta| = 1$, or $k \geq 2$.

To determine the symbols $\upsilon(t)$ we proceed as in the proof of [21, Lemma IV-29]. Using the fact that $V$ is constant in $(x, y)$ outside of a compact set we see that we can find a unique sequence $\{\upsilon_j\}_{j=0}^\infty \subset C^\infty\left([-T, T] \times \mathbb{R}^{2n-2d}\right)$ such that (35) is satisfied. These functions further have the property that for every $k \in \mathbb{N}_0$ and all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ and $\gamma, \delta \in \mathbb{N}_0^{n-2d}$ they satisfy $\partial_t^k \partial_y^\alpha \partial_\eta^\beta \partial_z^\gamma \partial_\theta^\delta \upsilon_j\,(t, y, z, \eta; \theta) = \mathcal{O}\left(\langle (z, \theta) \rangle^k\right)$ uniformly in $(t, y, \eta) \in [-T, T] \times \mathbb{R}^{2d}$ for all $j \in \mathbb{N}_0$.

Therefore for $t \in [-T, T]$ we have that for every $N \in \mathbb{N}_0$

$$\tau_N(t), \tau(t) \in S_{2d}^{0,1}\left(1; \cup_{(y,\eta) \in \mathbb{R}^{2d}} \mathcal{I}_h^0\left(\mathbb{R}^{n-2d} \times \mathbb{R}^{n-2d}, \left(graph \exp\left(tH_{p_{\eta y}}\right)\right)'\right)\right). \tag{37}$$

Moreover, from [14, Theorem 25.3.8] we have that $\tau_N^w\left(t, y, h^2 D_y\right) \in \mathcal{B}\left(L^2\left(\mathbb{R}^{n-d}\right)\right)$, $N \in \mathbb{N}_0$, and $\tau^w\left(t, y, h^2 D_y\right) \in \mathcal{B}\left(L^2\left(\mathbb{R}^{n-d}\right)\right)$.

Moreover we have that

$$\begin{cases} (hD_t + \mathcal{A}(h))\, \tau_N^w\left(t, y, h^2 D_y\right) = h^{N+1} r_N^w(t, y, h^2 D_y) \\ \tau_N^w\left(0, y, h^2 D_y\right) = I \end{cases}, \tag{38}$$

where $r_N \in C^\infty\left([-T, T]; S_{2d}^{0,1}\left(1; \mathcal{B}\left(L^2\left(\mathbb{R}^{n-2d}\right)\right)\right)\right)$. Therefore

$$\left\| r_N^w\left(t, y, h^2 D_y\right) \right\|_{\mathcal{B}(L^2(\mathbb{R}^{n-d}))} = \mathcal{O}(1) \tag{39}$$

uniformly with respect to $t \in [-T, T]$.

To estimate the error $\left\| U(t) - \tau_N^w\left(t, y, h^2 D_y\right) \right\|_{\mathcal{B}(L^2(\mathbb{R}^{n-d}))}$ of approximating $U(t)$ by $\tau_N^w\left(t, y, h^2 D_y\right)$ for $t \in [-T, T]$ we now set

$$X_N(t) = U(-t) \tau_N^w\left(t, y, h^2 D_y\right) - I.$$

From (38) we find that

$$\begin{cases} ihU(t)\, \partial_t X_N(t) & = h^{N+1} r_N^w\left(t, y, h^2 D_y\right) \\ X_N(0) = 0, & \end{cases}$$

from which we obtain

$$U(t) - \tau_N^w\left(t, y, h^2 D_y\right) = U(t) X_N(t) = ih^N \int_0^t U(t - s) r_N^w\left(s, y, h^2 D_y\right) ds.$$

From (39) it therefore follows that

$$\sup_{t\in[-T,T]}\left\|U(t)-\tau_N^w\left(t,y,h^2D_y\right)\right\|_{\mathcal{B}(L^2(\mathbb{R}^{n-d}))}=\mathcal{O}\left(h^N\right) \tag{40}$$

for every $N\in\mathbb{N}$ and therefore

$$\left\|\frac{i}{h}\int_0^t U(t)e^{-\frac{i}{h}(\lambda-\Lambda_Q)t}dt-\frac{i}{h}\int_0^t\tau^w\left(t,y,h^2D_y\right)e^{-\frac{i}{h}(\lambda-\Lambda_Q)t}dt\right\|_{\mathcal{B}(L^2(\mathbb{R}^{n-d}))}=\mathcal{O}(h^\infty) \tag{41}$$

for $|t|<T$.

Let, now, $|t|>T$ and let $T_1\in(0,T)$ and $m\in\mathbb{N}_0$ be such that $mT_1\le t<(m+1)T_1$ if $t>0$ or $-(m+1)T_1<t\le -mT_1$ if $t<0$. We then approximate $U(t)=U(t-mT_1)U(T_1)^m$ by

$$\tilde{U}(t)\stackrel{def}{=}\tau^w\left(t-mT_1,y,h^2D_y\right)\left(\tau^w\left(T_1,y,h^2D_y\right)\right)^m \tag{42}$$

if $t>0$ and we approximate $U(t)=U(t+mT_1)U(-T_1)^m$ by

$$\tilde{U}(t)\stackrel{def}{=}\tau^w\left(t+mT_1,y,h^2D_y\right)\left(\tau^w\left(-T_1,y,h^2D_y\right)\right)^m$$

if $t>0$. To estimate the errors in these approximations, we use the identity

$$\prod_{j=1}^m A_j-\prod_{j=1}^m B_j=\sum_{j=1}^m A_1\ldots A_{j-1}\left(A_j-B_j\right)B_{j+1}\ldots B_M$$

and estimate (40) to obtain that

$$\sup_{mT_1\le|t|\le(m+1)T_1}\left\|U(t)-\tilde{U}(t)\right\|_{\mathcal{B}(L^2(\mathbb{R}^{n-d}))}=\mathcal{O}\left(h^\infty\right), \tag{43}$$

which implies that (41) holds for all $t\in\mathbb{R}$.

To analyze, now, the microlocal structure of the operator $\frac{i}{h}\int_0^{T_0}\tilde{U}\left(t\right)e^{-\frac{i}{h}(\lambda-\Lambda_Q)t}dt$, we observe that we can assume that $T_0>T$, which we do for the simplicity of the presentation below. We further choose a function $\chi\in C_c^\infty\left((0,T_1)\right)$ such that for some $\nu>0$ it satisfies the condition $\sum_{k\in\mathbb{Z}}\chi\left(\cdot-\nu k\right)\equiv 1$. Then, after a change of variable, we obtain that for some $N\in\mathbb{N}$

$$\begin{aligned}\frac{i}{h}\int_0^{T_0}e^{-it(\mathcal{A}(h)-\lambda+\Lambda_Q)/h}dt&=\sum_{k\in\mathbb{Z}}\frac{i}{h}\int_0^{T_0}\chi(t-k\nu)e^{-it(\mathcal{A}(h)-\lambda+\Lambda_Q)/h}dt\\&=\sum_{k=-N}^{N}\frac{i}{h}\int_0^{T_0}\chi(t-k\nu)e^{-it(\mathcal{A}(h)-\lambda+\Lambda_Q)/h}dt\\&=\sum_{k=-N}^{N}e^{-ik\nu(\mathcal{A}(h)-\lambda+\Lambda_Q)/h}\circ\frac{i}{h}\int_0^{T_0}\chi(t)e^{-it(\mathcal{A}(h)-\lambda+\Lambda_Q)/h}dt,\end{aligned}$$

which from (40) and (41) for all $t\in\mathbb{R}$ is readily seen to be approximated within $\mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}^{n-d}))}\left(h^\infty\right)$ by

$$\sum_{k=-N}^{N}\tilde{U}\left(k\nu\right)e^{-\frac{i}{h}k\nu(\lambda-\Lambda_Q)}\circ\frac{i}{h}\int_0^{T_0}\chi(t)\tilde{U}(t)e^{-\frac{i}{h}t(\lambda-\Lambda_Q)}dt. \tag{44}$$

