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Structure of the Semi-Classical Amplitude for General Scattering Relations

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We consider scattering by general compactly supported semi-classical perturbations of the Euclidean Laplace-Beltrami operator. We show that if the suitably cut-off resolvent quantizes a Lagrangian relation on the product cotangent bundle, the scattering amplitude quantizes the natural scattering relation. When we work microlocally near a non-trapped ray, our result implies that the scattering amplitude defines a semiclassical Fourier integral operator associated to the scattering relation in a neighborhood of that ray. Compared to previous work, we allow this relation to have more general geometric structure.

Keywords Scattering amplitude; Scattering relation; Semi-classical analysis.

Mathematics Subject Classification Primary 35P25; Secondary 35S99.

1. Introduction and Statement of Results

We study the semi-classical scattering amplitude for compactly supported metric and potential perturbation $P(h)$ of the Euclidean Laplace-Beltrami operator $P_0(h)$ on $\mathbb{R}^n$. The scattering amplitude at energy $\lambda > 0$ is the amplitude of the leading term in the asymptotic expansion of an outgoing solution of $(P(h) - \lambda)u = 0$ as $\|x\| = r \to \infty$. We prove that near a non-trapped trajectory the scattering amplitude quantizes, in the sense of global semi-classical Fourier Integral Operators, the natural in this setting scattering relation. The latter is given here by the graph of the Hamiltonian flow of the principal symbol p of $P(h)$ between two hypersurfaces inside $\{p = \lambda\}$ transverse to the Hamiltonian vector field of p. We also prove a general Black Box theorem which states that the scattering amplitude has the structure of a semi-classical Fourier integral operator whenever the suitably cut-off resolvent is such an operator associated to a Lagrangian relation on $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$.

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1.1. *A Survey of Earlier Results*

The structure of the scattering amplitude has been of considerable interest to researchers in mathematical physics. To outline the earlier results, we begin by making some definitions. Let $P(h) = -\frac{1}{2}h^2\Delta + V$, where

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}V(x)\right| \leq C_\alpha(1 + \|x\|^2)^{-\frac{\mu+|\alpha|}{2}}, \qquad x \in \mathbb{R}^n, \quad \mu > 0. \tag{1}$$

Let $\omega_0 \in \mathbb{S}^{n-1}$ and for $z \in \omega_0^\perp$, with z its coordinate representation in $\mathbb{R}^{n-1}$, let

$$\{q_\infty(\cdot; z, \lambda), p_\infty(\cdot; z, \lambda)\}$$

be the unique phase trajectory such that

$$\lim_{t\to-\infty} \|q_\infty(t; z, \lambda) - \sqrt{2\lambda}\omega_0 t - z\|_{C^\infty(\omega_0^\perp)} = 0,$$
$$\lim_{t\to-\infty} \|p_\infty(t; z, \lambda) - \sqrt{2\lambda}\omega_0\|_{C^\infty(\omega_0^\perp)} = 0$$

in the C^∞ topology for the impact parameter z. If $\lim_{t\to\infty}\|q_\infty(t; z, \lambda)\| = \infty$, then there exist $U \subset \omega_0^\perp$, open, $z \in U$, $\xi_\infty(\cdot; \lambda) \in C^\infty(\omega_0^\perp \cap U; \mathbb{S}^{n-1})$ and $r_\infty(\cdot; \lambda) \in C^\infty(\omega_0^\perp \cap U; \mathbb{R}^n)$ such that

$$\lim_{t\to\infty} \|q_\infty(t; z, \lambda) - \sqrt{2\lambda}\xi_\infty(z; \lambda)t - r_\infty(z; \lambda)\|_{C^\infty(U)} = 0,$$
$$\lim_{t\to\infty} \|q_\infty(t; z, \lambda) - \sqrt{2\lambda}\xi_\infty(z; \lambda)\|_{C^\infty(U)} = 0.$$

The trajectory $\{q_\infty(t; z, \lambda), p_\infty(t; z, \lambda)\}$ is then said to have initial direction ω_0 and final direction $\theta_0 = \xi_\infty(z, \lambda)$ and θ_0 is said to be non-degenerate, or regular, for ω_0 if for all $z \in \omega_0^\perp$ with $\xi_\infty(z; \lambda) = \theta_0$, the angular density $\hat{\sigma}(z; \lambda)$ for the trajectory $\{q_\infty(t; z, \lambda), p_\infty(t; z, \lambda)\}$ satisfies

$$\hat{\sigma}(z; \lambda) \stackrel{\text{def}}{=} \left|\det\left(\xi_\infty, \frac{\partial}{\partial z_1}\xi_\infty, \ldots, \frac{\partial}{\partial z_{n-1}}\xi_\infty\right)\right| \neq 0. \tag{2}$$

The first asymptotic expansion of the semi-classical scattering amplitude was given by Vainberg (1977). He considers the semi-classical Schrödinger operator with a potential $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ and assumes (2). For the associated scattering amplitude at non-trapping energies $\lambda > \sup V$, he then proves an asymptotic expansion of the scattering amplitude in the form

$$f(\theta, \lambda) = \sum_{j=1}^{l} \hat{\sigma}(z_j; \lambda)^{-1/2} \exp(ih^{-1}S_j - i\mu_j\pi/2) + \mathcal{O}(h), \tag{3}$$

where $\{z_j\}_{j=1}^l \equiv (\xi_\infty^{-1}(\cdot; \lambda))(\theta_0)$, S_j is a modified action along the jth (ω_0, θ) trajectory, $\theta \in U$, where $U \subset \mathbb{S}^{n-1}$ is a sufficiently small open neighborhood of θ_0, and μ_j is the path index of the trajectory. The error term is estimated uniformly in $\theta \in U$.

Majda (1976) considers scattering processes defined by the classical wave equation in the presence of a convex obstacle in $\mathbb{R}^n$, where $n = 2, 3$. For Dirichlet, Neumann, and in the case of three-dimensional space, impedance boundary conditions, he proves an asymptotic expansion of the scattering amplitude, the leading term of which is the product of the Gauss curvature and the reflection coefficient evaluated at a point on the boundary of the obstacle. In this setting, the scattering amplitude is the coefficient of the leading term in the asymptotic expansion of an outgoing solution of the reduced wave equation. To establish the aforementioned result, he studies the radiation pattern of a solution which approximates this outgoing solution near the boundary of the obstacle. He also applies his main result to inverse scattering problems for convex bodies with the above boundary conditions. In particular, he proves that both the shape of the boundary of the obstacle and the nature of the boundary conditions are completely determined by the asymptotic limit of the scattering amplitude.

Guillemin (1977) discusses similar asymptotic expansions. He studies the behavior of the scattering matrix in several different settings: on a compact manifold, in obstacle scattering, for a compactly supported perturbation of the Euclidean metric on $\mathbb{R}^n$, and for a quotient Riemannian manifold. In each case, he presents formulas for the kernel of the scattering matrix at energy λ of the form

$$S(\lambda, \omega, \theta) = c(\lambda, n) \sum_{j=1}^{N} c(j)\, |J_j(\lambda)|^{-1/2}\, e^{i\lambda T_j} + \mathcal{O}\left(\frac{1}{\lambda}\right), \qquad \theta \neq \omega,$$

under the assumption that there are N scattering rays with initial direction ω and final direction θ, where T_j is the sojourn time of the jth scattering ray and J_j is the scattering differential cross-section, evaluated at the point of incidence of the jth scattering ray. In the case of the quotient, the scattering matrix is a unitary matrix of size depending on the topology of the manifold. For each energy level λ, its jkth entry has the form

$$S_{jk}(\lambda) = ac(\lambda) \sum e^{-T_k(i\lambda+1/2)},$$

where

$$c(\lambda) = \int_{-\infty}^{\infty} \frac{dq}{(1+q^2)^{1/2+i\lambda}}.$$

To derive these results, Guillemin uses the representation of the scattering operator in terms of the wave operators. He also derives a formula for the scattering differential cross-section in the case of scattering by a smooth convex obstacle from which he deduces that the asymptotic behavior of the scattering amplitude determines the shape of the scatterer.

A different form of the asymptotic expansion of the scattering amplitude for smooth compactly supported potentials was given by Protas (1982). He proves that at non-trapping energies and for fixed initial directions, the scattering amplitude can be expressed as the sum of two Maslov canonical operators associated to each of the Lagrangian submanifolds L_+ and L_0 of $T^*\mathbb{S}^{n-1}$ constructed in the following way. The manifold L_+ is the projection onto $T^*\mathbb{S}^{n-1}$ of the Hamiltonian flow of the symbol of P with fixed initial direction, while L_0 coincides with L_+ when $V \equiv 0$. This representation of the scattering amplitude is shown to hold uniformly in an open set containing the final direction and disjoint from the initial direction.

Yajima (1987) was the first to prove an asymptotic expansion of the form (3) of the scattering amplitude for potential perturbations $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ of the semi-classical Laplacian satisfying (1) for a constant $\mu > \max(1, \frac{n-1}{2})$. He also works with outgoing directions non-degenerate for a fixed initial direction and at non-trapping energy levels. His results, however, are only valid in the L^2 sense.

Robert and Tamura (1989) work in the same setting as Yajima (1987) with $\mu > 1$. For the scattering amplitude at non-trapping energies $\lambda > 0$, which now satisfies

$$f(\lambda, h) \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \operatorname{diag}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})),$$

they establish an asymptotic expansion of the form (3) with

$$S_j = \int_{-\infty}^{\infty} (|p_\infty(t; z_j, \lambda)|^2/2 - V(q_\infty(t; z_j, \lambda)) - \lambda) dt - \langle r_\infty(w_j; \lambda), \sqrt{2\lambda}\theta \rangle. \tag{4}$$

Michel (2004) works in the same setting as Robert and Tamura but he allows the energy level to be trapping while satisfying the condition

> There exists a neighborhood W of $\omega_0 \in \mathbb{S}^{n-1}$ such that $\forall \omega' \in W$, $\forall z \in (\omega')^\perp$, $\lim_{t\to\infty} \|q_\infty(t, z, \omega')\| = \infty$.

He further assumes that there exists $\epsilon > 0$ such that the resonances λ_j satisfy $|\Im \lambda_j| \geq Ch^q$ for $\Re \lambda_j \in [\lambda - \epsilon, \lambda + \epsilon]$. Under these assumptions, he establishes the same asymptotic expansion of the scattering amplitude (3). Like Robert and Tamura, Michel also uses Isozaki-Kitada's representation formula of the scattering amplitude. Also like Robert and Tamura, Michel applies the method of stationary phase to an appropriately modified formula for the scattering amplitude, which all of these authors obtain after the use of resolvent estimates and results on the propagation of singularities.

1.2. *Statement of Results*

In this article we analyze the semi-classical scattering amplitude from a different point of view and *without* making any geometric assumptions on the scattering relation such as (2). We divide our results in two parts. First, we prove a general "black-box" theorem, which states that, under a microlocal assumption on the resolvent, essentially the assumption that a suitably cut-off resolvent quantizes the flow relation, the scattering amplitude is a semi-classical Fourier Integral Operator associated to the classical scattering relation. In other words, it quantizes that canonical relation.

The abstract "black-box" framework of Sjöstrand and Zworski (1991) allows us to formulate our hypotheses independently of the structure of the scatterer. This theorem thus serves to highlight the property of the "black-box" resolvent, which would imply that the scattering amplitude is a semi-classical Fourier integral operator. In the remainder of this article we prove that the appropriately cut-off resolvent in the setting of a smooth compactly supported metric and potential perturbation of the Euclidean Laplacian does indeed possess this property, and we explore the consequences of that. In particular, we show that the representation (3), under the non-degeneracy assumption, follows from knowing that the scattering amplitude is a semi-classical Fourier integral operator.

To state our general "black-box" theorem now, we begin by introducing some definitions and notation. We let

$$\pi_1 : T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1}$$

and

$$\pi_2 : T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n \times T^*\mathbb{R}^n$$

denote the canonical projections, and we introduce the following Lagrangian submanifold of

$$T^*\left(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n\right)$$

depending on the real energy $\lambda > 0$:

$$\begin{gathered}\Lambda(\lambda) = \{(m, df(m)) : m \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n \backslash B(0, R_0) \times \mathbb{R}^n \backslash B(0, R_0)\}, \\ f(m) = \sqrt{2\lambda}\langle y, \theta\rangle - \sqrt{2\lambda}\langle x, \omega\rangle, \qquad m = (\theta, \omega, y, x).\end{gathered} \tag{5}$$

For a Lagrangian submanifold $\Lambda \subset T^*M$, where M is a smooth manifold, we let $-\Lambda = \{(x, \xi) : (x, -\xi) \in \Lambda\}$.

We refer to Section 2.3 for the definition of the scattering amplitude $A(\lambda, h)$. The classes of Fourier integral distributions I_h^ν and Fourier integral operators $\mathcal{I}_h^\nu$ are defined in Appendix B, while their complete characterization is given in Alexandrova (Preprint, Section 4.2). The notion of microlocal localization is also reviewed in Appendix B.

Lastly, we make two assumptions.

Assumption 1. There exists $s \in \mathbb{R}$ such that for some $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi \equiv 0$ on $B(0, R_0)$, $\|\varphi R(\lambda, h)\varphi\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(h^s)$.

Assumption 2. There exists a Lagrangian submanifold $\Lambda_R(\lambda) \subset \pi_2(-\Lambda(\lambda))$ of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ such that

$$\begin{gathered}\text{for every } \tilde{\chi}_j \in C_c^\infty(\mathbb{R}^n \backslash B(0, R_0)), \qquad j = 1, 2, \quad \operatorname{supp}\tilde{\chi}_1 \cap \operatorname{supp}\tilde{\chi}_2 = \emptyset, \\ DK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} \in I_h^r(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_R(\lambda)),\end{gathered}$$

where $D \in \Psi_h^0(1, \mathbb{R}^n \times \mathbb{R}^n)$ is a microlocal cut-off to a neighborhood of

$$\{(y_0 + s\theta_0, x_0 + t\omega_0, -\sqrt{2\lambda}\theta_0, \sqrt{2\lambda}\omega_0,) : s \in [s_1, s_2], t \in [t_1, t_2]\},$$

for some fixed $s_1 < s_2 < 0$, $t_1 > t_2 > 0$, ω_0, θ_0, x_0, and y_0, and $K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}$ denotes the Schwartz kernel of the cut-off resolvent.

