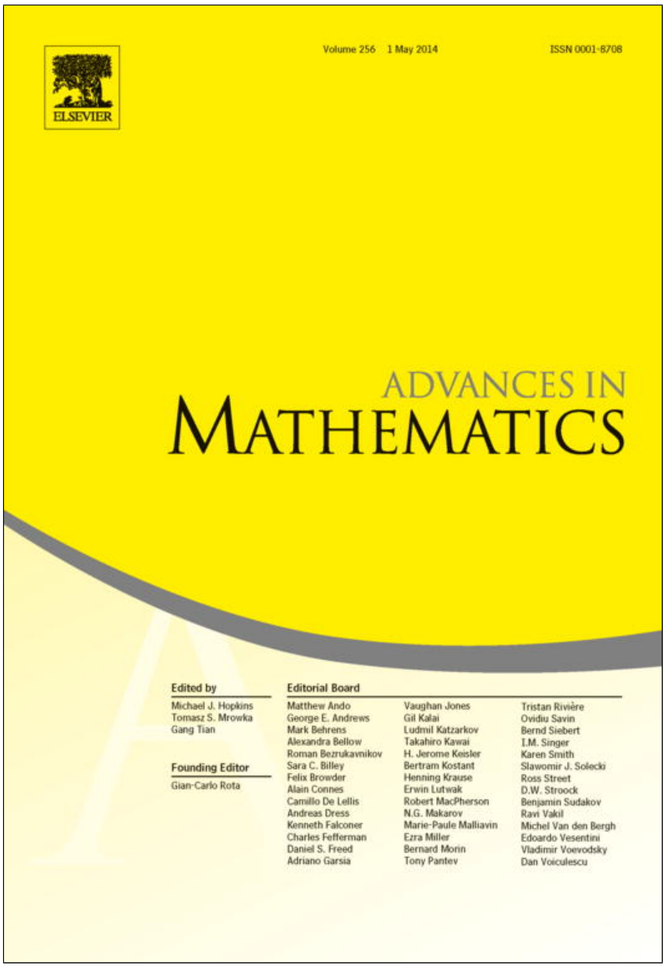
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Resonances in scattering by two magnetic fields at large separation and a complex scaling method

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ABSTRACT

We study the quantum resonances in magnetic scattering in two dimensions. The scattering system consists of two obstacles by which the magnetic fields are completely shielded. The trajectories trapped between the two obstacles are shown to generate the resonances near the positive real axis, when the distance between the obstacles goes to infinity. The location is described in terms of the backward amplitudes for scattering by each obstacle. A difficulty arises from the fact that even if the supports of the magnetic fields are largely separated from each other, the corresponding vector potentials are not expected to be well separated. To overcome this, we make use of a gauge transformation and develop a new type of complex scaling method. We can cover the scattering by two solenoids at large separation as a special case. The obtained result heavily depends on the magnetic fluxes of the solenoids. This indicates that the Aharonov–Bohm effect influences the location of resonances.

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1. Introduction

The resonance problem is one of the most active subjects in scattering theory at present. In this paper we develop a new method of complex scaling to study the resonances for scattering by magnetic fields with two compact supports at large separation in two dimensions, and we show that the resonances are generated near the real axis by the trajectories trapped between the two fields as the distance between the two supports goes to infinity.

The resonances are defined as the poles of the resolvent kernel (the Green function) meromorphically continued from the upper half plane of the complex plane to the lower half plane (the unphysical sheet) as a function of the spectral parameter. The complex scaling method has been initiated by [3] and further developed by [14,17] to study the location of resonances in quantum mechanics. The method has been successfully employed in the semiclassical theory of resonances near the real axis generated by closed classical trajectories. In particular, there is a large number of works devoted to the semiclassical problem of shape resonances. The upper or lower bounds on the resonance width (the imaginary part of the resonance) and its asymptotic expansion have been studied by many authors [7,8,11,15], etc., under various kinds of assumptions. We refer to the book [12] for an extensive list of references and to [18,22] for excellent surveys. We also refer to [10] for the recent progress on the subject.

The main body of the present work is occupied by constructing the resolvent kernel of magnetic Schrödinger operators with two compactly supported fields. This is done by combining the two resolvent kernels constructed for each field. To do this, we have to overcome two main difficulties. The first difficulty, which is proper to the two dimensional space, comes from the topological feature that $\boldsymbol{R}^2 \setminus \{0\}$ is not simply connected. For compactly supported magnetic fields, the corresponding vector potentials cannot be expected to fall off rapidly at infinity. This is the case where the magnetic fluxes do not vanish. In fact, vector potentials have the long range property. This makes it difficult to separate the vector potentials well, even if the supports of the two magnetic fields are largely separated from each other. The second difficulty, which is specific to the resonance problem, is that the resolvent kernel with spectral parameters in the unphysical sheet grows exponentially at infinity. Hence the integrals of such quantities cannot be controlled simply by integration by parts using oscillatory properties. We encounter this difficulty in the composition of the resolvent kernels constructed for each magnetic field. To overcome these difficulties, we present a new method of complex scaling and analyze the asymptotic properties of the resolvent kernel as the distance between the supports of the two fields goes to infinity. We are going to explain the differences between our complex scaling method and the existing ones in some details in Section 5.