Now, the operator

$$\frac{i}{h}\int_0^{T_0}\chi(t)\tilde{U}\left(t\right)e^{-\frac{i}{h}(\lambda-\Lambda_Q)t}dt=\frac{i}{h}\int_0^{T_0}\chi(t)\tau^w\left(t,y,h^2D_y\right)e^{-\frac{i}{h}(\lambda-\Lambda_Q)t}dt$$

has a symbol with values in operators with the Schwartz kernels

$$K_{\frac{i}{h}\int_0^{T_0}\chi(t)\tau(t,y,\eta)e^{-\frac{i}{h}\left(\lambda-\Lambda_Q\right)t}dt}(z,z')$$
$$=\frac{i}{(2\pi h)^{n-2d}h}\int_0^{T_0}\int_{\mathbb{R}^{n-2d}}e^{\frac{i}{h}\left(\phi(t,y,z,\eta;\theta)-z'\cdot\theta-(\lambda-\Lambda_Q)t\right)}\chi(t)\upsilon\left(t,y,z,z',\eta;\theta\right)d\theta dt.$$

Working as in the proof of [4, Lemma 3.3], we therefore have that

$$\frac{i}{h}\int_0^{T_0}\chi(t)\tau\left(t\right)e^{-\frac{i}{h}(\lambda-\Lambda_Q)t}dt\in S^{1,1}_{2d}\left(1;\cup_{(\eta,y)\in\mathbb{R}^{2d}}\mathcal{I}^{\frac{1}{2}}_h\left(\mathbb{R}^{n-2d}\times\mathbb{R}^{n-2d},\Lambda_{\eta y}\left(\lambda-\Lambda_Q\right)\right)\right). \tag{45}$$

Proceeding with the proof of Proposition 5.4, we first observe that since the potential $V$ depends on $(x,y)$ only in the compact set $\mathcal{K}$, it follows that the phase functions in $\tau(t)$ for $|t|<T$ and in $\int_0^{T_0}\chi(t)\tau(t)e^{-\frac{i}{h}t(\lambda-\Lambda_Q)}dt$ are independent of $(\eta,y)\in\mathcal{K}^c$ and therefore have bounded second order partial derivatives with respect to these variables. We also easily check that for every $t_1,t_2\in\mathbb{R}$ and for every $(y,\eta)\in\mathbb{R}^{2d}$ the intersection of the manifolds

$$graph\exp\left(t_1H_{p_{\eta y}}\right)\times graph\exp\left(t_2H_{p_{\eta y}}\right)\text{ and }T^*\mathbb{R}^{n-2d}\times diag\left(T^*\mathbb{R}^{n-2d}\times T^*\mathbb{R}^{n-2d}\right)\times T^*\mathbb{R}^{n-2d}$$

is transverse, i.e., clean with excess 0, at every point of intersection, as well as proper and connected. Proposition 4.2 together with (37) and (42) therefore implies that for every $k\in\mathbb{N}$

$$\tilde{U}\left(k\nu\right)\in\Psi^{0,1}_h\left(1,\mathbb{R}^d;\cup_{(y,\eta)\in\mathbb{R}^{2d}}\mathcal{I}^0_h\left(\mathbb{R}^{n-2d}\times\mathbb{R}^{n-2d},\left(graph\exp\left(k\nu H_{p_{\eta y}}\right)\right)'\right)\right). \tag{46}$$

Similarly, we easily see that for every $t\in\mathbb{R}$ and for every $(y,\eta)\in\mathbb{R}^{2d}$ the intersection of the manifolds

$$graph\exp\left(tH_{p_{\eta y}}\right)\times\Lambda_{\eta y}\left(\lambda-\Lambda_Q\right)'\text{ and }T^*\mathbb{R}^{n-2d}\times diag\left(T^*\mathbb{R}^{n-2d}\times T^*\mathbb{R}^{n-2d}\right)\times T^*\mathbb{R}^{n-2d}$$

is transverse at every point of intersection and it is proper and connected. Another application of Proposition 4.2 and equations (45) and (46) then imply that the operator given in (44) is an element of

$$\Psi^{1,1}_h\left(1,\mathbb{R}^d;\cup_{(\eta,y)\in\mathbb{R}^{2d}}\mathcal{I}^{\frac{1}{2}}_h\left(\mathbb{R}^{n-2d}\times\mathbb{R}^{n-2d},\Lambda_{\eta y}\left(\lambda-\Lambda_Q\right)\right)\right),$$

which concludes the proof of Proposition 5.4. □

Proposition 5.4 now allows us to analyze the operators $F_{\gamma\delta\kappa}(\lambda,h)$ given by (32) for $\delta,\gamma,\kappa\in\{1,\dots,N(\lambda)\}$.

For $\delta\in\{1,\dots,N(\lambda)\}$ let $c_{Q_\delta}\in S^{1,1}_{2d}\left(1;\cup_{(y,\eta)\in\mathbb{R}^{2d}}\mathcal{I}^{\frac{1}{2}}_h\left(\mathbb{R}^{n-2d}\times\mathbb{R}^{n-2d},\Lambda_{\eta y}\left(\lambda-\Lambda_{Q_\delta}\right)\right)\right)$ be as in Proposition 5.4. Then from [1, Lemma 4.3] we have that

$$K_{\left([h^2\Delta_z,\chi_1]\otimes[h^2\Delta_z,\chi_2]^t\right)c_{Q_\delta}(y,\eta)}\in I_h^{-\frac{3}{2}}\left(\mathbb{R}^{n-2d}\times\mathbb{R}^{n-2d},\Lambda_{\eta y}\left(\lambda-\Lambda_{Q_\delta}\right)\right).$$

Further, observe that for $Q\in\mathcal{L}(\lambda)$

$$K_{\mathcal{E}_Q(\lambda,h)}\in I_h^{\frac{1-2n+4d}{4}}\left(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{n-2d},C_Q\right), \tag{47}$$

where

$$C_Q=\left\{\left(\omega,-\sqrt{\lambda-\Lambda_Q}d^{\mathbb{S}^{n-2d-1}}_\omega\left\langle z,\omega\right\rangle;z,-\sqrt{\lambda-\Lambda_Q}\omega\right):z\in\mathbb{R}^{n-2d},\omega\in\mathbb{S}^{n-2d-1}\right\}.$$

Therefore

$$K_{\mathcal{E}^*_Q(\lambda,h)}\in I_h^{\frac{1-2n+4d}{4}}\left(\mathbb{R}^{n-2d}\times\mathbb{S}^{n-2d-1},C^*_Q\right)$$

with

$$C^*_Q=\left\{\left(z,\sqrt{\lambda-\Lambda_Q}\theta;\theta,-\sqrt{\lambda-\Lambda_Q}d^{\mathbb{S}^{n-2d-1}}_\theta\left\langle z,\theta\right\rangle\right):z\in\mathbb{R}^{n-2d},\theta\in\mathbb{S}^{n-2d-1}\right\}.$$