The last assumption means that the cut-off resolvent is a Fourier Integral Operator microlocally near some incoming and outgoing directions. The first assumption is made so that the notion of applying semi-classical pseudodifferential operators (which here are defined up to residual terms in $\mathcal{O}(h^\infty)$) makes sense. It will be used explicitly in the discussion of the resolvent near a non-trapped trajectory in Sections 5.2 and 5.3.

We also note that implicit in our assumptions is the fact that λ is non-resonant, in the sense that the resolvent does not have a pole at λ.

We can now state our

Theorem 1 (General Black Box Theorem). *Suppose that Assumptions* 1 *and* 2 *hold. Then*

$$\pi_1 \circ (\pi_2|_{\Lambda(\lambda)})^{-1}(-\Lambda_R(\lambda))$$

is a smooth Lagrangian submanifold of $T^*S^{n-1} \times T^*S^{n-1}$ *near*

$$\bar{p} = (\theta_0, \sqrt{2\lambda} d_\theta \langle y_0, \theta_0 \rangle; \omega_0, -\sqrt{2\lambda} d_\omega \langle x_0, \omega_0 \rangle)$$

and for every $C \in \Psi_h^0(1, \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ *with wavefront set in a sufficiently small neighborhood of* $\bar{p}$ *we have*

$$CA(\lambda, h) \in I_h^{r+\frac{1}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \pi_1 \circ (\pi_2|_{\Lambda(\lambda))})^{-1}(-\Lambda_R(\lambda))).$$

Alexandrova (Preprint, Theorem 1) gives a characterization of semi-classical Fourier integral distributions as oscillatory integrals. Applied to the scattering amplitude, this characterization roughly says that for every non-degenerate phase function ϕ which locally parameterizes $\pi_1 \circ (\pi_2|_{\Lambda(\lambda)})^{-1}\Lambda_R(\lambda)$, we can find a symbol a admitting an asymptotic expansion in h such that $CA(\lambda, h)$ can be represented as an oscillatory integral with phase ϕ and symbol a. From the discussion in Alexandrova (Preprint, Section 4.1) we further know that such a non-degenerate phase function always exists, and therefore we can always express $CA(\lambda, h)$ as an oscillatory integral admitting an asymptotic expansion in h. In the special case when the non-degeneracy assumption (2) holds, we recover the phases (4) in (3) (see Theorem 6) below. We expect that a finer analysis based on our method would give a precise description of the amplitudes as well. What is different here is the fact that we can handle the cases in which the non-degeneracy assumption fails. This always occurs at the transition between the perturbation and free propagation (see Figure 1).

This paper is organized as follows. In Section 2.1 we introduce some of the notation, which we will use throughout this article. The "black box" setup is recalled in Section 2.2, while the relevant part of semi-classical analysis is reviewed in Appendix B. The representation of the scattering amplitude, which we will use here, is given in Section 2.3. Its derivation is reviewed in Appendix A, where we also prove two lemmas on the structure of the cut-off resolvent. Section 3 is dedicated to the geometric aspects of the problem with the scattering relation defined and studied in Section 3.1, and the resolvent relation, i.e., the canonical relation which we will prove is quantized by the cut-off resolvent, described in Section 3.2. The proof of the General Black Box Theorem is given in Section 4. In Section 5, we discuss applications of our General Black Box Theorem to non-trapping (Section 5.2) and trapping (Section 5.3) smooth compactly supported perturbations of the Euclidean Laplace-Beltrami operator. For that, we prove, in Section 5.1, that the cut-off resolvent for such perturbations satisfies Assumption 2.

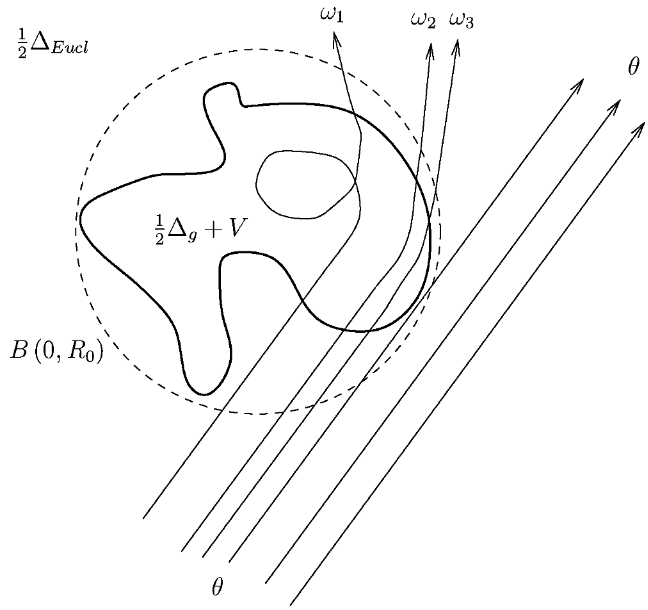

Figure 1. A typical trajectory in the presence of a perturbation. At the boundary between the perturbed trajectories and free trajectories the scattering relation has degenerate projections to the (θ, ω) variables.

The microlocal representation of the scattering amplitude analogous to (3) under the non-degeneracy assumption (2) on the angular density is given in Section 5.4. Our results are applied to an inverse problem in Section 5.2.

2. Preliminaries

In this section, we present some of the preliminary results we shall use throughout this work. We first introduce some of the notation.

2.1. *Notation*

We shall denote the Euclidean norm on $\mathbb{R}^n$ by $\|\cdot\|$ and we set $B(0,r) = \{x \in \mathbb{R}^n | \|x\| \leq r\}$ for $r > 0$. On any smooth manifold M, we denote by σ the canonical symplectic form on T^*M, and everywhere below we work with the canonical symplectic structure on T^*M. The canonical symplectic coordinates on $T^*\mathbb{R}^n$ will be denoted by (x, ξ). If $C \subset T^*M_1 \times T^*M_2$, where M_j, $j = 1, 2$, are smooth manifolds, we will use the notation $C' = \{(y, -\eta; x, \xi) : (y, \eta; x, \xi) \in C\}$. The Euclidean norm on $\mathbb{R}^n$ will be denoted by $\|\cdot\|$ and we set $B(a, R) = \{x \in \mathbb{R}^n | \|a - x\| < R\}$, for $R > 0$. For a sequentially continuous operator $T : C_c^\infty(\mathbb{R}^m) \to \mathcal{D}'(\mathbb{R}^n)$ we shall denote by K_T its Schwartz kernel. For such an operator T we use T^t to denote the operator with Schwartz kernel $K_{T^t}(x, y) = K_T(y, x)$. Unless otherwise specified, we will use $\langle \cdot, \cdot \rangle$ to denote the standard inner product on $\mathbb{R}^n$, $\mathbb{C}^n$, and $L^2(\mathbb{R}^n)$, and C to denote a positive constant, which will be allowed to change from line to line.

2.2. Black-Box Formalism

We review here the abstract "black-box" setup of Sjöstrand and Zworski (1991), which we will use below. Let $\mathscr{H}$ be a complex Hilbert space with the orthogonal decomposition

$$\mathscr{H} = \mathscr{H}_{R_0} \oplus L^2(\mathbb{R}^n \backslash B(0, R_0)),$$

where $R_0 > 0$. We consider a family of unbounded self-adjoint operators $P(h) : \mathscr{H} \to \mathscr{H}$, $0 < h \leq 1$, with domains $\mathscr{D}(h) \subset \mathscr{H}$, satisfying the following conditions:

$$1_{\mathbb{R}^n \backslash B(0,R_0)} \mathscr{D}(h) = H_h^2(\mathbb{R}^n \backslash B(0, R_0)),$$

$$(P(h)u(h))|_{\mathbb{R}^n \backslash B(0,R_0)} = -\frac{1}{2} h^2 \Delta(u(h)|_{\mathbb{R}^n \backslash B(0,R_0)}), \qquad \forall u(h) \in \mathscr{D}(h),$$

$$1_{B(0,R_0)}(i - P(h))^{-1} \qquad \text{is compact} : \mathscr{H} \to \mathscr{H}$$

$$P(h) > -C_0, \qquad C_0 > 0.$$

$$u(h) \in H_h^2(\mathbb{R}^n \backslash B(0, R_0)), \qquad u(h) = 0 \text{ near } \partial B(0, R_0) \Rightarrow u(h) \in \mathscr{D}(h),$$

where everywhere we identify $H_h^2(\mathbb{R}^n \backslash B(0, R_0))$, $u(h) = 0$ near $\partial B(0, R_0)$, with a subspace of $\mathscr{H}$ in the natural way. We equip the domain $\mathscr{D}(h)$ with the norm $\|(P(h) + i)u\|_{\mathscr{H}}$.

We also let $R(z, h) = (P(h) - z)^{-1}$, $z \in \mathbb{C} \backslash \mathbb{R}$ and we use the same notation for the meromorphic extension of $R(\cdot, h)$: $\mathscr{H}_{\text{comp}} \to \mathscr{D}_{\text{loc}}(h)$ from $\Im z > 0$ to a neighborhood of $\mathbb{R}$. The poles of the meromorphic extension are called resonances.

We now prove the following

Lemma 1. *Let $v : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be given by $v(x, y) = (y, x)$.*
Then, for $\lambda > 0$ not a resonance, $v^*(K_{R(\lambda,h)}|_{\mathbb{R}^n \backslash B(0,R_0)}) = K_{R(\lambda,h)}|_{\mathbb{R}^n \backslash B(0,R_0)}$.

Proof. For $u, v \in L^2(\mathbb{R}^n \backslash B(0, R_0))$ let $\langle u, v \rangle = \int uv$. Let u and v be further chosen with compact support and let $z \in \mathbb{C}$ be such that $\Im z > 0$. We then have

$$\begin{aligned}
&\langle (R(z, h)u)|_{\mathbb{R}^n \backslash B(0,R_0)}, v \rangle \\
&\quad = \langle (R(z, h)u)|_{\mathbb{R}^n \backslash B(0,R_0)}, (P(h) - z)R(z, h)v \rangle \\
&\quad = \left\langle (R(z, h)u)|_{\mathbb{R}^n \backslash B(0,R_0)}, \left(-\frac{1}{2}h^2\Delta - z\right)((R(z, h)v)|_{\mathbb{R}^n \backslash B(0,R_0)}) \right\rangle \\
&\quad = \left\langle \left(-\frac{1}{2}h^2\Delta - z\right)((R(z, h)u)|_{\mathbb{R}^n \backslash B(0,R_0)}), (R(z, h)v)|_{\mathbb{R}^n \backslash B(0,R_0)} \right\rangle \\
&\quad = \langle ((P(h) - z)(R(z, h)u))|_{\mathbb{R}^n \backslash B(0,R_0)}, (R(z, h)v)|_{\mathbb{R}^n \backslash B(0,R_0)} \rangle \\
&\quad = \langle u, (R(z, h)v)|_{\mathbb{R}^n \backslash B(0,R_0)} \rangle. \qquad (6)
\end{aligned}$$

Let, now, $(z_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ satisfy $\Im z_k > 0$, $k \in \mathbb{N}$ and $z_k \to \lambda$, $k \to \infty$. Then, from (6) we have that for every k,

$$\langle (R(z_k, h)u)|_{\mathbb{R}^n \backslash B(0,R_0)}, v \rangle = \langle u, (R(z_k, h)v)|_{\mathbb{R}^n \backslash B(0,R_0)} \rangle. \qquad (7)$$

Letting $k \to \infty$ in (7) and using the fact that $R(\cdot, h) : \mathcal{H}_{\text{comp}} \to \mathcal{H}_{\text{loc}}$ is analytic in the upper-half plane and up to the real axis, we obtain

$$\langle (R(\lambda, h)u)|_{\mathbb{R}^n \setminus B(0,R_0)}, v\rangle = \langle u, (R(\lambda, h)v)|_{\mathbb{R}^n \setminus B(0,R_0)}\rangle,$$

which completes the proof of the lemma. □

2.3. *The Scattering Amplitude*

Here we recall the definition of the scattering amplitude and its representation, which we will use in the proof of our General Black Box Theorem. We refer the reader to Appendix A for the derivation of this representation of the scattering amplitude.

We first recall that for any $\theta \in \mathbb{S}^{n-1}$ and $\lambda > 0$, there exists a unique, up to a compactly supported function, solution ψ to the problem $(P(h) - \lambda)\psi(\cdot, \theta; \lambda, h) = 0$, $\psi(\cdot, \theta; \lambda, h) \in \mathcal{D}_{\text{loc}}(P(h))$ such that

$$\psi(x, \theta; \lambda, h) = (1 - \chi(x))e^{\frac{i\sqrt{2\lambda}\theta\cdot x}{h}} + \psi_{sc}(x, \theta; \lambda, h),$$

where ψ_{sc} satisfies the Sommerfeld outgoing condition at infinity:

$$(\partial/\partial r - i\sqrt{2\lambda}/h)\psi_{sc} = \mathcal{O}(r^{-(n+1)/2}), \qquad \text{as } r = \|x\| \to \infty$$

and $\chi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 on $B(0, R_0)$. Then

$$\psi(x, \theta; \lambda, h) = e^{\frac{i\sqrt{2\lambda}\theta\cdot x}{h}} + \frac{e^{i\sqrt{2\lambda}r/h}}{r^{(n-1)/2}}A\left(\frac{x}{r}, \theta; \lambda, h\right) + \mathcal{O}\left(\frac{1}{r^{(n+1)/2}}\right), \qquad \text{as } r = \|x\| \to \infty.$$

The function $A(\cdot, \cdot\cdot; \lambda, h)$ is called the scattering amplitude at energy λ. We shall use $\mathcal{A}(\lambda, h)$ to denote the operator with integral kernel $A(\cdot, \cdot\cdot; \lambda, h)$ and, as is customary in the literature, we shall call the operator $\mathcal{A}(\lambda, h)$ the scattering amplitude at energy λ as well.