We set up our problem and formulate the obtained result precisely. We always work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$ and write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2} (-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with the vector potential $A = (a_1, a_2)$. The magnetic field b associated with A is defined by

$$b = \nabla \times A = \partial_1 a_2 - \partial_2 a_1 : \boldsymbol{R}^2 \to \boldsymbol{R}$$

and the integral

$$\alpha = (2\pi)^{-1} \int b(x)\, dx$$

is called the magnetic flux of b, where the integral without the domain attached is taken over the whole space. We often use this abbreviation throughout the whole exposition.

Let $b_\pm \in C_0^\infty(\boldsymbol{R}^2)$ be a given magnetic field with the magnetic flux

$$\alpha_\pm = (2\pi)^{-1} \int b_\pm(x)\, dx.$$

We assume that the support $\operatorname{supp} b_\pm$ of $b_\pm$ satisfies

$$\operatorname{supp} b_\pm \subset \mathcal{O}_\pm \tag{1.1}$$

for some simply connected bounded obstacle $\mathcal{O}_\pm$, where $\mathcal{O}_\pm$ is assumed to have the origin as an interior point and the smooth boundary $\partial\mathcal{O}_\pm$. We can take the vector potential $A_\pm(x)$ associated with $b_\pm$ to fulfill

$$A_\pm(x) = \Phi_\pm(x) = \alpha_\pm \Phi(x) \tag{1.2}$$

over $\Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm$, where $\Phi(x)$ is defined by

$$\Phi(x) = \bigl(-\partial_2 \log|x|, \partial_1 \log|x|\bigr) = \bigl(-x_2/|x|^2, x_1/|x|^2\bigr) \tag{1.3}$$

and generates the point-like field (solenoid)

$$\nabla \times \Phi = \Delta \log|x| = 2\pi\delta(x)$$

with center at the origin. For $d_\pm \in \boldsymbol{R}^2$, we set

$$\Omega_d = \boldsymbol{R}^2 \setminus (\overline{\mathcal{O}}_{-d} \cup \overline{\mathcal{O}}_{+d}), \qquad \mathcal{O}_{\pm d} = \{x\colon\ x - d_\pm \in \mathcal{O}_\pm\},$$

and define

$$\Phi_d(x) = \Phi_{-d}(x) + \Phi_{+d}(x) = \Phi_-(x - d_-) + \Phi_+(x - d_+). \tag{1.4}$$

We deal with $|d| = |d_+ - d_-| \gg 1$ as a large parameter with the direction $\hat{d} = d/|d|$ fixed. We consider the self-adjoint operator

$$H_d = H(\Phi_d), \qquad \mathcal{D}(H_d) = H^2(\Omega_d) \cap H_0^1(\Omega_d), \tag{1.5}$$

under the zero Dirichlet boundary conditions, where $H^2(\Omega_d)$ and $H_0^1(\Omega_d)$ stand for the usual Sobolev spaces over Ω_d.

We denote by $R(\zeta; T) = (T - \zeta)^{-1}$ the resolvent of an operator T acting on $L^2(\boldsymbol{R}^2)$ or $L^2(\Omega_d)$. It is well known that H_d has no positive eigenvalues and the continuous spectrum occupied by $[0, \infty)$ is absolutely continuous. We further know that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2(\Omega_d) \to L^2(\Omega_d), \quad \operatorname{Re} \zeta > 0,\ \operatorname{Im} \zeta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to a region in the lower half plane across the positive real axis where the continuous spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\operatorname{Im} \zeta \leqslant 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\Omega_d)$ to $L^2_{\mathrm{loc}}(\Omega_d)$ in the sense that $\chi R(\zeta; H_d)\chi : L^2(\Omega_d) \to L^2(\Omega_d)$ is bounded for every $\chi \in C_0^\infty(\overline{\Omega}_d)$, where $L^2_{\mathrm{comp}}(W)$ denotes the space of square integrable functions with compact support in the closure $\overline{W}$ of a region $W \subset \boldsymbol{R}^2$ and $L^2_{\mathrm{loc}}(W)$ denotes the space of locally square integrable functions over $\overline{W}$. This can be shown by an application of the complex scaling method (see [7,12,19]). We use the same notation $R(\zeta; H_d)$ to denote this meromorphic function with values in operators from $L^2_{\mathrm{comp}}(\Omega_d)$ to $L^2_{\mathrm{loc}}(\Omega_d)$. In fact, we can show that $R(\zeta; H_d)$ admits the meromorphic continuation to the lower half plane $\{\zeta \in \boldsymbol{C}\colon \operatorname{Re} \zeta > 0,\ \operatorname{Im} \zeta < 0\}$, but the argument here is restricted only to a neighborhood of the positive real axis. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane. Our aim is to study how the resonances are generated near the real axis by the trajectories trapped between $\mathcal{O}_{-d}$ and $\mathcal{O}_{+d}$ as $|d| \to \infty$.