From this we also find that for $\tilde{\chi}_j \in C_c^\infty\left(\mathbb{R}^{n-2d}\right)$, $\chi_j = 1$ on $supp\sum_{k=1}^{n-2d}\partial_{z_k}\chi_j$, $j=1,2$,

$$WF_h^f\left(\tilde{\chi}_2 K_{\mathcal{E}_Q^*(\lambda,h)}\right) = \left\{\left(z, \sqrt{\lambda-\Lambda_Q}\,\omega; \omega, \sqrt{\lambda-\Lambda_Q}\, d_\omega^{\mathbb{S}^{n-2d-1}}\langle z,\omega\rangle\right) : z \in supp\,\tilde{\chi}_2, \omega \in \mathbb{S}^{n-2d-1}\right\} \tag{48}$$

and

$$WF_h^f\left(\tilde{\chi}_1 K_{\mathcal{E}_Q(\lambda,h)}\right) = \left\{\left(\theta, -\sqrt{\lambda-\Lambda_Q}\, d_\theta^{\mathbb{S}^{n-2d-1}}\langle w,\theta\rangle; w, -\sqrt{\lambda-\Lambda_Q}\,\theta\right) : w \in supp\,\tilde{\chi}_1, \theta \in \mathbb{S}^{n-2d-1}\right\}. \tag{49}$$

Then, unless $\Lambda_{Q_\delta} = \Lambda_{Q_\gamma} = \Lambda_{Q_\kappa}$, from (48), (49), [1, Lemma 4.5], and [1, Lemma 3.7 (iii)] we have that for all $\alpha, \beta \in \mathbb{N}_0^d$

$$WF_h^f\left(\partial_y^\alpha \partial_\eta^\beta K_{\mathcal{E}_{Q_\kappa}[h^2\Delta_z,\chi_1]c_{Q_\delta}(y,\eta)[h^2\Delta_z,\chi_2]\mathcal{E}_{Q_\gamma}^*}\right) = \emptyset \tag{50}$$

for all $(y,\eta) \in \mathbb{R}^{2d}$.

On the other hand, since for every $Q \in \mathcal{L}(\lambda)$ we have

$$supp\, K_{\tilde{\chi}_2 \mathcal{E}_Q(\lambda,h)} \subset supp\,\tilde{\chi}_2 \times \mathbb{S}^{n-2d-1}$$

and the support of $\tilde{\chi}_2$ is compact, it is easy to see that $WF_h^i\left(K_{\tilde{\chi}_2\mathcal{E}_Q(\lambda,h)}\right) = \emptyset$. Similarly, $WF_h^i\left(K_{\tilde{\chi}_1\mathcal{E}_Q^*(\lambda,h)}\right) = \emptyset$. Therefore, from [1, Theorem 3.7 (iii)], it follows that for all $\alpha, \beta \in \mathbb{N}_0^d$

$$WF_h^i\left(\partial_y^\alpha \partial_\eta^\beta K_{\mathcal{E}_{Q_\kappa}[h^2\Delta_z,\chi_1]c_{Q_\delta}(y,\eta)[h^2\Delta_z,\chi_2]\mathcal{E}_{Q_\gamma}^*}\right) = \emptyset \tag{51}$$

for all $(\eta, y) \in \mathbb{R}^{2d}$. Now, since the estimates in (50) and (51) can be made uniformly in $(\eta, y) \in \mathcal{K}$ and since the operator $\mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right] c_{Q_\delta}(y,\eta)\left[h^2\Delta_z,\chi_2\right]\mathcal{E}_{Q_\gamma}^*$ is independent of $(\eta, y) \in \mathcal{K}^c$, it follows that the estimates in (50) and (51) can be made uniformly in $(\eta, y) \in \mathbb{R}^{2d}$.

Therefore, from (50) and (51), we have that, unless $\Lambda_{Q_\delta} = \Lambda_{Q_\gamma} = \Lambda_{Q_\kappa}$,

$$\left\|\mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right] c_{Q_\delta}^w\left(y, h^2 D_y\right)\left[h^2\Delta_z,\chi_2\right]\mathcal{E}_{Q_\gamma}^*\right\|_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^d))} = \mathcal{O}\left(h^\infty\right). \tag{52}$$

From (31), (32), Proposition 5.4, and (52) we therefore obtain that

$$\begin{aligned}\mathcal{T}(\lambda b, b) &= c_0(h)\sum_{\Lambda_{Q_\delta}=\Lambda_{Q_\kappa}=\Lambda_{Q_\gamma}} \tilde{U}^*\tilde{\Pi}_{\gamma\kappa}\otimes F_{\gamma\delta\kappa}(\lambda,h)\,\tilde{U} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{2d}))}(h) \\ &= c_0(h)\sum_{k=1}^{K(\lambda)}\sum_{i=1}^{n_k}\sum_{j=1}^{n_k}\tilde{U}^*\tilde{\Pi}_{kij}\otimes G_k(\lambda,h)\,\tilde{U} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{2d}))}(h)\end{aligned} \tag{53}$$

with

$$G_k(\lambda,h) = \sum_{\left\{\gamma,\kappa,\delta:\Lambda_{Q_\gamma}=\Lambda_{Q_\kappa}=\Lambda_{Q_\delta}=\lambda_k\right\}} F_{\gamma\delta\kappa}(\lambda,h). \tag{54}$$

We now analyze the microlocal structure of the operators $G_k(\lambda,h)$. Let $k \in \{1,\ldots,K(\lambda)\}$ and let $\gamma,\delta,\kappa \in \{1,\ldots,N(\lambda)\}$ be such that $\Lambda_{Q_\gamma} = \Lambda_{Q_\delta} = \Lambda_{Q_\kappa} = \lambda_k$ and assume, without loss of generality, that

$$Q_\gamma = Q_\delta = Q_\kappa = Q. \tag{55}$$

Then for every $(y,\eta) \in \mathbb{R}^{2d}$ we have that the manifolds

$$C_Q' \times \Lambda_{\eta y}\left(\lambda-\lambda_k\right)'$$

and

$$T^*\mathbb{S}^{n-2d-1} \times diag\left(T^*\mathbb{R}^{n-2d}\times T^*\mathbb{R}^{n-2d}\right)\times T^*\mathbb{R}^{n-2d}$$

intersect transversely. This, together with (47) and (55), implies that for every $(y,\eta)\in\mathbb{R}^{2d}$

$$\mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right]c_{Q_\delta}(y,\eta)\left[h^2\Delta_z,\chi_2\right]\in\mathcal{I}_h^{\frac{4d-2n-5}{4}}\left(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{n-2d},C_Q'\circ\Lambda_{\eta y}\left(\lambda-\lambda_k\right)\right).$$

Therefore, using also (25) and [1, Lemma 4.3], we have that

$$\mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right]c_{Q_\delta}\left[h^2\Delta_z,\chi_2\right]\in S_{2d}^{d-\frac{n+1}{2},1}\left(1;\cup_{(y,\eta)\in\mathbb{R}^{2d}}\mathcal{I}_h^{\frac{4d-2n-5}{4}}\left(\mathbb{S}^{n-2d-1}\times\mathbb{R}^{n-2d},C_Q'\circ\Lambda_{\eta y}(\lambda-\lambda_k)\right)\right).$$