To give the representation of the scattering amplitude, which we will use in this paper, we first introduce the following operators, as in Petkov and Zworski (2001),

$$[\mathbb{E}_\pm(\lambda, h)f](\omega) = \int e^{\pm\frac{i\sqrt{2\lambda}\omega\cdot x}{h}} f(x)dx = \hat{f}\left(\mp\frac{\sqrt{2\lambda}\omega}{h}\right), \qquad \omega \in S^{n-1},$$

where f has compact support. Then, using (49) and Lemma 5, we can express the scattering amplitude as

$$A(\lambda, h) = c(n, \lambda, h)(\mathbb{E}_-(\lambda, h) \otimes \mathbb{E}_+(\lambda, h))([h^2\Delta, \chi_2] \otimes [h^2\Delta, \chi_1]^t)K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1},$$

where $c(n, \lambda, h) = e^{-i\pi\frac{n-3}{4}}\lambda^{\frac{n-3}{4}}2^{-\frac{n+9}{4}}(\pi h)^{-\frac{n+1}{2}}$, $\chi_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, 2$, are such that $\chi_1 = 1$ on $B(0, R_0)$ and $\chi_2 = 1$ on $\operatorname{supp}\chi_1$, $\tilde{\chi}_j \in C_c^\infty(\mathbb{R}^n \setminus B(0, R_0))$ are such that $\tilde{\chi}_j = 1$ on $\operatorname{supp}\nabla\chi_j$, $j = 1, 2$, and $\operatorname{supp}\tilde{\chi}_1 \cap \operatorname{supp}\tilde{\chi}_2 = \emptyset$.

3. Scattering Geometry

Here we collect the geometric results that we will use in the applications of our General Black Box Theorem. We begin by introducing the setting in which we shall work.

Let X be a smooth manifold of dimension $n > 1$ such that X coincides with $\mathbb{R}^n$ outside of $B(0, R_0)$ for some $R_0 > 0$. Let g be a Riemannian metric on X, which satisfies the condition

$$g_{ij}(x) = \delta_{ij} \qquad \text{for } \|x\| > R_0.$$

Let $V \in C_c^\infty(X \backslash (\mathbb{R}^n \backslash B(0, R_0)); \mathbb{R})$. Let $P(h) = \frac{1}{2}h^2\Delta_g + V$, $0 < h \leq 1$, with $p(x, \xi) = \frac{1}{2}\|\xi\|_g^2 + V(x)$ denoting its semi-classical principal symbol. We assume that for some $\lambda > 0$, the operator $P(h) - \lambda$ is of principal type. This implies that the energy surface $\Sigma_\lambda \stackrel{\text{def}}{=} p^{-1}(\lambda)$ is a smooth $2n - 1$ dimensional manifold.

Let H_p be the Hamiltonian vector field of p and let $\gamma(\cdot; x_0, \xi_0) = (x(\cdot; x_0, \xi_0), \xi(\cdot; x_0, \xi_0))$ denote the integral curve of H_p, or (phase) trajectory, with initial conditions $(x_0, \xi_0) \in T^*X$. We define a non-trapping energy level as follows.

Definition 1. The energy $\lambda > 0$ is non-trapping if for every $(x_0, \xi_0) \in \Sigma_\lambda$ there exists $t_0 > 0$ such that $x(s; x_0, \xi_0) \in \mathbb{R}^n \backslash B(0, R_0)$ for every $|s| > t_0$.

A phase trajectory $\gamma(\cdot; x_0, \xi_0)$ is non-trapped if there exists $t > 0$ such that $x(s; x_0, \xi_0) \in \mathbb{R}^n \backslash B(0, R_0)$ for all $|s| > t$.

3.1. *Scattering Relation*

We now define the canonical relation, which we will prove to be quantized by the scattering amplitude in the sense of semi-classical Fourier integral operators. We shall call this canonical relation the scattering relation. Below we also prove several lemmas which allow us to parameterize the scattering relation when the non-degeneracy assumption holds.

The idea for the scattering relation is that it relates the incoming and the outgoing data near a non-trapped trajectory in the way suggested by Figure 2.

To give a precise definition, let $\gamma_0 \subset \Sigma_\lambda$ be a non-trapped trajectory. Let $i : T^*\mathbb{S}^{n-1} \hookrightarrow T^*\mathbb{R}^n$ denote the inclusion map. Let $\psi : T^*\mathbb{R}^n \ni (x, \xi) \mapsto (\xi, x) \in T^*(\mathbb{R}^n)^*$. Then $L = (\psi \circ i)(T^*\mathbb{S}^{n-1})$ is a smooth submanifold of $T^*\mathbb{R}^n$ and therefore

$$L_-(\lambda) \stackrel{\text{def}}{=} \left\{(x, \xi) \,\middle|\, \left(\frac{\xi}{\sqrt{2\lambda}}, x + \frac{(1 + R_0)}{\sqrt{2\lambda}}\xi\right) \in L\right\} \subset \Sigma_\lambda$$

and

$$L_+(\lambda) \stackrel{\text{def}}{=} \left\{(x, \xi) \,\middle|\, \left(\frac{\xi}{\sqrt{2\lambda}}, x + \frac{(1 - R_0)}{\sqrt{2\lambda}}\xi\right) \in L\right\} \subset \Sigma_\lambda$$

are hypersurfaces in Σ_λ. Let $p_1 = \gamma_0 \cap L_-(\lambda)$. Since γ_0 is non-trapped, we have that for every $(x, \xi) \in L_-(\lambda)$ in a sufficiently small neighborhood of p_1, there exists a unique $T(x, \xi) > 0$ such that

$$\exp(T(x, \xi)H_p)(x, \xi) \in L_+(\lambda).$$

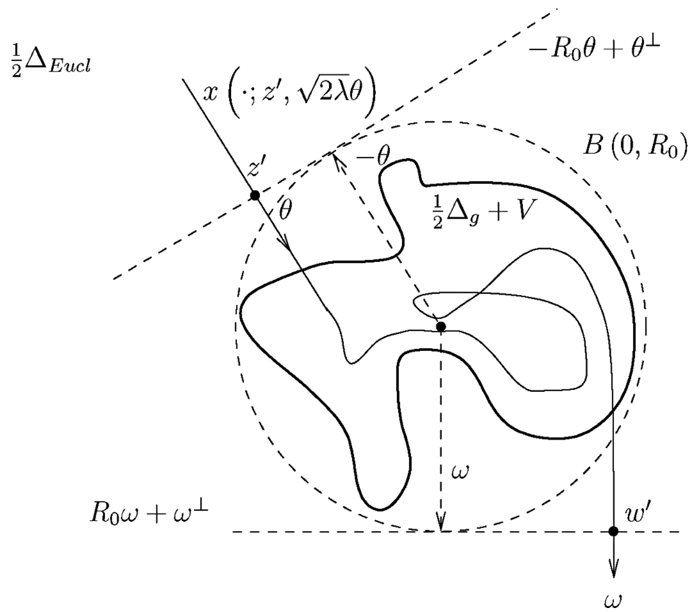

Figure 2. The scattering relation at energy λ consists of the points $(\theta, z' + (R_0 - 1)\theta; \omega, -w' + (R_0 - 1)\omega) \in T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1}$ with $\gamma(\cdot; z', \sqrt{2\lambda}\theta)$ non-trapped and $(w', \sqrt{2\lambda}\omega) \in \gamma(\cdot; z', \sqrt{2\lambda}\theta)$.

Therefore, since $L_-(\lambda)$ and $L_+(\lambda)$ are hypersurfaces in Σ_λ transverse to H_p near γ, we have that there exists an open set $U \subset L_-(\lambda)$, $p_1 \in U$, such that

$$SR_{\overline{U}}(\lambda) = \big\{(\eta, y; \xi, -x) \,|\, (z, \zeta) = (y - (R_0 + 1)\eta, \sqrt{2\lambda}\eta) \in \overline{U},$$
$$(x + (R_0 - 1)\xi, \sqrt{2\lambda}\xi) = \exp(T(z, \zeta)H_p)(z, \zeta)\big\}$$

is a closed Lagrangian submanifold of $(T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1}, \pi_1^*\sigma + \pi_2^*\sigma)$, where $\pi_j : T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$, $j = 1, 2$, is the canonical projection onto the jth factor. We shall call $SR_{\overline{U}}(\lambda)$ a scattering relation at energy λ (see Figure 2). If λ is a non-trapping energy, we let $SR(\lambda) = SR_{L_-(\lambda)}(\lambda)$.

We now show how, under a certain geometric assumption, we can find a non-degenerate phase function which parameterizes the scattering relation near a non-trapped trajectory. To state the assumption, let us first introduce some notation. For $\theta \in \mathbb{S}^{n-1}$ and $z \in \theta^\perp - R_0\theta$, if $\gamma(\cdot; z, \sqrt{2\lambda}\theta)$ is a non-trapped trajectory, then, as we saw above, there exist

$$x_\infty, \xi_\infty \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{R}^n), \qquad \|\xi_\infty(\theta, z)\| = 1,$$

such that

$$\gamma(t; z, \sqrt{2\lambda}\theta) = (x_\infty(\theta, z) + t\sqrt{2\lambda}\xi_\infty(\theta, z), \sqrt{2\lambda}\xi_\infty(\theta, z)), \qquad t \gg 0.$$

We shall call such a non-trapped phase trajectory with initial direction θ and final direction $\omega = \xi_\infty(\theta, z)$, a (θ, ω)-trajectory. We, now, make the following

Definition 2. If $\theta_0, \omega_0 \in \mathbb{S}^{n-1}$ are such that

$$\xi_\infty(\theta_0, z') = \omega_0$$

implies that the map

$$\theta_0^\perp - R_0\theta_0 \ni z \mapsto \xi_\infty(\theta_0, z) \in \mathbb{S}^{n-1} \qquad \text{is non-degenerate at } z', \tag{8}$$

then we shall say that ω_0 is regular for θ_0.

We remark that this definition is a rephrasing of the condition stated in Equation (2).

We then have the following

Lemma 2. *If $\omega_0 \in \mathbb{S}^{n-1}$ is regular for $\theta_0 \in \mathbb{S}^{n-1}$, then*

(a) $\theta_0 \neq \omega_0$.
(b) *There exist $O_j \subset \mathbb{S}^{n-1}$, $j = 1, 2$, open, $\theta_0 \in O_1$, $\omega_0 \in O_2$, and a number $L \in \mathbb{N}$ such that for every $(\theta, \omega) \in O_1 \times O_2$, there exist at least L $(\theta, \omega)-$ trajectories.*

Proof. We shall work in a local trivialization of $T^*\mathbb{S}^{n-1}_{R_0}$ near $T^*_{R_0\theta_0}\mathbb{S}^{n-1}_{R_0}$ and $T^*_{R_0\omega_0}\mathbb{S}^{n-1}_{R_0}$, where $\mathbb{S}^{n-1}_{R_0} = \{x \in \mathbb{R}^n : \|x\| = R_0\}$. We first prove that $\theta_0 \neq \omega_0$. Assume that $\theta_0 = \omega_0$. Then for every $z \in \mathbb{R}^{n-1}$ with $\|z\| > R_0$ we have that $\xi_\infty(\theta_0, z) = \theta_0$. Therefore $\det(\frac{\partial \xi_\infty(\theta_0, \cdot)}{\partial z}(z)) = 0$ for $z \in \theta_0^\perp$, $\|z\| > R_0$, which is a contradiction with the regularity assumption. Thus it follows that $\theta_0 \neq \omega_0$, which establishes (a).

Let, now, $z' \in (\xi_\infty(\theta_0, \cdot))^{-1}(\omega_0)$. Then, by the Inverse Function Theorem, there exist open sets $O_2' \subset \mathbb{S}^{n-1}$, $\omega_0 \in O_2'$ and $\mathcal{Z}' \subset \mathbb{R}^{n-1}$, $z' \in \mathcal{Z}'$, such that $\xi_\infty|_{\mathcal{Z}'}(\theta_0, \cdot)$ is a diffeomorphism onto O_2'. Therefore, the set $(\xi_\infty(\theta_0, \cdot))^{-1}(\omega_0)$ is discrete. From the first part of this proof, it follows that $(\xi_\infty(\theta_0, \cdot))^{-1}(\omega_0)$ is also bounded. Therefore, it is finite and we shall denote its elements by $\{z_1, \ldots, z_L\}$, $L \in \mathbb{N}$.

By the Implicit Function Theorem and the regularity assumption, we now have that there exist open sets $O_1, O_2 \subset \mathbb{S}^{n-1}$ with $\theta_0 \in O_1$ and $\omega_0 \in O_2$ and functions $z_l \in C^\infty(O_1 \times O_2; \mathbb{R}^{n-1})$, $l = 1, \ldots, L$, such that $z_l(\theta_0, \omega_0) = z_l$ and $\xi_\infty(\theta, z_l(\theta, \omega)) = \omega$, $(\theta, \omega) \in O_1 \times O_2$, which completes the proof of (b). $\square$

For the (θ, ω) trajectory defined by $z_l(\theta, \omega)$ as in the proof of the lemma, we shall use the subscript l to distinguish it from all other (θ, ω) trajectories.

Let $w_l(\theta, \omega) = \gamma(\cdot; z_l(\theta, \omega), \sqrt{2\lambda}\theta) \cap (R_0\omega + \omega^\perp) - R_0\omega$, $(\theta, \omega) \in O_1 \times O_2$. We also have the following

Lemma 3. *Let $\theta_0 \in \mathbb{S}^{n-1}$ be regular for $\omega_0 \in \mathbb{S}^{n-1}$.*

Then there exist $O_j \subset \mathbb{S}^{n-1}$, $j = 1, 2$, open, $\omega_0 \in O_1$, $\theta_0 \in O_2$, such that the map

$$\omega^\perp \ni w \mapsto \xi_\infty(-\omega, w) \in \mathbb{S}^{n-1}$$

is non-degenerate in a neighborhood of $w_l(\theta, \omega)$, $(\theta, \omega) \in O_1 \times O_2$, $l = 1, \ldots, L$.

Proof. As above, we have that

$$SR^l_{O_1 \times O_2}(\lambda) = \{(\theta, \omega, z_l(\theta, \omega), -w_l(\theta, \omega)) | (\theta, \omega) \in O_1 \times O_2\} \tag{9}$$

is a Lagrangian submanifold of $(T^*(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), \pi_1^*\sigma + \pi_2^*\sigma)$. Furthermore, $\pi|_{SR^l_{O_1 \times O_2}(\lambda)}$ is a surjection, where $\pi : T^*(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \to \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ is the canonical projection. Therefore, after decreasing, if necessary, $O_1 \times O_2$ around (θ_0, ω_0) we have that there exists a function $\mathcal{W}_l \in C^\infty(O_1 \times O_2)$ such that

$$SR^l_{O_1 \times O_2}(\lambda) = \{(m, d\mathcal{W}_l(m)) : m \in O_1 \times O_2\}.$$

Since $SR^l_{O_1 \times O_2}(\lambda)$ is also a canonical relation, after decreasing further $O_1 \times O_2$ near (θ_0, ω_0), if necessary, we can assume that

$$\det(\partial_\theta \partial_\omega \mathcal{W}_l(\theta, \omega)) \neq 0, \qquad (\theta, \omega) \in O_1 \times O_2,$$

which implies the conclusion of the lemma. □

Everywhere below we shall assume that O_1 and O_2 are chosen in such a way that the conclusions of Lemmas 2 and 3 hold in some open neighborhoods of $\overline{O}_1$ and $\overline{O}_2$.

The same proof as in Robert and Tamura (1989, Lemma 3.2) now shows that there exist $0 < S_0 < S_1$, $T_0 \gg 0$, and open sets $U^l_{\theta,\omega} \subset \mathbb{R}^{n-1}$, $z_l(\theta, \omega) \in U^l_{\theta,\omega}$, $l = 1, \ldots, L$, $(\theta, \omega) \in \overline{O}_1 \times \overline{O}_2$, such that

$$\det\left(\frac{\partial x(t; \cdot, \sqrt{2\lambda}\theta)}{\partial y}(y) \right) \neq 0 \tag{10}$$

for $y \in \{z - (s\sqrt{2\lambda} + R_0)\theta : z \in U^l_{\theta,\omega}, s \in [S_0, S_1]\}$, $t > T_0$.

For $(\theta, \omega) \in \overline{O}_1 \times \overline{O}_2$ we now define the (modified) action along the segment of the lth (θ, ω)-trajectory, $\gamma(\cdot; z_l(\theta, \omega), \sqrt{2\lambda}\theta)$, between the points $y_l(s; \theta, \omega) = z_l(\theta, \omega) - (s\sqrt{2\lambda} + R_0)\theta$, $s \in [S_1, S_0]$, and $x_l(t, s, \theta, \omega) = x_l(t; y_l(s; \theta, \omega), \sqrt{2\lambda}\theta)$, $t > T_0$. We choose $t_0 > T_0$ and we set

$$S_l(\theta, \omega) = \langle y_l(s; \theta, \omega), \sqrt{2\lambda}\theta \rangle + \int_0^{t_0} L(x_l, \dot{x}_l) dt - \langle x_l(t_0, s, \theta, \omega), \sqrt{2\lambda}\omega \rangle - \lambda t_0, \tag{11}$$

where $L(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|_g^2 - V(x)$ is the Lagrangian, and the integral is taken over the lth bicharacteristic curve connecting $y_l(s; \theta, \omega)$ and $x_l(t_0, s, \theta, \omega)$. We observe that, since the support of the perturbation is compact, $S_l(\theta, \omega)$ is independent of s for $s \in [S_1, S_0]$.

Setting

$$SR_l(\lambda) = SR^l_{\overline{O}_1 \times \overline{O}_2}(\lambda), \tag{12}$$

we now have the following

Lemma 4. *Let $\omega_0 \in \mathbb{S}^{n-1}$ be regular for $\theta_0 \in \mathbb{S}^{n-1}$.*

Then $SR_l(\lambda) = \Lambda_{S_l}$, where $\Lambda_{S_l} = \{(\theta, \omega, d_\theta S_l, d_\omega S_l) : (\theta, \omega) \in \overline{O}_1 \times \overline{O}_2\}$, $l = 1, \ldots, L$.

Proof. For $(\theta, \omega) \in \overline{O}_1 \times \overline{O}_2$, $s \in [S_1, S_0]$, let $i(s, \theta)_\omega : \mathbb{S}^{n-1} \to \mathbb{R}^n \times \mathbb{S}^{n-1}$, $\varphi \mapsto (x_l(t_0, s, \theta, \omega), \varphi)$. Let $f : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$, $(x, \varphi) \mapsto \langle x, \varphi\rangle$. We consider

$$\begin{aligned} d_\omega S_l(\theta, \omega) &= d_\omega\left(\langle y_l(s; \theta, \omega), \sqrt{2\lambda}\theta\rangle + \int_0^{t_0} L(x_l, \dot{x}_l)dt\right) \\ &\quad - d_\omega(\langle x_l(t_0, s, \theta, \cdot), \sqrt{2\lambda}\cdot\rangle)(\omega) \\ &= \langle\sqrt{2\lambda}\omega, d_\omega x_l(t_0, s, \theta, \cdot)(\omega)\rangle - \langle\sqrt{2\lambda}\omega, d_\omega x_l(t_0, s, \theta, \cdot)(\omega)\rangle \\ &\quad - \sqrt{2\lambda}\, i(s, \theta)^*_\omega df_{(x_l(t_0,s,\theta,\omega),\omega)} \\ &= -\sqrt{2\lambda}\, i(s, \theta)^*_\omega df_{(x_l(t_0,s,\theta,\omega),\omega)}, \end{aligned} \tag{13}$$

where (10) has allowed us to use Arnold (1980, Theorem 46.D.2) to obtain the second equality.

To compute $d_\theta S_l$, we first reparameterize the phase trajectories in the reverse direction, which is equivalent to considering the reverse of the initial and final directions. We further rewrite $S_l(\theta, \omega)$ in the following way

$$S_l(\theta, \omega) = -\langle x_l(s; \theta, \omega), \sqrt{2\lambda}\omega\rangle + \int_0^{t_0} L(x_l, \dot{x}_l)dt + \langle y_l(t_0, s, \omega, \theta), \sqrt{2\lambda}\theta\rangle - \lambda t_0,$$

where $x_l(s; \theta, \omega) = w_l(\theta, \omega) + (s\sqrt{2\lambda}+R_0)\omega$, $y_l(t_0, s, \theta, \omega) = x_l(t_0; x_l(s; \theta, \omega), -\sqrt{2\lambda}\omega)$, $s \in [S_0, S_1]$, and the integral is taken over the lth bicharacteristic curve connecting $x_l(s; \theta, \omega)$ and $y_l(t, s, \theta, \omega)$. We observe that this bicharacteristic curve is uniquely defined by Lemma 3 and (10).

As above, for $(\theta, \omega) \in \overline{O}_1 \times \overline{O}_2$, $s \in [S_1, S_0]$, let $i(s, \omega)_\theta : \mathbb{S}^{n-1} \to \mathbb{R}^n \times \mathbb{S}^{n-1}$, $\varphi \mapsto (y_l(t_0, s, \theta, \omega), \varphi)$, $(\theta, \omega) \in \overline{O}_1 \times \overline{O}_2$. Lemma 3 and (10) allow us to proceed as in (13) and we obtain

$$\begin{aligned} d_\theta S_l(\omega, \theta) &= d_\theta\left(-\langle x_l(s; \theta, \omega), \sqrt{2\lambda}\omega\rangle + \int_0^{t_0} L(x_l, \dot{x}_l)dt\right) \\ &\quad + d_\theta(\langle y_l(t_0, s, \omega, \cdot), \sqrt{2\lambda}\cdot\rangle)(\theta) \\ &= \sqrt{2\lambda}\, i(s, \omega)^*_\theta df_{(y_l(t_0,s,\theta,\omega),\theta)}. \end{aligned} \tag{14}$$

From (13) and (14), we therefore have that S_l is a non-degenerate phase function such that $SR_l(\lambda) = \Lambda_{S_l}$. □

3.2. *Resolvent Relation*

We now define the Lagrangian submanifold, which we will prove is quantized by the cut-off resolvent in the sense of semi-classical Fourier integral operators. We work with the same assumptions and definitions as in the previous subsection.

We assume that for every $\rho \in \Sigma_\lambda$ and every $t \in J$, where J is an open interval,

$$\exp(tH_p)(\rho) \neq \rho.$$

We set

$$\widetilde{\Lambda}_R(\lambda, J) = \bigcup_{t\in J} \operatorname{graph} \exp(tH_p)|_{\Sigma_\lambda}.$$

Then $\widetilde{\Lambda}_R(\lambda, J)$ is a Lagrangian submanifold of $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \tilde{\sigma})$, where $\tilde{\sigma} = \pi_1^*\sigma - \pi_2^*\sigma = d\eta \wedge dy - d\xi \wedge dx$ with $\pi_j : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, $j = 1, 2$, denoting the canonical projection onto the jth factor. We define the resolvent relation at energy $\lambda > 0$ as

$$\Lambda_R(\lambda) = \widetilde{\Lambda}'_R(\lambda, \mathbb{R}^+) \cap T^*(\operatorname{supp} \tilde{\chi}_1 \times \operatorname{supp} \tilde{\chi}_2).$$

4. Proof of the General Black Box Theorem

We now turn to the proof of our General Black Box Theorem. This proof relies on the generalization of Egorov's Theorem to manifolds of unequal dimensions, which we have proven in Alexandrova (Preprint, Theorem 2).

Proof. We first prove that $A(\lambda, h) \in \mathscr{D}'_h(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. For that, we begin by showing that $K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} \in \mathscr{D}'_h(\mathbb{R}^{2n})$. Let $\chi \in C_c^\infty(\mathbb{R}^{2n})$ and let $\rho_j \in C_c^\infty(\mathbb{R}^n \backslash B(0, R_0))$, $j = 1, 2$, be such that $\rho_2 \times \rho_1 = 1$ on $\operatorname{supp} \tilde{\chi}_2 \times \operatorname{supp} \tilde{\chi}_1 \cap \operatorname{supp} \chi$. Then, since $K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} \in C^\infty(\mathbb{R}^{2n})$ (see Lemma 5), we have

$$\begin{aligned}
\left|\mathscr{F}_h(\chi K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1})(\xi, \eta)\right| &= \left| \iint K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}(x, y)\chi(x, y) e^{-\frac{i}{h}(\langle x,\xi\rangle + \langle y,\eta\rangle)} dx\, dy \right| \\
&\le \iint \left|K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}(x, y)\chi(x, y)\right| dx\, dy \\
&\le C \iint \left|K_{\rho_2 R(\lambda,h)\rho_1}(x, y)(\rho_2 \otimes \rho_1)(x, y)\right| dx\, dy \\
&\le C \|\rho_2\|_{L^2(\mathbb{R}^n)} \|\rho_1\|_{L^2(\mathbb{R}^n)} \|\rho_2 R(\lambda, h)\rho_1\|_{\mathscr{B}(L^2(\mathbb{R}^n))} \\
&= \mathscr{O}(h^s),
\end{aligned} \tag{15}$$

which verifies the assertion.

Let now $\psi \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ have support in a coordinate chart on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ with local coordinates $(\tilde{\omega}, \tilde{\theta})$. Then, since $A(\lambda, h) \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, we have

$$\begin{aligned}
\left|\mathscr{F}_h(\psi A(\lambda, h))(\tilde{\xi}, \tilde{\eta})\right| &= \left| \iint A(\lambda, h)(\tilde{\omega}, \tilde{\theta})\psi(\tilde{\omega}, \tilde{\theta}) e^{-\frac{i}{h}(\langle\tilde{\omega},\tilde{\xi}\rangle + \langle\tilde{\theta},\tilde{\eta}\rangle)}\, d\tilde{\theta}\, d\tilde{\omega} \right| \\
&\le \iint \left|A(\lambda, h)(\tilde{\omega}, \tilde{\theta})\psi(\tilde{\omega}, \tilde{\theta})\right| d\tilde{\theta}\, d\tilde{\omega} \\
&\le C \iint \left|A(\lambda, h)(\omega, \theta)\right| d\theta\, d\omega \\
&\le C h^{-\frac{n+1}{2}} \iint \left|\mathscr{F}_h(K_{\tilde{\chi}_1 R(\lambda,h)\tilde{\chi}_2})(\omega, -\theta)\right| d\theta\, d\omega \\
&= \mathscr{O}\big(h^{s-\frac{n+1}{2}}\big),
\end{aligned}$$

where the last equality follows from (15). Therefore $A(\lambda, h) \in \mathscr{D}'_h(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$.