Similarly, for every $(y,\eta)\in\mathbb{R}^{2d}$ the manifolds

$$\left(C_Q'\circ\Lambda_{\eta y}\left(\lambda-\lambda_k\right)\right)'\times\left(C_Q^*\right)'$$

and

$$T^*\mathbb{S}^{n-2d-1}\times diag\left(T^*\mathbb{R}^{n-2d}\times T^*\mathbb{R}^{n-2d}\right)\times T^*\mathbb{S}^{n-2d-1}$$

intersect transversely. Since also

$$C_Q'\circ\Lambda_{\eta y}\left(\lambda-\lambda_k\right)'\circ C_Q^*=SR_{\eta y}\left(\lambda-\lambda_k\right)$$

with $SR_{\eta y}\left(\lambda-\lambda_k\right)$ defined as in (10) with respect to the symbol $p_{\eta y}$, we have that for every $(y,\eta)\in\mathbb{R}^{2d}$

$$\mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right]c_{Q_\delta}(y,\eta)\left[h^2\Delta_z,\chi_2\right]\mathcal{E}_{Q_\gamma}^*\in\mathcal{I}_h^{2d-n-1}\left(\mathbb{S}^{n-2d-1}\times\mathbb{S}^{n-2d-1},SR_{\eta y}\left(\lambda-\lambda_k\right)\right)$$

and similarly as above

$$\begin{aligned}g_k\overset{def}{=}&\sum_{\left\{\gamma,\kappa,\delta:\Lambda_{Q_\gamma}=\Lambda_{Q_\kappa}=\Lambda_{Q_\delta}=\lambda_k\right\}}\mathcal{E}_{Q_\kappa}\left[h^2\Delta_z,\chi_1\right]c_{Q_\delta}\left[h^2\Delta_z,\chi_2\right]\mathcal{E}_{Q_\gamma}^*\\&\in S_{2d}^{2d-n,1}\left(1;\cup_{(y,\eta)\in\mathbb{R}^{2d}}\mathcal{I}_h^{2d-n-1}\left(\mathbb{S}^{n-2d-1}\times\mathbb{S}^{n-2d-1},SR_{\eta y}\left(\lambda-\lambda_k\right)\right)\right).\end{aligned}\tag{56}$$

From (54), (32), Proposition 5.4, and (56) we now easily see that

$$\left\|G_k(\lambda,h)-g_k^w(y,h^2D_y)\right\|_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^d))}=\mathcal{O}(h^\infty),$$

which, together with (53) and the second part of (56), completes the proof of the theorem.

**Remark:** We remark that in the setting treated in [17] ($d=1$ under the assumption that the final direction $\theta_0$ is non-degenerate for the initial direction $\omega_0$) the fact that

$$K_{F_{\gamma\delta\kappa}}\left(\theta_0,\omega_0\right)=\mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}))}\left(h^\infty\right)\ \text{for}\ \gamma\neq\kappa.\tag{57}$$

is already proven in [17, Section 4.2] using a different approach which is also applicable to our current situation, namely, as a consequence of Lemma 5.3. However, the fact that

$$K_{F_{\kappa\delta\kappa}}\left(\theta_0,\omega_0\right)=\mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}))}\left(h^\infty\right)\ \text{for}\ \delta\neq\kappa\tag{58}$$

has not been proven in [17]. Equations [17, (5.10)] and [17, (5.11)] establish only that

$$K_{F_{\kappa\delta\kappa}}\left(\theta_0,\omega_0\right)=\mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}))}\left(h\right)\ \text{for}\ \delta\neq\kappa$$

and, as also remarked just above [17, (5.10)] and [17, (5.11)], this is the only estimate proven in [17] for the operators $K_{F_{\kappa\delta\kappa}}\left(\theta_0,\omega_0\right)$ with $\kappa\neq\delta$. For [17, Theorem 2] it is, however, necessary to establish the estimate (58) in order to eliminate the possibility that the terms of the form $K_{F_{\kappa\delta\kappa}}\left(\theta_0,\omega_0\right)$ with $\kappa\neq\delta$ make a non-negligible contribution to the asymptotic expansion of the scattering amplitude stated in [17, Theorem 2]. We remark further that the estimate (58) is not implied by any of the methods employed in [17] and we believe that our approach leading to establishing (52) is required in [17].

## 6 Proof of Theorem 3.1

Let $R>0$ be such that $supp\,\chi_2 \subset B^{n-2d}(0,R)$ and let $k\in\{1,\dots,K(\lambda)\}$. Now, let $\Psi_k : T^*\mathbb{S}^{n-2d-1}\times\mathcal{K}\to\mathbb{S}^{n-2d-1}$ be such that for every $(\eta,y)\in\mathcal{K}$ we have

$$SR_{T^*\mathbb{S}^{n-2d-1},\eta y}(\lambda-\lambda_k) = graph\ \Psi_k(\cdot;y,\eta).$$

Let $(\eta_0,y_0)\in\mathcal{K}$ and let $(\omega',z')\in T^*\mathbb{S}^{n-2d-1}$. Then [11, Lemma 9.5] implies that there exist open sets $\mathcal{U}'_k\times\mathcal{Y}'_k\subset T^*\mathbb{S}^{n-2d-1}\times\mathbb{R}^{2d}$ containing $(\omega',z',y_0,\eta_0)$ and two partitions $\{I,II\}$ and $\{III,IV\}$ of the indices $\{1,\dots,n-2d-1\}$ such that the map

$$\begin{aligned}\pi_{I,II,III,IV}:\quad & SR_{T^*\mathbb{S}^{n-2d-1},\eta y}(\lambda-\lambda_k) && \to \mathbb{R}^{2n-4d-2}\\ & (\omega,a;\theta,b) && \mapsto \left(\omega^I,a^{II};\theta^{III},b^{IV}\right)\end{aligned}$$

is a diffeomorphism on $graph\ \Psi_k(\cdot;y,\eta)|_{\mathcal{U}'_k}$ for every $(\eta,y)\in\mathcal{Y}'_k$. Here $\omega^I\equiv(\omega_i)_{i\in I}$ and the other terms on the right are defined analogously.

Then, perhaps after reducing the set $\mathcal{U}'_k$ around $(\omega',z')$, we can find an integer $m'_k\in\mathbb{N}_0$, an open set $V'_k\subset\mathbb{R}^{2n-4d-2+m'_k}$, and a function $\varphi'_k\in C^\infty(V'_k\times\mathcal{Y}'_k)$, such that for every $(y,\eta)\in\mathcal{Y}'_k$ the function $\varphi'_k(\cdot;y,\eta)$ is a non-degenerate phase function locally parameterizing the scattering relation at energy $\lambda-\lambda_k$ with parameters $(\eta,y)$ is the sense that

$$\Lambda_{\varphi'_k(\cdot;y,\eta)}\equiv SR_{\mathcal{U}'_k,\eta y}(\lambda-\lambda_k).$$

Now, since

$$g_k\in S^{1,1}_{2d}\left(1;\cup_{(y,\eta)\in\mathbb{R}^{2d}}\mathcal{I}^{2d-n-1}_h\left(\mathbb{S}^{n-2d-1}\times\mathbb{S}^{n-2d-1},SR_{y\eta}(\lambda-\lambda_k)\right)\right)$$

we can apply [1, Theorem 4.4] to obtain that there exist an open set $\mathcal{U}''_k\Subset\mathcal{U}'_k$ with $(\omega',z')\in\mathcal{U}''_k$ and a symbol $a'_k\in C^\infty\left((0,1]_h;C^\infty(V'_k\times\mathcal{Y}'_k)\right)\cap S^{d-\frac{n-m'_k+3}{2},0}_{2n-2d-2+m'_k}(1)$ such that for every $(\eta,y)\in\mathcal{Y}'_k$ we have $supp\ a'_k(\cdot;y,\eta)\Subset V'_k$ and

$$K_{g_k(y,\eta)}(\theta,\omega)\equiv\int_{\mathbb{R}^{m'_k}}e^{\frac{i}{h}\varphi'_k(\theta,\omega,\tau;y,\eta)}a'_k(\theta,\omega,\tau;y,\eta)d\tau, \tag{59}$$

microlocally near $graph\,\Psi_k(\cdot;y,\eta)|_{\overline{\mathcal{U}''_k}}$. Possibly after decreasing the set $\mathcal{Y}'_k$ around $(y_0,\eta_0)$ we can estimate the error of the approximation in (59) uniformly in $(y,\eta)\in\mathcal{Y}'_k$.