A direct calculation now shows that $\pi_1 \circ (\pi_2|_{\Lambda(\lambda)})^{-1}(-\Lambda_R(\lambda))$ is a Lagrangian submanifold of $T^*\mathbb{S}^{n-1} \times T^*\mathbb{S}^{n-1}$, where we recall that $\Lambda(\lambda)$ was defined in (5). To prove the theorem, we now first note that

$$\mathbb{E}_-(\lambda, h) \otimes \mathbb{E}_+(\lambda, h) \in \mathcal{I}_h^{-n+\frac{1}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n, \Lambda(\lambda)) \tag{16}$$

and we easily see that

$$\pi_2|_{\Lambda(\lambda)} \qquad \text{is an immersion.} \tag{17}$$

Let now

$$A_j \in \Psi_h^0(1, \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), \qquad j = 1, \dots, N$$

have symbols with compact essential support contained in a neighborhood of $\bar{p}$ and principal symbols vanishing on

$$\pi_1 \circ (\pi_2|_{\Lambda(\lambda)})^{-1}(-\Lambda_R(\lambda)).$$

With (16) and (17), the hypotheses of the generalization of Egorov's Theorem to manifolds of unequal dimensions (Alexandrova, Preprint, Theorem 2) are satisfied, and we deduce from Alexandrova (Preprint, Theorem 2) that there exist $D, B_j \in \Psi_h^0(1, \mathbb{R}^{2n})$, $j = 1, \dots, N$, with compact wavefront sets near the point

$$\bar{q} = (y_0 + s\theta_0, x_0 + t\omega_0, -\sqrt{2\lambda}\theta_0, \sqrt{2\lambda}\omega_0,) \qquad \text{for some } s \in [s_1, s_2], t \in [t_1, t_2],$$

such that

$$\begin{gathered}(\pi_2|_{\Lambda(\lambda)})^* \upsilon^* \sigma_0(B_j) = -(\pi_1|_{\Lambda(\lambda)})^* \sigma_0(A_j), \qquad j = 1, \dots, N,\\ (\pi_2|_{\Lambda(\lambda)})^* \upsilon^* \sigma_0(D) = -(\pi_1|_{\Lambda(\lambda)})^* \sigma_0(C),\end{gathered} \tag{18}$$

where $\upsilon(z, \zeta) = (z, -\zeta)$, $(z, \zeta) \in T^*\mathbb{R}^{2n}$, and

$$\left(\prod_{j=1}^N A_j\right) C(\mathbb{E}_-(\lambda, h) \otimes \mathbb{E}_+(\lambda, h))\chi \equiv (\mathbb{E}_-(\lambda, h) \otimes \mathbb{E}_+(\lambda, h))\chi \cdot \left(\prod_{j=1}^N B_j\right) D \tag{19}$$

near $(\bar{p}, \bar{q})$, where $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is equal to 1 on $\operatorname{supp} \tilde{\chi}_2 \times \operatorname{supp} \tilde{\chi}_1$.

Let now $Y = [h^2\Delta, \chi_2] \otimes [h^2\Delta, \chi_1]^t$. Since $Y \in \Psi_h^{-1,1}(\mathbb{R}^{2n})$, we obtain from Alexandrova (Preprint, Lemma 5) that

$$YK_{\tilde{\chi}_2 R(\lambda, h)\tilde{\chi}_1} \in I_h^{r-1}(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_R(\lambda)). \tag{20}$$

Therefore, from (19), we have

$$\begin{aligned}\left(\prod_{j=1}^N A_j\right) CA(\lambda, h) = c(n, \lambda, h)(\mathbb{E}_-(\lambda, h) \otimes \mathbb{E}_+(\lambda, h))\chi \cdot \left(\prod_{j=1}^N B_j\right) DYK_{\tilde{\chi}_2 R(\lambda, h)\tilde{\chi}_1}\\ + \mathcal{O}_{L^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})}(h^\infty).\end{aligned} \tag{21}$$

The choice of the operators A_j and (18) now imply

$$\sigma_0(B_j)|_{\Lambda_R(\lambda)} = 0, \qquad j = 1, \dots, N.$$

Therefore,

$$\left(\prod_{j=1}^{N} B_j\right) DYK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = \mathcal{O}_{L^2(\mathbb{R}^{2n})}(h^{N-r+1-\frac{n}{2}}). \tag{22}$$

Lastly, as in the proof of Burq (2002b, Proposition 2.1), we have that

$$\|\mathbb{E}^{\phi}_{\pm}\|_{\mathscr{B}(L^2(\mathbb{R}^n),L^2(\mathbb{S}^{n-1}))} = \mathcal{O}(h^{\frac{n-1}{2}}), \tag{23}$$

where $\mathbb{E}^{\phi}_{\pm}(\lambda, h)$ are the operators with Schwartz kernels

$$K_{\mathbb{E}^{\phi}_{\pm}(\lambda,h)}(\omega, x) = \phi(x)\exp(\pm i\sqrt{2\lambda}\langle\omega, x\rangle/h)$$

for $\phi \in C^{\infty}_c(\mathbb{R}^n)$.

We now substitute (22) into (21) and use (23) to obtain

$$\left(\prod_{j=1}^{N} A_j\right) CA(\lambda, h) = \mathcal{O}_{L^2(\mathbb{R}^{2(n-1)})}(h^{N-r-\frac{n}{2}}).$$

Therefore $CA(\lambda, h) \in I_h^{r+\frac{1}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \pi_1(\pi_2|_{\Lambda(\lambda)})^{-1}(-\Lambda_R(\lambda)))$. □

5. Applications

Here we discuss several applications of the General Black Box Theorem to compactly supported potential and metric perturbations of the Euclidean Laplacian. More precisely, let the setting be as in Section 3. Then the operators $P(h) = \frac{1}{2}h^2\Delta_g + V$, $0 < h \leq 1$, acting on $\mathscr{H} = L^2(\mathbb{R}^n, d\mathrm{vol}_g)$ and equipped with the domains $\mathscr{D}(h) = H^2_h(\mathbb{R}^n, d\mathrm{vol}_g)$, admit unique self-adjoint extensions, which we denote by the same notation. Melrose (1995, Proposition 2.3) states that there are no resonances in $\mathbb{R}\backslash\{0\}$ in the case of a smooth compactly supported potential perturbation.

5.1. *The Cut-off Resolvent as a Semi-Classical Fourier Integral Operator*

We now prove that the second assumption of our General Black Box Theorem is satisfied in the setting we have just described. In the following theorem, we let $\pi_1 : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ denote the canonical projection onto the first factor.

Theorem 2. *Let* $\|\tilde{\chi}_2 R(\lambda, h)\tilde{\chi}_1\|_{\mathscr{B}(L^2(\mathbb{R}^n))} \leq C_\chi h^s$ *for some* $s \in \mathbb{R}$. *Let* $\rho_0 \in \Lambda_R(\lambda)'$ *be such that* $\gamma(\cdot; \pi_1(\rho_0))$ *is a non-trapped trajectory.*

Then there exists an open set $V \subset \Lambda_R(\lambda)$, $\rho_0 \in W'$, *such that*

$$\tilde{\chi}_2 R(\lambda, h)\tilde{\chi}_1 \in \mathscr{I}^1_h(\mathbb{R}^{2n}, \overline{W} \cap \Lambda_R(\lambda)).$$

Proof. There are two main ideas in the proof of this theorem. The first one is the representation formula for the resolvent (24) and the second one is the decomposition of functions vanishing on $\Lambda_R(\lambda)$ given by (32) and (33).

By (15) we have that $K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} \in \mathscr{D}'_h(\mathbb{R}^{2n})$. We now recall the derivation of a representation of the resolvent, which we shall use to prove the theorem. For that, let $f \in C_c^\infty(\mathbb{R}^n)$ and $T > 0$. Then

$$-\frac{i}{h}(P(h)-\lambda)\int_0^T e^{\frac{i}{h}t\lambda}U(t)f\,dt = e^{\frac{i}{h}T\lambda}U(T)f - f,$$

where $U(t) = e^{-\frac{i}{h}tP(h)}$, $t \in \mathbb{R}$ is the unitary group of $P(h)$. The same proof as in Michel (2002, Lemma B.1), shows that $(1-\chi_0)U(T)f \in \mathscr{S}(\mathbb{R}^n)$, $\chi_0 \in C_c^\infty(\mathbb{R}^n)$, $\chi_0 = 1$ on $B(0, R_0)$. Since we can also think of $R(\lambda, h)$ as $\lim_{\epsilon\to 0,\epsilon>0} R(\lambda \pm i\epsilon, h)$ in the spaces of bounded operators $\mathscr{B}(L^2_\alpha(X), L^2_{-\alpha}(X))$, $\alpha > \frac{1}{2}$, where

$$L^2_\alpha(X) = \{f : (\chi_0 + (1-\chi_0)\langle x\rangle^\alpha)f \in L^2(X, d\mathrm{vol}_g\mathbb{R}^n)\},$$

we obtain

$$\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1 = \frac{i}{h}\int_0^T e^{\frac{i}{h}t\lambda}\tilde{\chi}_2 U(t)\tilde{\chi}_1\,dt + e^{\frac{i}{h}T\lambda}\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1 \tag{24}$$

and this is the representation of the resolvent, which we shall use in this proof.

Since $\rho_0 \in \Lambda_R(\lambda)'$ is such that $\gamma(\cdot;\pi_1(\rho_0))$ is a non-trapped trajectory, there exists an open set $W \subset \Lambda_R(\lambda)$, $\rho_0 \in W'$, such that for every $\rho \in W'$, $\gamma(\cdot;\pi_1(\rho))$ is a non-trapped trajectory. By adjusting W, if necessary, we can assume that the same holds for all points in $\overline{W}'$. Let $Q \in \Psi_h^0(1, \mathbb{R}^{2n})$ be a microlocal cut-off to the neighborhood W as in (51) with a compactly supported symbol. First, we shall prove that $QK_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1} = \mathscr{O}_{L^2(\mathbb{R}^{2n})}(h^\infty)$ for $T > 0$ sufficiently large. By Alexandrova (Preprint, Lemma 4(a)) and the choice of Q, we have that

$$WF_h^i(QK_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1}) = \emptyset. \tag{25}$$

Therefore, by Michel (2004, Proposition 7.1(i)), it is sufficient to prove that there exists $\overline{T} > 0$ such that for every $T > \overline{T}$, $WF_h^f(QK_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1}) = \emptyset$. This will follow, if we prove that

$$WF_h^f(K_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1}) \cap WF_h^f(Q) = \emptyset.$$

To prove the latter, consider

$$\langle K_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1}, (\psi_2 \otimes \psi_1)e^{-\frac{i}{h}(\langle\cdot,\xi\rangle + \langle\cdot\cdot,\eta\rangle)}\rangle,$$

where $\operatorname{supp}\psi_1 \times \{\eta\} \times \operatorname{supp}\psi_2 \times \{\xi\} \subset W$. Let $W_1 \subset \mathbb{R}^n$ be a bounded open set such that $\operatorname{supp}\psi_1 \times W_1 \times \operatorname{supp}\psi_2 \times \{\xi\} \subset W$, $\eta \in W_1$. Now, for every η we have that

$$WF_h^f(\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle}) = \operatorname{supp}\psi_1 \times \{-\eta\} \tag{26}$$

is compact. Recalling also the well-known fact that $U(t) \in \mathscr{I}_h^0(\mathbb{R}^{2n}, \Lambda_t')$, $t \in \mathbb{R}$, where $\Lambda_t = \operatorname{graph}\exp(tH_p)$, we obtain from Alexandrova (Preprint, Lemma 6), that

$$WF_h^f(K_{U(t)}) \subset \Lambda_t'. \tag{27}$$

From (26), (27), and the lemma on propagation of the semi-classical wave front set Alexandrova (Preprint, Lemma 4(c)), we therefore have that

$$WF_h^f(U(T)\tilde{\chi}_1\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle}) \subset \exp(TH_p)(WF_h^f(\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle})), \qquad \eta \in W_1. \tag{28}$$

After decreasing W_1, if necessary, we have, by the proof of Alexandrova (Preprint, Lemma 3), that the estimates in (28) can be made uniform in $\eta \in W_1$. Since $\exp(TH_p)$ is a diffeomorphism, it follows that $\bigcup_{\eta\in\overline{W}_1}\exp(TH_p)(WF_h^f(\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle}))$ is compact. Further, as we are working with the outgoing resolvent, we have that

$$\begin{aligned} WF_h^f(R(\lambda,h)U(T)\tilde{\chi}_1\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle}) &\subset \bigcup_{t>0}\exp(tH_p)\big(WF_h^f(U(T)\tilde{\chi}_1\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle})\big) \\ &\subset \bigcup_{t>0}\exp((t+T)H_p)\big(WF_h^f(\psi_1 e^{-\frac{i}{h}\langle\cdot,\eta\rangle})\big). \end{aligned}$$

By the non-trapping assumption, there exists $\overline{T} > 0$ such that for every $T > \overline{T}$ and every $(y,\eta) \in \operatorname{supp}\psi_1 \times W_1$, we have $x(T; y, \eta) \in (\operatorname{supp}\tilde{\chi}_2)^c$. We now let $T > \overline{T}$ be fixed, and we have

$$\big\langle K_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1}, (\psi_2 \otimes \psi_1)e^{-\frac{i}{h}(\langle\cdot,\xi\rangle+\langle\cdot,\eta\rangle)}\big\rangle = \mathcal{O}(h^\infty)$$

for every $\eta \in W_1$ and uniformly in $\xi \in U$, where $U \subset \mathbb{R}^n$ a bounded open set such that $\operatorname{supp}\psi_1 \times W_1 \times \psi_2 \times U \subset V$. The proof of Alexandrova (Preprint, Lemma 3), now shows again that the estimate here can be made uniform in $\eta \in W_1$. We thus have that, after decreasing the support of $\sigma(Q)$, if necessary, for every $T > \overline{T}$

$$WF_h^f(QK_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1}) = \emptyset, \tag{29}$$

which, together with (25), gives

$$QK_{\tilde{\chi}_2 R(\lambda,h)U(T)\tilde{\chi}_1} = \mathcal{O}_{L^2(\mathbb{R}^{2n})}(h^\infty). \tag{30}$$

Let, now, $B_j \in \Psi_h^0(1,\mathbb{R}^{2n})$, $j = 1,\dots,k$, have symbols with compact essential support and principal symbols b_j vanishing on $\Lambda_R(\lambda)$ and consider

$$\left(\prod_{j=1}^k B_j\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}. \tag{31}$$

We have that $\Lambda_R(\lambda) = (\bigcup_{s\in\mathbb{R}^+}\Lambda'_s) \cap (\Sigma_\lambda \times \Sigma_\lambda)$, and the intersection at every point is clean. Therefore, by Hörmander (1980, Proposition C.3.1, Vol. 3), we can choose local coordinates around ρ such that $\bigcup_{s\in\mathbb{R}^+}\Lambda'_s$ and $\Sigma_\lambda \times \Sigma_\lambda$ are given there by linear equations in the local coordinates. This implies that for every $j = 1,\dots,k$ we can find functions $c_j, g_j, h_j \in C_c^\infty(T^*\mathbb{R}^n)$ with

$$c_j|_{\cup_{s\in\mathbb{R}^+}\Lambda'_s} = 0, \qquad j = 1,\dots,k, \tag{32}$$

such that after decreasing the essential support of b_j around p, if necessary, we have that

$$b_j = c_j + ((p-\lambda)\otimes 1)g_j + (1\otimes(p-\lambda))h_j + \mathcal{O}(h^\infty) \qquad \text{in } S_{4n}^0(1), \quad j = 1,\dots,k. \tag{33}$$

Now, for $a, b \in S^0_{2n}(1)$, we have that $Op_h(a)Op_h(b) = Op_h(ab) + \mathcal{O}_{\mathcal{B}(L^2(\mathbb{R}^n))}(h)$ and we can therefore rewrite (31) as follows

$$\left(\prod_{j=1}^{k} B_j\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = \left(\prod_{j=1}^{k} (Op_h(c_j) + Op_h(g_j)((P(h)-\lambda)\otimes I) + Op_h(h_j)(I\otimes(P(h)-\lambda)) + hS_j)\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1},$$

where $S_j \in \Psi^0_h(1, \mathbb{R}^{2n})$ and ess-supp$_h\sigma(S_j)$ is compact, $j = 1, \ldots, k$. We further rewrite this as

$$\left(\prod_{j=1}^{k} B_j\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = \sum_{l=1}^{4^k} \left(\prod_{j=1}^{k} T^l_j\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}, \tag{34}$$

where

$$T^l_j \in \left\{Op_h(c_j), Op_h(g_j)((P(h)-\lambda)\otimes I), Op_h(h_j)(I\otimes(P(h)-\lambda)), hS_j\right\},$$

$l = 1, \ldots, 4^k$, $j = 1, \ldots, k$. We now turn to analyzing the individual summands. As the superscript will not be important, we will omit it from the notation.

We consider the case $k > 1$. The case $k = 1$ will be implicit in the discussion below. Let $m \in \{2, \ldots, k\}$ be the largest index such that

$$T_m \in \{Op_h(c_m), hS_m\}$$

and

$$T_{m-1} \in \{Op_h(g_{m-1})((P(h)-\lambda)\otimes I), Op_h(h_{m-1})(I\otimes(P(h)-\lambda))\}.$$

We first assume that $T_m = Op_h(c_m)$. Since

$$\sigma_0([T_m, T_{m-1}]) = \frac{h}{i}\{\sigma_0(T_m), \sigma_0(T_{m-1})\}$$

and $\sigma_0(T_m)$ and $\sigma_0(T_{m-1})$ vanish on the Lagrangian submanifold $\Lambda_R(\lambda)$ near ρ, it follows that $\sigma_0([T_m, T_{m-1}])$ also vanishes on $\Lambda_R(\lambda)$ near ρ. Therefore we have, as before, that

$$\sigma_0([T_m, T_{m-1}]) = \tilde{c} + \tilde{g}((p-\lambda)\otimes 1) + \tilde{h}(1\otimes(p-\lambda)) + \mathcal{O}(h^\infty) \qquad \text{in } S^0_{4n}(1),$$

for some $\tilde{c}, \tilde{g}, \tilde{h} \in C^\infty_c(\mathbb{R}^{4n})$ with supports in a sufficiently small neighborhood of ρ and $\tilde{c}|_{\cup_{s\in\mathbb{R}^+}\Lambda'_s} = 0$. Thus

$$\begin{aligned} T_{m-1}T_m = {} & Op_h(\tilde{c}) + Op_h(\tilde{g})((P(h)-\lambda)\otimes I) \\ & + Op_h(\tilde{h})(I\otimes(P(h)-\lambda)) + T_m T_{m-1} + h\widetilde{T}, \end{aligned} \tag{35}$$

where $\widetilde{T} \in \Psi^0_h(1, \mathbb{R}^{2n})$ and ess-supp$_h\sigma(\widetilde{T})$ is compact, and we use the latter expression in (35) to replace $T_{m-1}T_m$ in the product above.

If now $T_m = hS_m$, we rewrite $T_{m-1}T_m = T_mT_{m-1} + [T_{m-1}, T_m]$ in the product above and observe that $[T_{m-1}, T_m] \in \Psi_h^{-2}(1, \mathbb{R}^{4n})$.

We iterate this process until each product which appears in (34), where we may now have more than 4^k products, is of the form

$$h^{k-k_1} \prod_{j=1}^{k_1} Q_j, \qquad k_1 \in \{1, \ldots, k\},$$

where for some $j_0 \in \{0, \ldots, k_1\}$, we have that

for $j_0 < j \leq k_1$, $\quad Q_j \in \{Op_h(g_j^1)((P(h) - \lambda) \otimes I), Op_h(h_j^1)(I \otimes (P(h) - \lambda))\}$,

for some $g_j^1,\ h_j^1 \in C_c^\infty(\mathbb{R}^{4n})$,

for $1 \leq j \leq j_0$, $\quad Q_j \in \{Op_h(c_j^1), hS_j^1\}$,

for some $c_j^1 \in C_c^\infty(T^*\mathbb{R}^n)$, $\quad c_j^1|_{\cup_{t\in\mathbb{R}^+}\Lambda_t'} = 0$, $\quad S_j^1 \in \Psi_h^0(1, \mathbb{R}^{2n})$,
ess-supp$_h\sigma(S_j^1)$ is compact,

with all symbols having essential supports in a sufficiently small neighborhood of ρ.

Next, we let $\tilde{m} \in \{2, \ldots, k_1\}$ denote the largest index for which we have $T_{\tilde{m}-1} = Op_h(c_{\tilde{m}-1}^1)$ and $T_{\tilde{m}} = hS_{\tilde{m}}^1$. We then replace $T_{\tilde{m}-1}T_{\tilde{m}}$ by $T_{\tilde{m}}T_{\tilde{m}-1} + [T_{\tilde{m}-1}, T_{\tilde{m}}]$ in the product above and observe that $[T_{\tilde{m}-1}, T_{\tilde{m}}] \in \Psi_h^{-1}(1, \mathbb{R}^{2n})$ and ess-supp$_h$ $\sigma([T_{\tilde{m}-1}, T_{\tilde{m}}])$ is compact.

We repeat this procedure until every product which appears in (34) is of the form

$$h^{k-k_2} \prod_{j=1}^{k_2} V_j, \qquad k_2 \in \{1, \ldots, k\},$$

where for some $j_1, j_2 \in \{0, \ldots, k_2\}$, $j_1 \leq j_2$, we have that

for $j_2 < j \leq k_2$, $\quad V_j \in \{Op_h(g_j^2)((P(h) - \lambda) \otimes I), Op_h(h_j^2)(I \otimes (P(h) - \lambda))\}$,

for some $g_j^2,\ h_j^2 \in C_c^\infty(\mathbb{R}^{4n})$,

for $j_1 < j \leq j_2$, $\quad V_j = Op_h(c_j^2)$, $\quad$ for some $c_j^2 \in C_c^\infty(T^*\mathbb{R}^n)$, $\quad c_j^2|_{\cup_{t\in\mathbb{R}^+}\Lambda_t'} = 0$,

for $1 \leq j \leq j_1$, $\quad V_j = hS_j^2$, $\quad$ for some $S_j^2 \in \Psi_h^0(1, \mathbb{R}^{2n})$,
ess-supp$_h\sigma(S_j^2)$ is compact,

where again all symbols have essential supports in a sufficiently small neighborhood of ρ.

We shall again omit the superscripts from the notation below. We also observe that the symmetry of $K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}$ (see Lemma 1) allows us to assume that

$$V_j = Op_h(g_j)((P(h) - \lambda) \otimes I), \qquad g_j \in C_c^\infty(\mathbb{R}^{4n}),\ \ j_2 < j \leq k_2.$$

We now analyze

$$\begin{aligned}&((P(h) - \lambda) \otimes I)QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}\\ &\quad = [(P(h) - \lambda) \otimes I, Q]K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} + Q((P(h) - \lambda) \otimes I)K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}. \qquad (36)\end{aligned}$$

To analyze the second term, consider

$$\begin{aligned}
&\Big(-\frac{1}{2}h^2\Delta-\lambda\Big)\tilde{\chi}_2R(\lambda,h)\tilde{\chi}_1\\
&\quad=\Big(-\frac{1}{2}h^2\Delta\tilde{\chi}_2\Big)R(\lambda,h)\tilde{\chi}_1-\frac{1}{2}\langle h\nabla\tilde{\chi}_2,h\nabla\rangle R(\lambda,h)\tilde{\chi}_1+\tilde{\chi}_2(P(h)-\lambda)R(\lambda,h)\tilde{\chi}_1\\
&\quad=\Big(-\frac{1}{2}h^2\Delta\tilde{\chi}_2\Big)R(\lambda,h)\tilde{\chi}_1-\frac{1}{2}\langle h\nabla\tilde{\chi}_2,h\nabla\rangle R(\lambda,h)\tilde{\chi}_1
\end{aligned}$$

and therefore

$$\begin{aligned}
&\Big\|Q\Big(\Big(-\frac{1}{2}h^2\Delta-\lambda\Big)\otimes I\Big)K_{\tilde{\chi}_2R(\lambda,h)\tilde{\chi}_1}\Big\|_{L^2(\mathbb{R}^{2n})}\\
&\quad\le Ch(h\|QK_{(\Delta\tilde{\chi}_2)R(\lambda,h)\tilde{\chi}_1}\|_{L^2(\mathbb{R}^{2n})}+\|QK_{\langle\nabla\tilde{\chi}_2,h\nabla\rangle R(\lambda,h)\tilde{\chi}_1}\|_{L^2(\mathbb{R}^{2n})}).
\end{aligned}\tag{37}$$

Now,

$$\begin{aligned}
QK_{\langle\nabla\tilde{\chi}_2,h\nabla\rangle R(\lambda,h)\tilde{\chi}_1}&\equiv\frac{i}{h}\int_0^T e^{\frac{i}{h}t\lambda}QK_{\langle\nabla\tilde{\chi}_2,h\nabla\rangle U(t)\tilde{\chi}_1}\,dt\\
&=\frac{i}{h}\int_0^T e^{\frac{i}{h}t\lambda}Q(\langle\nabla\tilde{\chi}_2,h\nabla\rangle\otimes\tilde{\chi}_1)K_{U(t)}\,dt
\end{aligned}\tag{38}$$

Since $U(t)\in\mathscr{I}_h^0(\mathbb{R}^{2n},\Lambda_t')$, $t\in\mathbb{R}$, by Alexandrova (Preprint, Lemma 5) it follows that

$$\|Q(\langle\nabla\tilde{\chi}_2,h\nabla\rangle\otimes\tilde{\chi}_1)K_{U(t)}\|_{L^2(\mathbb{R}^{2n})}=\mathscr{O}\big(h^{-\frac{n}{2}}\big)$$

with the norm depending on t continuously. Therefore, from (38), we obtain

$$\|QK_{\langle\nabla\tilde{\chi}_2,h\nabla\rangle R(\lambda,h)\tilde{\chi}_1}\|_{L^2(\mathbb{R}^{2n})}=\mathscr{O}(h^{-\frac{n}{2}-1}).\tag{39}$$

From

$$QK_{(\Delta\tilde{\chi}_2)R(\lambda,h)\tilde{\chi}_1}\equiv\frac{i}{h}\int_0^T e^{\frac{i}{h}t\lambda}QK_{(\Delta\tilde{\chi}_2)U(t)\tilde{\chi}_1}\,dt$$

we conclude similarly that

$$\|QK_{(\Delta\tilde{\chi}_2)R(\lambda,h)\tilde{\chi}_1}\|_{L^2(\mathbb{R}^{2n})}=\mathscr{O}(h^{-\frac{n}{2}-1}).\tag{40}$$

In the same way we also obtain

$$[(P(h)-\lambda)\otimes I,Q]K_{(\Delta\tilde{\chi}_2)R(\lambda,h)\tilde{\chi}_1}=\mathscr{O}_{L^2(\mathbb{R}^{2n})}(h^{-\frac{n}{2}})\tag{41}$$

From (36), (37), (39)–(41), and the fact that $Op_h(g)=\mathscr{O}_{\mathscr{B}(L^2(\mathbb{R}^{2n}))}(1)$ for any $g\in C_c^\infty(\mathbb{R}^{4n})$, we then obtain the estimate

$$\|Op_h(g)((P(h)-\lambda)\otimes I)QK_{\tilde{\chi}_2R(\lambda,h)\tilde{\chi}_1}\|_{L^2(\mathbb{R}^{2n})}=\mathscr{O}(h^{-\frac{n}{2}}),\tag{42}$$

Let now $f \in C_c^\infty(\mathbb{R}^{4n})$ also have support near ρ. Then

$$
\begin{aligned}
&Op_h(f)((P(h)-\lambda)\otimes I)Op_h(g)((P(h)-\lambda)\otimes I)QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}\\
&\quad = Op_h(f)Op_h(g)((P(h)-\lambda)\otimes I)^2 QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}\\
&\qquad + Op_h(f)[(P(h)-\lambda)\otimes I, Op_h(g)]((P(h)-\lambda)\otimes I)QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}.
\end{aligned}
$$