Then, since $\mathbb{S}^{n-2d-1}\times\overline{B^{n-2d-1}(0,R)}$ is compact, for every $k\in\{1,\dots,K(\lambda)\}$, it can be covered by finitely many open sets $\{\mathcal{U}_{k0\sigma}\}_{\sigma=1}^{\sigma_{k0}}$, $\sigma_k\in\mathbb{N}$, having the properties of the sets $\mathcal{U}'_k$ above. Let further $\mathcal{Y}_{k0}=\cap_{\sigma=1}^{\sigma_{k0}}\mathcal{Y}_{k0\sigma}$. Since $\mathcal{K}$ is compact, for every $k\in\{1,\dots,K(\lambda)\}$, it can be covered by finitely many open sets $\{\mathcal{Y}_{kq}\}_{q=1}^{q_k}$ for some $q_k\in\mathbb{N}$, having the properties of the set $\mathcal{Y}_{k0}$. We further choose the sets $\left\{\{\mathcal{Y}_{kq}\}_{q=1}^{q_k}\right\}_{k=1}^{K(\lambda)}$ to be bounded. For every $q\in\{1,\dots,q_k\}$ we denote the corresponding open cover of $\mathbb{S}^{n-2d-1}\times\overline{B^{n-2d-1}(0,R)}$ by $\{\mathcal{U}_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}$. We further denote the dimensions of the fiber variables $\tau$, their domains, the phase functions, and the symbols in the associated oscillatory integrals (59) by $\left\{\{\tilde m_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, $\left\{\{V_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, $\left\{\{\tilde\varphi_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, and $\left\{\{\tilde a_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, respectively.

Therefore from (56), (48), (49), and [1, Lemma 3.7 (ii), (iii)], it follows that for every $(\eta,y)\in\mathcal{K}$ we have

$$WF^f_h\left(K_{g_k(y,\eta)}\right)\subset\mathbb{S}^{n-2d-1}\times B^{n-2d-1}(0,R)\times\mathbb{S}^{n-2d-1}\times B^{n-2d-1}(0,R). \tag{60}$$

Furthermore, from (56) and [1, Lemma 4.5] we have that for all $(\eta,y)\in\mathcal{K}$

$$WF^f_h\left(K_{g_k(y,\eta)}\right)\subset SR_{\mathbb{S}^{n-2d-1}\times B^{n-2d-1}(0,R),\eta y}(\lambda-\lambda_k). \tag{61}$$

We now consider the infinite semi-classical wavefront set of the Schwartz kernel of the operator $g_k(y,\eta)$ defined in (56). Since

$$supp\,\tilde\chi_1 K_{\mathcal{E}_{Q_\kappa}(\lambda,h)}\subset supp\,\tilde\chi_1\times\mathbb{S}^{n-2d-1}$$

and $supp\,\tilde{\chi}_1$ is compact it is easy to see that $WF_h^i\left(\tilde{\chi}_1 K_{\mathcal{E}_{Q_\kappa}(\lambda,h)}\right)=\emptyset$. Similarly we have that

$$WF_h^i\left(\tilde{\chi}_2 K_{\mathcal{E}^*_{Q_\gamma}(\lambda,h)}\right)=\emptyset.$$

Therefore, from [1, Theorem 3.7 (iii)], it follows that

$$WF_h^i\left(K_{g_k(y,\eta)}\right)=\emptyset \tag{62}$$

for all $(\eta,y)\in\mathbb{R}^{2d}$. Since the estimates in (62) can be made uniformly in $(\eta,y)\in\mathcal{K}$ and $d_k$ is constant in $(\eta,y)$ for $(\eta,y)\notin\mathcal{K}$, it follows that the estimates in (62) can be made uniformly in $(\eta,y)\in\mathbb{R}^{2d}$.

Let, now, $(\chi_{kq})_{q=1}^{q_k}\subset C_c^\infty\left(\mathbb{R}^{2d}\right)$ be a partition of unity subordinate to the cover $(\mathcal{Y}_{kq})_{q=1}^{q_k}$ of $\mathcal{K}$ and let $\chi_k\in C^\infty\left(\mathbb{R}^{2d}\right)$ be given by $\chi_k=1-\sum_{q=1}^{q_k}\chi_{kq}$. From (68), (59), the remark following it, (60), (61), (62), and the remark following that, we see that an appropriately chosen microlocal partition of unity constructed for every $q=1,\dots,q_k$, subordinate to the open covers

$$\Big\{\left\{graph\,\Psi_k\left(\cdot;y,\eta\right)|_{\mathcal{U}_{kq\sigma}}\right\}_{\sigma=1}^{\sigma_{kq}},\left(SR_{\mathbb{S}^{n-2d-1}\times B^{n-2d-1}(0,R),y\eta}\left(\lambda-\lambda_k\right)\right)^c, \\ \mathbb{S}^{n-2d-1}\times\overline{B^{n-2d-1}(0,R)}^c\times\mathbb{S}^{n-2d-1}\times\overline{B^{n-2d-1}(0,R)}^c\Big\}$$

of $T^*\mathbb{S}^{n-2d-1}\times T^*\mathbb{S}^{n-2d-1}$ with $(\eta,y)\in\mathcal{Y}_{kq}$ and with symbols depending smoothly on $(y,\eta)$ reveals that for some dimensions $\left\{\{m_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, functions $\left\{\{\varphi_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, and symbols $\left\{\{a_{kq\sigma}\}_{\sigma=1}^{\sigma_{kq}}\right\}_{q=1}^{q_k}$, with properties as above we have that for $q\in\{1,\dots,q_k\}$

$$g_k\chi_{kq}\equiv\sum_{\sigma=1}^{\sigma_{kq}}\int_{\mathbb{R}^{m_{kq\sigma}}}e^{\frac{i}{h}\varphi_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)}a_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)\,d\tau+\mathcal{O}_{C^0(\mathbb{R}^{2d};\mathcal{B}(L^2(\mathbb{S}^{n-2d-1})))}\left(h^\infty\right).$$

Now, since $g_k\sum_{q=1}^{q_k}\chi_{kq}$ has compact support, it follows that

$$\left(g_k\sum_{q=1}^{q_k}\chi_{kq}-\sum_{q=1}^{q_k}\sum_{\sigma=1}^{\sigma_{kq}}\int_{\mathbb{R}^{m_{kq\sigma}}}e^{\frac{i}{h}\varphi_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)}a_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)\,d\tau\right)^w\left(y,h^2D_y\right)=Op_{h^2}\left(\tilde{s}_k\right)+\mathcal{H}_k, \tag{63}$$

where

$$\tilde{s}_k\in S^{0,1}_{3d}\left(1;\mathcal{B}\left(L^2\left(\mathbb{S}^{n-2d-1}\right)\right)\right)\ \text{ and }K_{Op_{h^2}(\tilde{s}_k)}\in C_c^\infty\left(\mathbb{R}^{2d};\,\mathcal{B}\left(L^2\left(\mathbb{S}^{n-2d-1}\right)\right)\right) \tag{64}$$

and

$$\|\mathcal{H}_k\|_{\mathcal{B}(L^2(\mathbb{R}^d\times\mathbb{S}^{n-2d-1}))}=\mathcal{O}\left(h^\infty\right). \tag{65}$$

From (63) we further obtain that

$$\left\|K_{Op_{h^2}(\tilde{s}_k)}\right\|_{C^0(\mathbb{R}^{2d};\mathcal{B}(L^2(\mathbb{S}^{n-2d-1})))}=\mathcal{O}\left(h^\infty\right). \tag{66}$$

Therefore, from (64), (66), and Schur's Lemma [15, Lemma 2.8.4] we obtain that

$$\|Op_{h^2}\left(\tilde{s}_k\right)\|_{\mathcal{B}(L^2(\mathbb{R}^d\times\mathbb{S}^{n-2d-1}))}=\mathcal{O}\left(h^\infty\right),$$

which, together with (65), implies

$$\left(g_k\sum_{q=1}^{q_k}\chi_{kq}\right)^w\left(y,h^2D_y\right)=\sum_{q=1}^{q_k}\sum_{\sigma=1}^{\sigma_{kq}}\left(\int_{\mathbb{R}^{m_{kq\sigma}}}e^{\frac{i}{h}\varphi_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)}a_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot\cdot)d\tau\right)^w\left(y,h^2D_y\right) \\ +\mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}^d\times\mathbb{S}^{n-2d-1}))}\left(h^\infty\right). \tag{67}$$

where $\theta,\omega\in\mathbb{S}^{n-2d-1}$.

We now observe that $g_k$ is constant in $(\eta, y)$ for $(\eta, y) \in \mathcal{K}^c$. As above we therefore have that there exist $N_k \in \mathbb{N}$, $\{m_{kN}\}_{N=1}^{N_k} \subset \mathbb{N}_0$, open sets $V_{kN} \subset \mathbb{R}^{m_{kN}}$, non-degenerate phase functions $\psi_{kN} \in C^\infty\left(\mathbb{S}^{n-2d-1} \times \mathbb{S}^{n-2d-1} \times V_{kN}\right)$ for $N \in \{1, \ldots, N_k\}$, locally parameterizing the scattering relation $SR_{\eta y}(\lambda - \lambda_k)$, and symbols $a_{kN} \in S^{d-\frac{n-m_{kN}+3}{2},0}_{2n-4d-2+m_{kN}}(1)$ for $N \in \{1, \ldots, N_k\}$ such that

$$g_k \chi_k = \chi_k \sum_{N=1}^{N_k} \int_{\mathbb{R}^{m_N}} e^{\frac{i}{h}\psi_{kN}(\theta,\omega,\tau)} a_{kN}(\theta, \omega, \tau)\, d\tau + \mathcal{O}_{S^{1,1}_{2d}(1;\mathcal{B}(L^2(\mathbb{S}^{n-2d-1})))}(h^\infty). \tag{68}$$

From (67) and (68) we therefore obtain

$$\begin{aligned} g_k^w\left(y, h^2 D_y\right) = &\sum_{q=1}^{q_k} \sum_{\sigma=1}^{\sigma_k} \left(\int_{\mathbb{R}^{m_{kq\sigma}}} e^{i\sqrt{b}\varphi_{kq\sigma}(\theta,\omega,\tau;\cdot,\cdot)} a_{kq\sigma}(\theta, \omega, \tau; \cdot, \cdot)\, d\tau\right)^w \left(y, h^2 D_y\right) \\ &+ \chi_k^w\left(y, h^2 D_y\right) \sum_{N=1}^{N_k} \int_{\mathbb{R}^{m_N}} e^{\frac{i}{h}\psi_{kN}(\theta,\omega,\tau)} a_{kN}(\theta, \omega, \tau)\, d\tau + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{S}^{n-2d-1}\times\mathbb{R}^d))}(h^\infty), \end{aligned}$$

where $\theta, \omega \in \mathbb{S}^{n-2d-1}$.

The conclusion of the theorem now follows from the Main Theorem.

# 7 Proof of Corollary 3.2

With the notation introduced in Section 3 immediately preceding the statement of Corollary 3.2, let $k \in \{1, \ldots, K(\lambda)\}$, $l \in \{1, \ldots, l_k\}$. Let $(\eta', y') \in \mathbb{R}^{2d}$. For $(\omega, \theta) \in O_- \times O_+$ let $w_{kl}(\omega, \theta; \eta', y') \in \theta^\perp$ be such that

$$\gamma_\infty\left(\cdot, w_{kl}(\omega, \theta; \eta', y'), -\sqrt{\lambda - \lambda_k}\theta; \eta', y'\right) = \gamma_\infty\left(\cdot, z_{kl}(\omega, \theta; \eta', y'), \sqrt{\lambda - \lambda_k}\omega; \eta', y'\right).$$

Then, working as in the proof of [2, Lemma 3], we see that there exist open neighborhoods $O'_- \subset O_-$ and $O'_+ \subset O_+$ in $\mathbb{S}^{n-2d-1}$ of $\omega_0$ and $\theta_0$, respectively, such that the map

$$\theta^\perp \ni w \mapsto \xi_\infty\left(-\sqrt{\lambda - \lambda_k}\theta, w; \eta', y'\right) \in \mathbb{S}^{n-2d-1}$$

is non-degenerate at $w_{kl}(\omega, \theta; \eta', y')$ for all $(\omega, \theta) \in O'_- \times O'_+$.

Let $SR_{l,\eta' y'}\left(O'_- \times O'_+; \lambda - \lambda_k\right)$ denote the scattering relation

$$SR_{l,\eta' y'}\left(O'_- \times O'_+; \lambda - \lambda_k\right) = \left\{(\omega, z_{kl}(\omega, \theta; \eta', y'); \theta, w_{kl}(\omega, \theta; \eta', y')) : (\omega, \theta) \in O'_- \times O'_+\right\}.$$

We can now apply [2, Lemma 4] to conclude that under the non-degeneracy assumption on $\theta_0$ and $\omega_0$

$$\Lambda_{S_{kl}(\cdot;y',\eta')|_{O'_-\times O'_+}} \equiv SR_{l,\eta' y'}\left(O'_- \times O'_+; \lambda - \lambda_k\right). \tag{69}$$

Now, using Proposition 5.2 we see that we can estimate the remainder term in (30) as follows

$$\sup_{\substack{(\omega,\theta)\in \\ \mathbb{S}^{n-2d-1}\times\mathbb{S}^{n-2d-1}}} \left\| K_{\tilde{U}^*\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)R_-[\Delta_z,\chi_1]E_0(\lambda)^{-1}e^{-\frac{i}{h}T_0E_0(\lambda)}[\Delta_z,\chi_2]R_+\tilde{\mathcal{F}}_0\left(\frac{\lambda}{h^2}\right)^*\tilde{U}}(\omega, \theta)\right\|_{\mathcal{B}(L^2(\mathbb{R}^{2d}))} = \mathcal{O}\left(\frac{1}{h}\right).$$

From (50) and (51) we further have that the estimate given in (52) can also be made in this new norm and proceeding as in the proof of the Main Theorem we then obtain from Proposition 5.4 that

$$\sup_{\substack{(\omega,\theta)\in \\ \mathbb{S}^{n-2d-1} \\ \times\mathbb{S}^{n-2d-1}}} \left\| K_{\mathcal{T}(\lambda b,b)}(\theta, \omega) - c_0(h) \sum_{k=1}^{K(\lambda)} \sum_{i,j=1}^{n_k} g_k^w\left(\theta, \omega, \frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b}\right) \Pi_{kij} \right\|_{\mathcal{B}(L^2(\mathbb{R}^{2d}))} = \mathcal{O}(h).$$

It is now clear that the remainder in Theorem 3.1 can also be estimated in this new norm and the conclusion of the Corollary then follows from (69).

# 8 Two Special Cases

In this section we discuss the microlocal structure of the scattering amplitude in the current setting in two special cases: when the potential $V$ depends only on $z$ and when the total dimension equals $n = 2d + 1$.