From (42) and the fact that $[(P(h)-\lambda)\otimes I, Op_h(g)] \in \Psi_h^{-1}(1, \mathbb{R}^{2n})$, we obtain

$$
Op_h(f)[(P(h)-\lambda)\otimes I, Op_h(g)]((P(h)-\lambda)\otimes I)QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = \mathcal{O}_{L^2(\mathbb{R}^{2n})}(h^{1-\frac{n}{2}}). \quad (43)
$$

The same argument as in (42) also implies that

$$
Op_h(f)Op_h(g)((P(h)-\lambda)\otimes I)^2 QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = \mathcal{O}_{L^2(\mathbb{R}^{2n})}(h^{1-\frac{n}{2}}) \quad (44)
$$

and we obtain, from (43) and (44),

$$
\|Op_h(f)((P(h)-\lambda)\otimes I)Op_h(g)((P(h)-\lambda)\otimes I)QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}\|_{L^2(\mathbb{R}^{2n})} = \mathcal{O}(h^{1-\frac{n}{2}}).
$$

Iterating this argument, we then have that

$$
\left\| \left(\prod_{j=j_2+1}^{k_2} V_j \right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} \right\|_{L^2(\mathbb{R}^{2n})} = \mathcal{O}(h^{k_2-j_2-1-\frac{n}{2}}), \qquad h \to 0.
$$

We now observe that

$$
\begin{aligned}
\left(\prod_{j=j_2+1}^{k_2} V_j \right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} &= h^{k_2-j_2} QK_{\chi_3(h)R(\lambda,h)\tilde{\chi}_1} + h^{k_2-j_2} K_{\chi_4 \widetilde{P} R(\lambda,h)\tilde{\chi}_1}\\
&\quad + h^{k_2-j_2} Op_h(e_{j_2}) K_{\tilde{\chi}_2 \widetilde{P} R(\lambda,h)\tilde{\chi}_1},
\end{aligned}
$$

where $\chi_3(h)$ is polynomial in h with smooth coefficients with supports contained in $\operatorname{supp}\tilde{\chi}_2$, $\chi_4 \in C_c^\infty(\mathbb{R}^n)$ is such that $\operatorname{supp}\chi_4 \subseteq \operatorname{supp}\tilde{\chi}_2$, $\widetilde{P} \in \Psi_h^0(1, \mathbb{R}^{2n})$ has compact ess-supp$_h\sigma(\widetilde{P})$ in a sufficiently small neighborhood of ρ, ess-supp$_h e_{j_2}$ is compact. Therefore

$$
\begin{aligned}
&\left(\prod_{i=j_1+1}^{j_2} V_i \right)\left(\prod_{j=j_2+1}^{k_2} V_j \right) K_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1}\\
&\quad \equiv h^{k_2-j_2-1} i \int_0^T e^{\frac{i}{h}t\lambda} \left(\prod_{i=j_1+1}^{j_2} V_i \right) Q(\chi_3(h)\otimes\tilde{\chi}_1)K_{U(t)}\, dt\\
&\qquad + h^{k_2-j_2-1} i \int_0^T e^{\frac{i}{h}t\lambda} \left(\prod_{i=j_1+1}^{j_2} V_i \right)(\chi_4 \otimes \tilde{\chi}_1)(\widetilde{P}\otimes I)K_{U(t)}\, dt\\
&\qquad + h^{k_2-j_2-1} i \int_0^T e^{\frac{i}{h}t\lambda} \left(\prod_{i=j_1+1}^{j_2} V_i \right) Op_h(e_{j_2})(\tilde{\chi}_2 \otimes \tilde{\chi}_1)K_{U(t)}\, dt\\
&\quad = \mathcal{O}_{L^2(\mathbb{R}^{2n})}(h^{k_2-1-j_1-\frac{n}{2}}),
\end{aligned}
$$

where we have again used (29) and the fact that $U(t) \in \mathcal{I}_h^0(\mathbb{R}^{2n}, \Lambda_t')$, $t \in \mathbb{R}$.

Lastly, from the fact that $V_j \in \Psi_h^{-1}(1, \mathbb{R}^{2n})$, $1 \leq j \leq j_1$, we have that

$$\left(\prod_{j=1}^{k} B_j\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = h^{k-k_2}\left(\prod_{j=1}^{k_2} V_j\right) QK_{\tilde{\chi}_2 R(\lambda,h)\tilde{\chi}_1} = \mathcal{O}_{L^2(\mathbb{R}^{2n})}(h^{k-1-\frac{n}{2}}),$$

which completes the proof of the theorem. □

5.2. *Non-Trapping Energy Level*

Theorem 3. *Let $\lambda > 0$ be a non-trapping energy level for P. Then $\mathcal{A}(\lambda, h) \in \mathcal{I}_h^{\frac{3}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, SR(\lambda))$.*

Proof. We recall from Vasy and Zworski (2000) that $\|\chi R(\lambda, h)\chi\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(\frac{1}{h})$ for every $\chi \in C_c^\infty(\mathbb{R}^n)$. Then the result follows from Theorems 2 and 1. □

An Inverse Problem. Following a suggestion of Plamen Stefanov, we include a discussion of an inverse problem motivated by Theorem 3. Suppose that $P = -h^2\Delta + V$, where $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$, satisfies the general assumptions of this article and $X \equiv \mathbb{R}^n$. Suppose that $\lambda > \max_{x\in\mathbb{R}^n} V_+(x)$. Let V further be such that the metric $g_1(x) = (\lambda - V(x))dx^2$ is simple in $B(0, R_0)$, in the sense that the boundary $\partial B(0, R_0)$ is strictly convex with respect to g_1 and for any $x \in B(0, R_0)$, the exponential map $\exp_x : \exp_x^{-1}(B(0, R_0)) \to B(0, R_0)$ is a diffeomorphism. We observe that since the metric g_1 is simple, it follows that the energy level λ is non-trapping for $P(h)$.

We have the following

Theorem 4. *For P and λ as above, the scattering relation, and hence, by Theorem 3, the scattering amplitude, determine V uniquely.*

Outline of Proof. We compare this with the problem of the wave equation with variable wave speed defined as

$$c(x) = (\lambda - V(x))^{-\frac{1}{2}}.$$

Then the function c is equal to $1/\sqrt{\lambda}$ for large x. We have a new Hamiltonian $\tilde{p} - 1 = c^{-2}(x)\|\xi\|^2 - 1$. These Hamiltonians have the same integral curves, but they are parameterized in different ways. Hence the scattering relation for this Schrödinger equation is that related to the metric $g_1(x) = c^{-2}(x)dx^2$. It is now implicit in Michel (1981) that the scattering relation for the metric g_1 determines the boundary distance function uniquely. The results of Mukhometov and Romanov (1978) and Mukhometov (1981) further imply that the boundary distance function determines uniquely a simple metric conformal to the Euclidean, in particular, it determines c and therefore V. □

5.3. *Trapping Energy Level*

Here we consider a trapping energy level $\lambda > 0$. We let $W \subset \Lambda_R(\lambda)$ be as in Theorem 2. Then for every $(x, \xi) \in \pi_1(W')$ there exist unique $t_j(x, \xi) \in \mathbb{R}$ such that $\exp(t_j(x,\xi)H_p)(x,\xi) \in L_j(\lambda)$, $j \in \{-,+\}$. As in Section 3.1, we have that there exists an open set $U \subset \bigcup_{(x,\xi)\in\pi_1(W')} \exp(t_-(x,\xi)H_p)(x,\xi)$ such that we can define the scattering relation $SR_{\overline{U}}(\lambda)$ as in Section 3.1.

Under these assumptions, we have the following

Theorem 5. $\mathscr{A}(\lambda, h) \in \mathcal{I}_h^{\frac{3}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, SR_{\overline{U}}(\lambda))$.

Proof. Let $C \in \Psi_h^0\left(1, \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}\right)$ have compact wavefront set in a neighborhood of a point $\rho \in SR_{\overline{U}}(\lambda)$. From Burq (2002a), we have that

$$\left\| 1_{R_1 \le \|x\| \le R_2} R(\lambda, h) 1_{R_1 \le \|x\| \le R_2} \right\|_{\mathscr{B}(L^2(\mathbb{R}^n))} = \mathcal{O}\left(\frac{1}{h}\right)$$

for some $R_1 > 0$ and every $R_2 > R_1$. Therefore, from Theorem 2 we obtain that $\tilde{\chi}_2 R(\lambda, h)\tilde{\chi}_1 \in \mathcal{I}_h^1(\mathbb{R}^{2n}, \overline{W} \cap \Lambda_R(\lambda))$ and the assertion of the theorem now follows by Theorem 1. $\square$

5.4. *Microlocal Representation of the Scattering Amplitude*

Here we show how, under the non-degeneracy assumption, the expansion (3) follows from the results we have proven in this article and the characterization of semi-classical Fourier integral operators as oscillatory integrals, which we have developed in Alexandrova (Preprint, Theorem 1). More precisely, we have the following

Theorem 6. *Let $\omega_0 \in \mathbb{S}^{n-1}$ be regular for $\theta_0 \in \mathbb{S}^{n-1}$ and $L \in \mathbb{N}$ be the number of (θ_0, ω_0) phase trajectories. Let $\lambda > 0$ be such that $P - \lambda$ is of principal type and $\|\tilde{\chi}_2 R(\lambda, h)\tilde{\chi}_1\|_{\mathscr{B}(L^2(\mathbb{R}^n))} = \mathcal{O}(h^m)$, $m \in \mathbb{R}$.*

Then there exist $a_l \in S_{2n-2}^{\frac{n}{2}+1}(1)$, $l = 1, \dots, L$, such that

$$A(\lambda, h) = e^{\frac{i}{h}S_l} a_l, \qquad l = 1, \dots, L,$$

microlocally near $SR_l(\lambda)$, $l = 1, \dots, L$, where S_l, $l = 1, \dots, L$, are given by (11).

Proof. By Theorems 1 and 2, $\mathscr{A}(\lambda, h) \in \mathcal{I}_h^{\frac{3}{2}}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \bigcup_{l=1}^L SR_l(\lambda))$. With this and Lemma 4 the hypotheses of Alexandrova (Preprint, Theorem 1) are satisfied, and we obtain that there exist $a_l \in S_{2n-2}^{\frac{n}{2}+1}(1)$, $l = 1, \dots, L$, such that $A(\lambda, h) = e^{\frac{i}{h}S_l} a_l$ microlocally near $SR_l(\lambda)$, $l = 1, \dots, L$. $\square$

We remark here that the phase function in this microlocal representation of the scattering amplitude is the same as the one given by Michel (2004) and Robert and Tamura (1989). This theorem further points out that such a microlocal representation always holds.

Appendix

A. *Representation of the Scattering Amplitude*

In this appendix we define the semi-classical scattering amplitude and derive the representation of the scattering amplitude, which was used in the proof of

Theorem 1. The derivation is similar to the one presented in Stefanov (2002). We also prove a lemma that gives additional information on the structure of the scattering amplitude.

Let $\chi_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, 2$, be as defined in Section 2.3. Let

$$\psi = (1-\chi_1)e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}} + \psi_{sc}$$

be such that

$$(P(h)-\lambda)\psi = 0.$$

Then

$$\begin{aligned}(P(h)-\lambda)\psi_{sc} &= -(P(h)-\lambda)(1-\chi_1)e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}}\\ &= -\Big(-\frac{1}{2}h^2\Delta-\lambda\Big)(1-\chi_1)e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}} = -\frac{1}{2}[h^2\Delta,\chi_1]e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}} \qquad (45)\end{aligned}$$

and therefore

$$\psi_{sc} = -\frac{1}{2}R(\lambda,h)[h^2\Delta,\chi_1]e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}}. \qquad (46)$$

Then

$$\Big(-\frac{1}{2}h^2\Delta-\lambda\Big)(1-\chi_2)\psi_{sc} = \frac{1}{2}[h^2\Delta,\chi_2]\psi_{sc} + (1-\chi_2)(P(h)-\lambda)\psi_{sc}. \qquad (47)$$

Substituting (45) into (47), we obtain

$$\begin{aligned}\Big(-\frac{1}{2}h^2\Delta-\lambda\Big)(1-\chi_2)\psi_{sc} &= \frac{1}{2}[h^2\Delta,\chi_2]\psi_{sc} - \frac{1}{2}(1-\chi_2)[h^2\Delta,\chi_1]e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}}\\ &= \frac{1}{2}[h^2\Delta,\chi_2]\psi_{sc} \qquad (48)\end{aligned}$$

since $\operatorname{supp}\nabla\chi_1 \cap \operatorname{supp}(1-\chi_2) = \emptyset$. Substituting (46) into (48), we obtain

$$(1-\chi_2)\psi_{sc} = -\frac{1}{4}R_0(\lambda,h)[h^2\Delta,\chi_2]R(\lambda,h)[h^2\Delta,\chi_1]e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}},$$

where $R_0(\lambda, h)$ denotes the meromorphic extension of $(P_0(h)-z)^{-1}$ from $\Im z > 0$ to $\mathbb{C}$. By Melrose (1995, Proposition 1.1), which establishes the asymptotic expansion of $(R_0(\lambda, h)f)(x)$ as $\|x\| \to \infty$ for $f \in C_c^\infty(\mathbb{R}^n)$, we now obtain that the scattering amplitude A is given by

$$A(\omega,\theta;\lambda,h) = c(n,\lambda;h)\int e^{-\frac{i\sqrt{2\lambda}\langle\omega,x\rangle}{h}}\Big([h^2\Delta,\chi_2]R(\lambda,h)[h^2\Delta,\chi_1]e^{\frac{i\sqrt{2\lambda}\langle\theta,\cdot\rangle}{h}}\Big)(x)dx, \qquad (49)$$

where

$$c(n,\lambda,h) = e^{-i\pi\frac{n-3}{4}}\lambda^{\frac{n-3}{4}}2^{-\frac{n+9}{4}}(\pi h)^{-\frac{n+1}{2}}$$

and this is the representation of the scattering amplitude, which we use in the proof of the General Black Box Theorem. We remark that the proof of Melrose (1995, Proposition 1.1) relies mostly on the residue theorem and stationary phase. The independence of this representation of the choice of the functions χ_j, $j = 1, 2$, with the properties stated in Section 2.3 is proven in Petkov and Zworski (2001). Petkov and Zworski (2001, Proposition 2.1) further show that the scattering amplitude with the constant $c(n, \lambda, h)$ replaced by $\tilde{c}(n, \lambda, h) = i\pi\lambda^{\frac{n-2}{2}}(2\pi)^{-n}h^{-n}$ is also the kernel of $S(\lambda, h) - I$, where $S(\lambda, h)$ is the scattering matrix at energy λ.