## 8.1 Independent Electric and Magnetic Fields

Suppose that the electric potential $V$ depends only on the variable $z$. Then, as in [17] we have that the scattering amplitude $\mathcal{T}$ takes the form

$$\mathcal{T}(\lambda b, b) = \sum_{k=1}^{K(\lambda)} T_E(\lambda - \lambda_k, h)\,\Pi_k,$$

where $T_E(\lambda - \lambda_k, h)$ is the scattering amplitude for the pair $\left(-h^2\Delta_z, -h^2\Delta_z + V(z)\right)$ at energy $\lambda - \lambda_k$. The microlocal structure of the scattering amplitude $T_E(\lambda - \lambda_k, h)$ without the non-degeneracy assumption at non-trapping energies and at trapping energies under an assumption on the absence of the resonances near the real axis has been studied in [2] for compactly supported potentials $V$ and in [3] for short range potentials $V$. At an energy which is a non-degenerate global maximum of the potential $V$ when it is short range or compactly supported the microlocal structure of the scattering amplitude has been analyzed in [4].

## 8.2 Total Dimension $n = 2d + 1$

The scattering matrix in this case is known to be given by

$$S(E, b) = \begin{pmatrix} S_{11}(E,b) & S_{12}(E,b) \\ S_{21}(E,b) & S_{22}(E,b) \end{pmatrix},$$

where the operators $S_{ij}(E,b) \in \mathcal{B}\left(L^2\left(\mathbb{R}^{2d}\right)\right)$, $i, j = 1, 2$, satisfy the relations $S_{11}(E,b) = S_{11}(E,b)$ and $S_{12}(E,b)S_{22}(E,b)^* = -S_{11}(E,b)S_{21}(E,b)^*$.

Under the assumption that $\lambda > 0$ is a non-trapping energy level, we have, as in [17], that there exists $k_1 \in \{1, \ldots, K(\lambda)\}$ such that

- for all $k \in \{1, \ldots, k_1\}$, $\lambda_k > \sup V$, and
- for all $k \in \{k_1 + 1, \ldots, K(\lambda)\}$ and all $(x, y) \in \mathbb{R}^{2d}$ the equation $V(x, y, z) = \lambda_k$ has exactly two solutions $\alpha_k(x, y) < \beta_k(x, y)$.

We remark that $\alpha_k$ and $\beta_k$ are smooth functions of $(x, y)$, which are independent of $(x, y) \notin \mathcal{K}$.

Analogously to [17, Theorem 1] we then have

**Theorem 8.1.** *Let $n = 2d + 1$ and let $\lambda > 0$ be non-trapping.*

*Then*

$$S_{11}(\lambda b, b) = \sum_{k=1}^{k_1} \sum_{i,j=1}^{n_k} s_{11,k}^w\left(\frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b}, \lambda\right)\Pi_{kij} + \sum_{k=K(\lambda)+1}^{\infty} \sum_{i,j=1}^{n_k} \Pi_{kij} + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}))}(h),$$

*where the symbols $s_{11,k} \in S_{2d}^{0,1}(1)$, $k \in \{1, \ldots, k_1\}$, satisfy*

$$s_{11,k}(y, \eta, \lambda) = \exp\left(\frac{i}{h}\int_{-\infty}^{\infty} \sqrt{\lambda - \lambda_k - V(\eta, y, z)} - \sqrt{\lambda - \lambda_k}dz\right) + hr_{11,k}(y, \eta, \lambda)$$

*with $r_{11,k}(y, \eta, \lambda) \in S_{2d}^{0,1}(1)$, $k \in \{1, \ldots, k_1\}$,*

$$S_{21}(\lambda b, b) = \sum_{k=k_1+1}^{\infty} \sum_{i,j=1}^{n_k} s_{21,k}^{w} \left( \frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b}, \lambda \right) + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}))}(h),$$

*where the symbols* $s_{21,k} \in S_{2d}^{0,1}(1),\ k \in \{1,\ldots,k_1\}$ *, satisfy*

$$s_{21,k}(y,\eta,\lambda) = i \exp\left( \frac{2i}{h} \left( \sqrt{\lambda - \lambda_k}\alpha_k(\eta,y) + \int_{-\infty}^{\alpha_k(\eta,y)} \sqrt{\lambda - \lambda_k - V(\eta,y,z)} - \sqrt{\lambda - \lambda_k} dz \right)\right) + h r_{11,k}(\lambda, y, \eta)$$

*with* $r_{11,k}(y,\eta,\lambda) \in S_{2d}^{0,1}(1),\ k \in \{1,\ldots,k_1\}$ *, and*

$$S_{12}(\lambda b, b) = \sum_{k=k_1+1}^{\infty} \sum_{i,j=1}^{n_k} s_{12,k}^{w} \left( \frac{y}{2} - \frac{D_x}{\mu b}, \frac{x}{2} + \frac{D_y}{\mu b}, \lambda \right) + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}))}(h),$$

*where the symbols* $s_{12,k} \in S_{2d}^{0,1}(1),\ k \in \{1,\ldots,k_1\}$ *, satisfy*

$$s_{12,k}(y,\eta,\lambda) = i \exp\left( \frac{2i}{h} \left( \sqrt{\lambda - \lambda_k}\beta_k(\eta,y) - \int_{\beta_k(\eta,y)}^{\infty} \sqrt{\lambda - \lambda_k - V(\eta,y,z)} - \sqrt{\lambda - \lambda_k} dz \right)\right) + h r_{12,k}(y, \eta, \lambda)$$

*with* $r_{12,k}(y,\eta,\lambda) \in S_{2d}^{0,1}(1),\ k \in \{1,\ldots,k_1\}$ *.*

*Proof.* To prove this theorem we can apply the same analysis as in the proof of [17, Theorem 1] to the operators $G_k(\lambda, h)$ in (53). Alternatively, one can use the analysis of the unitary group we have presented in the proof of the Main Theorem above. □

We observe that again the error term in the analogous result [17, Theorem 1] is asserted to be of order $\mathcal{O}(h^\infty)$ but this is due to the mistake made in [17] in estimating the error term in the analog to (31) to be $\mathcal{O}(h^\infty)$. The correct estimate of these error terms both here and in [17] is $\mathcal{O}(h)$.

# 9 Appendix: Semi-Classical Essentials

Here we recall some of the elements of semi-classical analysis which we use in this paper.

A family $(u_h)_{h\in]0,h_0]}$ of distributions in $\mathcal{D}'(\mathbb{R}^n)$ is called a semiclassical distribution when

$$\forall \chi \in C_0^\infty(\mathbb{R}^n), \quad \exists N \in \mathbb{N}, \quad |\mathcal{F}_h(\chi u)(\xi)| \le C_N h^{-N} \langle \xi \rangle^N,$$

where $\mathcal{F}_h$ is the $h$-Fourier transform

$$\mathcal{F}_h(\chi u)(\xi) = \left\langle u, \chi e^{-\frac{i}{h}\langle \cdot, \xi\rangle} \right\rangle_{\mathcal{D}'(\mathbb{R}^n)}.$$

The space of semiclassical distributions is denoted $\mathcal{D}'_h(\mathbb{R}^n)$. We define the semiclassical wavefront set of $u = (u_h) \in \mathcal{D}'_h(\mathbb{R}^n)$ as follows:

**Definition 9.1.** *Let* $u \in \mathcal{D}'_h(\mathbb{R}^n)$ *and let* $(x_0, \xi_0) \in T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n$*, where we remark that the latter union is the fiber-compactified cotangent bundle of* $\mathbb{R}^n$*. We shall say that* $(x_0,\xi_0)$ *does not belong to the semiclassical wavefront set of* $u$ *if:*

- *If* $(x_0,\xi_0) \in T^*\mathbb{R}^n$*: there exist* $\chi \in C_c^\infty(\mathbb{R}^n)$ *with* $\chi(x_0) \neq 0$ *and an open neighborhood* $U$ *of* $\xi_0$*, such that* $\forall N \in \mathbb{N}$*,* $\forall \xi \in U$*,*
$$|\mathcal{F}_h(\chi u)(\xi)| \le C_{N,U} h^N.$$
*We shall denote the complement of the set of all such points by* $WF_h^f(u)$*.*

- *If* $(x_0, \xi_0) \in T^*\mathbb{S}^{n-1}$*: there exist* $\chi \in C_c^\infty(\mathbb{R}^n)$ *with* $\chi(x_0) \neq 0$ *and a conic neighborhood* $U$ *of* $\xi_0$*, such that* $\forall N \in \mathbb{N}$, $\forall \xi \in U \cap \left\{|\xi| \geq \frac{1}{K}\right\}$ *for some* $K > 0$,

$$|\mathcal{F}_h(\chi u)(\xi)| \leq C_{N,U,K} h^N \langle \xi \rangle^{-N}.$$

*We shall denote the complement of the set of all such points by* $WF_h^i(u)$.

*We shall further use* $WF_h(u) = WF_h^f(u) \sqcup WF_h^i(u)$ *to denote the semiclassical wavefront set of* $u$.

Semi-classical symbols are defined as follows:

**Definition 9.2.** *We say that* $g : \mathbb{R}^n \to [0, \infty)$ *is an order function if there are constants* $C > 0$ *and* $N > 0$ *such that* $g(x) \leq C\langle x - y\rangle^N g(y)$.

*For* $d \in \mathbb{N}$, $m \in \mathbb{R}$, $\delta \in [0, 1]$, *and an order function* $g$ *on* $\mathbb{R}^{2d}$ *we define the symbol class*

$$S_{2d}^{m,\delta}(g) = \left\{a \in C^\infty\left(\mathbb{R}^{2d} \times (0, h_0]\right) : \left\|g^{-1}\partial^\alpha a\right\|_{L^\infty(\mathbb{R}^{2d})} \leq C_\alpha h^{-m-\delta|\alpha|}\ \forall \alpha \in \mathbb{N}_0^{2d}\right\}.$$

The finite (semi-classical) essential support of a symbol is given by:

**Definition 9.3.** *We say that the point* $(x_0, \xi_0) \in T^*\mathbb{R}^k$ *does not belong the finite (semi-classical) essential support of the symbol* $a \in S_{2k}^{m,\delta}(g)$ *if there exists an open neighborhood* $\mathcal{G}$ *of* $(x_0, \xi_0)$ *in* $T^*\mathbb{R}^k$ *such that for all* $\alpha, \beta \in \mathbb{N}_0^k$

$$\partial_x^\alpha \partial_\xi^\beta a(x, \xi) = \mathcal{O}(h^\infty) \text{ uniformly in } (x, \xi) \in \mathcal{G}.$$

*The complement of all such points* $(x_0, \xi_0)$ *will be denoted by* $ess - supp_h^f a$.

If $u, v \in \mathcal{D}_h'(\mathbb{R}^k)$ and $U \subset T^*\mathbb{R}^k$ is an open set, we say that $u = v$ microlocally near $U$ if there exists an open set $\tilde{U} \subset T^*\mathbb{R}^k$ such that $U \subset \tilde{U}$ and for every $a \in S_{2k}^{0,0}(1)$ with $ess - supp_h^f a \Subset \tilde{U}$ we have that $Op_h(a)(u - v) = \mathcal{O}(h^\infty)$ in $C^\infty(\mathbb{R}^k)$.

Let $\varphi \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\theta^m)$. The function $\varphi$ is said to be a non-degenerate phase function if

$$\varphi_\theta'(x, \theta) = 0 \text{ implies that } (\varphi_{\theta x}''\ \varphi_{\theta\theta}'') \text{ has maximum rank at } (x, \theta).$$

We let

$$C_\varphi = \{(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^m : \varphi_\theta'(x, \theta) = 0\} \text{ and } \Lambda_\varphi = \{(x, \varphi_x'(x, \theta)) : (x, \theta) \in C_\varphi\}$$

for $m > 0$ and

$$\Lambda_\varphi = \{(x, \varphi'(x)) : x \in \mathbb{R}^n\}$$

for $m = 0$. Then $\Lambda_\varphi$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$.

We say that the non-degenerate phase function $\varphi$ locally parameterizes the Lagrangian submanifold $\Lambda \subset T^*\mathbb{R}^n$ is there exists an open set $U \subset T^*\mathbb{R}^n$ such that

$$\Lambda_\varphi \cap U \equiv \Lambda \cap U \neq \emptyset.$$

We can now define semi-classical Fourier integral operators (h-FIOs) as follows (see also [1, Section 4]):

**Definition 9.4.** *Let* $M$ *be a smooth* $n$*-dimensional manifold and let* $\Lambda \subset T^*M$ *be a smooth closed Lagrangian submanifold with respect to the canonical symplectic structure on* $T^*M$. *Let* $r \in \mathbb{R}$. *Then the space* $I_h^r(M, \Lambda)$ *of semi-classical Fourier integral distributions of order* $r$ *associated to* $\Lambda$ *is defined as the set of all* $u \in \mathcal{D}_h'(M)$ *such that*

$$\left(\prod_{j=0}^{N} A_j\right)(u) = \mathcal{O}_{L^2(M)}\left(h^{N-r-\frac{n}{4}}\right), \ h \to 0,$$

*for all* $N \in \mathbb{N}_0$ *and for all* $A_j \in \Psi_h^0(1, M)$, $j = 0, \ldots, N-1$, *with compact wavefront set near* $\Lambda$ *and principal symbols vanishing on* $\Lambda$, *and any* $A_N \in \Psi_h^0(1, M)$ *with compact wavefront set near* $\Lambda$.

[1, Theorem 4.4] proves that this definition of h-FIOs is equivalent to the following

**Definition 9.5.** *Let $M$ be a smooth $n-$dimensional manifold and let $\Lambda \subset T^*M$ be a smooth closed Lagrangian submanifold with respect to the canonical symplectic structure on $T^*M$. Let $r \in \mathbb{R}$. Then the space $I_h^r(M, \Lambda)$ of semi-classical Fourier integral distributions of order $r$ associated to $\Lambda$ is defined as the set of all $u \in \mathcal{D}_h'(M)$ such that for every non-degenerate phase function defined in an open set $V \subset \mathbb{R}^{n+m}$, $m \in \mathbb{N}_0$, such that $\Lambda = \Lambda_\varphi$ near $\gamma$, there exists $a \in S_{n+m}^{r+\frac{m}{2}+\frac{n}{4}}(1)$ with $\operatorname{supp} a \Subset V$ such that microlocally near $\gamma$, $u$ is equal to the oscillatory integral $\int_{\mathbb{R}^m} e^{\frac{i}{h}\varphi(x,\theta)} a\,(x, \theta)\, d\theta$.*

We further have

**Definition 9.6.** *Let $M_1$ and $M_2$ be smooth manifolds, let $\pi_j : T^*M_2 \times T^*M_1 \to T^*M_j$, $j = 1, 2$, denote the canonical projection and $\sigma_j$ the canonical symplectic form on $T^*M_j$, $j = 1, 2$. A continuous linear operator $C_c^\infty(M_1) \to \mathcal{D}_h'(M_2)$, whose Schwartz kernel is an element of $I_h^r(M_2 \times M_1, \Lambda)$ for some Lagrangian submanifold $\Lambda$ of $(T^*M_2 \times T^*M_1, \pi_2^*\sigma_2 + \pi_1^*\sigma_1)$ and some $r \in R$ will be called a microlocal semi-classical Fourier integral operator (h-FIO) of order $r$ associated to $\Lambda$. We denote the space of these operators by $\mathcal{I}_h^r(M_2 \times M_1, \Lambda)$.*

*We also say that $T \in \mathcal{I}_h^r(M_2 \times M_1, \Lambda)$ is a global h-FIO associated to $\Lambda$ if $WF_h^f(K_T) \subset \Lambda$.*

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