From this representation of the scattering amplitude it is clear that it can be extended meromorphically everywhere where the resolvent can be extended meromorphically, and that the poles of the scattering amplitude are among the resonances.

We now prove the following

Lemma 5. *If $\lambda > 0$ is not a resonance and $\psi_j \in C_c^\infty(\mathbb{R}^n \backslash B(0, R_0))$, $j = 1, 2$, have disjoint supports, then $K_{\psi_1 R(\lambda,h)\psi_2} \in C^\infty(\mathbb{R}^{2n})$.*

Proof. Let $\varphi \in C^\infty(\mathbb{R}^n)$ and consider

$$\begin{aligned}
&\left(-\frac{1}{2}h^2\Delta - \lambda\right)\psi_1 R(\lambda, h)\psi_2\varphi \\
&= \left[-\frac{1}{2}h^2\Delta, \psi_1\right]R(\lambda, h)\psi_2\varphi + \psi_1\left(-\frac{1}{2}h^2\Delta - \lambda\right)((R(\lambda, h)\psi_2\varphi)|_{\mathbb{R}^n\backslash B(0,R_0)}) \\
&= \left[-\frac{1}{2}h^2\Delta, \psi_1\right]R(\lambda, h)\psi_2\varphi + \psi_1((P(h) - \lambda)R(\lambda, h)\psi_2\varphi)|_{\mathbb{R}^n\backslash B(0,R_0)} \\
&= \left[-\frac{1}{2}h^2\Delta, \psi_1\right]R(\lambda, h)\psi_2\varphi + \psi_1\psi_2\varphi \\
&= \left[-\frac{1}{2}h^2\Delta, \psi_1\right]R(\lambda, h)\psi_2\varphi \in H^{-1}(\mathbb{R}^n).
\end{aligned}$$

Therefore,

$$\psi_1 R(\lambda, h)\psi_2\varphi \in H^1(\mathbb{R}^n).$$

Similarly, for every $k \in \mathbb{N}$, we have that

$$\begin{aligned}
&\left(-\frac{1}{2}h^2\Delta - \lambda\right)^k \psi_1 R(\lambda, h)\psi_2\varphi \\
&= \underbrace{\left[-\frac{1}{2}h^2\Delta, \left[\dots, \left[-\frac{1}{2}h^2\Delta, \psi_1\right]\dots\right]\right]}_{k} R(\lambda, h)\psi_2\varphi \in H^{-k}(\mathbb{R}^n)
\end{aligned}$$

and therefore

$$\psi_1 R(\lambda, h)\psi_2\varphi \in H^k(\mathbb{R}^n) \qquad \text{for every } k.$$

Thus

$$\psi_1 R(\lambda, h)\psi_2\varphi \in C^\infty(\mathbb{R}^n). \tag{50}$$

This, together with Lemma 1, implies that $K_{\psi_1 R(\lambda,h)\psi_2} \in C^\infty(\mathbb{R}^{2n})$. $\square$

B. Elements of Semi-Classical Analysis

Here we review some of the elements of semi-classical analysis which we use in this article. First, we define two classes of symbols,

$$S^m_{2n}(1) = \{a \in C^\infty(\mathbb{R}^{2n} \times (0, h_0]) : \forall\alpha, \beta \in \mathbb{N}^n, |\partial^\alpha_x \partial^\beta_\xi a(x, \xi; h)| \le C_{\alpha,\beta} h^{-m}\}$$

and

$$\begin{aligned} S^{m,k}(T^*\mathbb{R}^n) = \{a \in C^\infty(T^*\mathbb{R}^n \times (0, h_0]) : \forall\alpha, \beta \in \mathbb{N}^n, |\partial^\alpha_x \partial^\beta_\xi a(x, \xi; h)| \\ \le C_{\alpha,\beta} h^{-m} \langle\xi\rangle^{k-|\beta|}\}, \end{aligned}$$

where $h_0 \in (0, 1]$ and $m, k \in \mathbb{R}$. For $a \in S^m_{2n}(1)$ or $a \in S^{m,k}(T^*\mathbb{R}^n)$ we define the corresponding semi-classical pseudodifferential operator of class $\Psi^m_h(1, \mathbb{R}^n)$ or $\Psi^{m,k}_h(\mathbb{R}^n)$, respectively, by setting

$$Op_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i\langle x-y,\xi\rangle}{h}} a(x, \xi; h)u(y)dy\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n),$$

and extending the definition to $\mathcal{S}'(\mathbb{R}^n)$ by duality (see Dimassi and Sjöstrand, 1999). Here we work only with symbols which admit asymptotic expansions in h and with pseudodifferential operators which are quantizations of such symbols. For $A \in \Psi^m_h(1, \mathbb{R}^n)$ or $A \in \Psi^{m,k}_h(\mathbb{R}^n)$, we use $\sigma_0(A)$ and $\sigma(A)$ to denote its principal symbol and its complete symbol, respectively. A semi-classical pseudodifferential operator will be called of principal type if its principal symbol a_0 satisfies

$$a_0 = 0 \implies da_0 \neq 0.$$

For $a \in S^{m,k}(T^*\mathbb{R}^n)$ or $a \in S^m_{2n}(1)$ we define:

$$\begin{aligned} &\text{ess-supp}_h\, a \\ &\quad = \{(x, \xi) \in T^*\mathbb{R}^n |\, \exists\, \epsilon > 0\ \partial^\alpha_x \partial^\beta_\xi a(x', \xi') = \mathcal{O}_{C(B((x,\xi),\epsilon))}(h^\infty), \quad \forall\alpha, \beta \in \mathbb{N}^n\}^c \\ &\quad \cup \Bigg(\Bigg\{(x, \xi) \in T^*\mathbb{R}^n\backslash\{0\} |\, \exists\, \epsilon > 0\ \partial^\alpha_x \partial^\beta_\xi a(x', \xi') = \mathcal{O}(h^\infty\langle\xi\rangle^{-\infty}), \\ &\qquad \text{uniformly in } (x', \xi') \text{ such that } \|x - x'\| + \frac{1}{\|\xi'\|} \\ &\qquad + \left|\frac{\xi}{\|\xi\|} - \frac{\xi'}{\|\xi'\|}\right| < \epsilon\Bigg\}/\mathbb{R}_+\Bigg)^c \\ &\quad \subset T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n, \end{aligned}$$

where we define $S^*\mathbb{R}^n = (T^*\mathbb{R}^n \backslash \{0\})/\mathbb{R}_+$. For $A \in \Psi_h^{m,k}(\mathbb{R}^n)$ or $A \in \Psi_h^m(1, \mathbb{R}^n)$ we then define

$$WF_h(A) = \text{ess-supp}_h\, a,\, A = Op_h(a).$$

We also define the class of semi-classical distributions $\mathscr{D}'_h(\mathbb{R}^n)$ with which we will work here

$$\mathscr{D}'_h(\mathbb{R}^n) = \{u \in C_h^\infty((0,1]; \mathscr{D}'(\mathbb{R}^n)) : \forall \chi \in C_c^\infty(\mathbb{R}^n) \exists\, N \in \mathbb{N} \text{ and } \\ C_N > 0 : |\mathscr{F}_h(\chi u)(\xi)| \leq C_N h^{-N} \langle \xi \rangle^N\}$$

where

$$\mathscr{F}_h(\chi u)(\xi) = \langle e^{-\frac{i}{h}\langle \cdot, \xi \rangle}, \chi u \rangle$$

and $\langle \cdot, \cdot \rangle$ denotes the distribution pairing. We also extend this definition in the obvious way to $\mathscr{E}'_h(\mathbb{R}^n)$.

The L^2-based semi-classical Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, consist of the distributions $u \in \mathscr{E}'_h(\mathbb{R}^n)$ such that

$$\|u\|^2_{H^s(\mathbb{R}^n)} \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\mathscr{F}_h(u)(\xi)|^2 d\xi < \infty.$$

We shall say that $u = v$ *microlocally* (or $u \equiv v$) near an open or closed set $U \subset T^*\mathbb{R}^n$, if $P(u - v) = \mathscr{O}(h^\infty)$ in $C_c^\infty(\mathbb{R}^n)$ for every $P \in \Psi_h^0(1, \mathbb{R}^n)$ such that

$$WF_h(P) \subset \widetilde{U}, \qquad \overline{U} \Subset \widetilde{U} \Subset T^*\mathbb{R}^n, \quad \widetilde{U} \text{ open.} \tag{51}$$

We shall also say that u satisfies a property $\mathscr{P}$ *microlocally* near an open set $U \subset T^*\mathbb{R}^n$ if there exists $v \in \mathscr{D}'_h(\mathbb{R}^n)$ such that $u = v$ microlocally near U and v satisfies property $\mathscr{P}$.

For $u \in \mathscr{D}'_h(\mathbb{R}^n)$ we define its semi-classical wavefront set as follows.

Definition 3. Let $u \in \mathscr{D}'_h(\mathbb{R}^n)$ and let $(x_0, \xi_0) \in \widehat{T}^*(\mathbb{R}^n)$. We shall say that (x_0, ξ_0) does not belong to $WF_h(u)$ if:

(i) If (x_0, ξ_0) is finite: there exist $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and an open neighborhood U of ξ_0, such that $\forall N \in \mathbb{N}$, $\forall \xi \in U$, $|\mathscr{F}_h(\chi u)(\xi)| \leq C_N h^N$. We shall denote the complement of the set of all such points by $WF_h^f(u)$.

(ii) If (x_0, ξ_0) is infinite: there exist $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and a conic neighborhood U of ξ_0, such that $\forall N \in \mathbb{N}$, $\forall \xi \in U \cap \{\|\xi\| \geq \frac{1}{C}\}$,

$$|\mathscr{F}_h(\chi u)(\xi)| \leq C_N h^N \langle \xi \rangle^{-N}.$$

We shall denote the complement of the set of all such points by $WF_h^i(u)$.

We extend these notions to compact manifolds through the following definition of semi-classical pseudodifferential operators on compact manifolds. Let M be a smooth compact manifold and $\kappa_j : M_j \to X_j$, $j = 1, \ldots, N$, a set of local charts.

A linear continuous operator $A : C^\infty(M) \to \mathscr{D}'_h(M)$ belongs to $\Psi_h^m(1, M)$ or $\Psi_h^{m,k}(T^*M)$ if for all $j \in \{1, \dots, N\}$ and $u \in C_c^\infty(M_j)$ we have $Au \circ \kappa_j^{-1} = A_j(u \circ \kappa_j^{-1})$ with $A_j \in \Psi_h^m(1, X_j)$ or $A_j \in \Psi_h^{m,k}(X_j)$, respectively, and $\chi_1 A \chi_2 : \mathscr{D}'_h(M) \to h^\infty C^\infty(M)$ if $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$.

We now define global semi-classical Fourier integral operators.

Definition 4. Let M be a smooth k-dimensional manifold and let $\Lambda \subset T^*M$ be a smooth closed Lagrangian submanifold with respect to the canonical symplectic structure on T^*M. Let $r \in \mathbb{R}$. Then the space $I_h^r(M, \Lambda)$ of semi-classical Fourier integral distributions of order r associated to Λ is defined as the set of all $u \in \mathscr{D}'_h(M)$ such that

$$\left(\prod_{j=0}^{N} A_j\right)(u) = \mathscr{O}_{L^2(M)}(h^{N-r-\frac{k}{4}}), \qquad h \to 0,$$

for all $N \in \mathbb{N}_0$ and for all $A_j \in \Psi_h^0(1, M)$, $j = 0, \dots, N-1$, with compact wavefront set and principal symbols vanishing on Λ, and any $A_N \in \Psi_h^0(1, M)$ with compact wavefront set.

A continuous linear operator $C_c^\infty(M_1) \to \mathscr{D}'_h(M_2)$, where M_1, M_2 are smooth manifolds, whose Schwartz kernel is an element of $I_h^r(M_1 \times M_2, \Lambda)$ for some Lagrangian submanifold $\Lambda \subset T^*M_1 \times T^*M_2$ and some $r \in \mathbb{R}$ will be called a global semi-classical Fourier integral operator of order r associated to Λ. We denote the space of these operators by $\mathscr{I}_h^r(M_1 \times M_2, \Lambda)$.

Lastly, we define the microlocal equivalence of two semi-classical Fourier integral operators.

Definition 5. Let M_j, $j = 1, 2$, be smooth manifolds, $\Lambda \subset T^*M_1 \times T^*M_2$— a Lagrangian submanifold, and $T, T' \in \mathscr{I}_h^r(M_1 \times M_2, \Lambda)$ for some $r \in \mathbb{R}$. For open or closed sets $U \subset T^*M_1$ and $V \subset T^*M_2$ the operators T and T' are said to be *microlocally equivalent* near $U \times V$ if there exist open sets $\widetilde{U} \Subset T^*M_1$ and $\widetilde{V} \Subset T^*M_2$ with $\overline{U} \Subset \widetilde{U}$ and $\overline{V} \Subset \widetilde{V}$ such that for any $A \in \Psi_h^0(1, M_1)$ and $B \in \Psi_h^0(1, M_2)$ with $WF_h(A) \subset \widetilde{U}$ and $WF_h(B) \subset \widetilde{V}$ we have that

$$B(T - T')A = \mathscr{O}(h^\infty) : \mathscr{D}'_h(M_1) \to C^\infty(M_2).$$

We shall also write $T \equiv T'$ near $V \times U$.

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