The obtained result is formulated in terms of the backward amplitudes for the pairs $(H_0, H_\pm)$, where $H_0 = -\Delta$ denotes the free Hamiltonian with domain $\mathcal{D}(H_0) = H^2(\boldsymbol{R}^2)$, and $H_\pm$ is the self-adjoint operator defined by

$$H_\pm = H(\Phi_\pm), \qquad \mathcal{D}(H_\pm) = H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm), \qquad \Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm, \tag{1.6}$$

with $\Phi_\pm$ defined by (1.2). We denote by $f_\pm(\omega \to \theta; E)$ the scattering amplitude from the incident direction $\omega \in S^1$ to the final one θ at energy $E > 0$ for $(H_0, H_\pm)$. As is shown in Section 3, these amplitudes admit the analytic extensions $f_\pm(\omega \to \theta; \zeta)$ in a complex neighborhood of the positive real axis as a function of E.

We now fix $E_0 > 0$ and take a complex neighborhood

$$D_d = \left\{ \zeta \in \boldsymbol{C}\colon\ |\operatorname{Re} \zeta - E_0| < \delta_0 E_0,\ |\operatorname{Im} \zeta| < (1 + 2\delta_0) E_0^{1/2} \frac{\log |d|}{|d|} \right\} \tag{1.7}$$

for δ_0, $0 < \delta_0 \ll 1$, small enough. We define

$$h(\zeta; d) = \frac{e^{2ik|d|}}{|d|} f_-(-\hat{d} \to \hat{d}; \zeta) f_+(\hat{d} \to -\hat{d}; \zeta), \quad k = \zeta^{1/2}, \tag{1.8}$$

over D_d, where the branch k is taken in such a way that $\operatorname{Re} k > 0$ for $\operatorname{Re} \zeta > 0$. We always use the notation k with the meaning ascribed here. Since

$$2 \operatorname{Im} k = 2 \operatorname{Im}(\operatorname{Re} \zeta + i \operatorname{Im} \zeta)^{1/2} = \operatorname{Im} \zeta / (\operatorname{Re} \zeta)^{1/2} + O\big(|\operatorname{Im} \zeta|^3\big) \tag{1.9}$$

for $\zeta \in D_d$ and since

$$(\operatorname{Re} \zeta)^{1/2} = E_0^{1/2}\big(1 + (\operatorname{Re} \zeta - E_0)/(2E_0) + O\big(\delta_0^2\big)\big) \tag{1.10}$$

with $|\operatorname{Re} \zeta - E_0| < \delta_0 E_0$, we can take $\delta_0 > 0$ so small that

$$|d|^{\delta_0} < \big|\exp\big(2ik|d|\big)\big|/|d| < |d|^{3\delta_0}, \quad |d| \gg 1,$$

on the bottom of D_d ($\operatorname{Im} \zeta = -(1 + 2\delta_0) E_0^{1/2}((\log |d|)/|d|)$). This implies that the curve defined by $|h(\zeta; d)| = 1$ with $|\operatorname{Re} \zeta - E_0| < \delta_0 E_0$ is completely contained in D_d. We denote by

$$\big\{\zeta_j(d)\big\}_{j=1}^{N_d}, \quad \zeta_j(d) \in D_d, \qquad \operatorname{Re} \zeta_1(d) < \operatorname{Re} \zeta_2(d) < \cdots < \operatorname{Re} \zeta_{N_d}(d),$$

the solutions to the equation

$$h(\zeta; d) = 1. \tag{1.11}$$

We can show (see Lemma 4.1) that $\zeta_j(d)$ behaves like

$$\operatorname{Im} \zeta_j(d) \sim -E_0^{1/2}\big((\log |d|)/|d|\big), \qquad \operatorname{Re}\big(\zeta_{j+1}(d) - \zeta_j(d)\big) \sim 2\pi E_0^{1/2}/|d|,$$

for $|d| \gg 1$. We remark that an easy calculation using the latter equation (see Lemma 4.1 for its precise form) and the definition of D_d in (1.7) gives that

$$N_d \sim \frac{\delta_0 E_0^{1/2} |d|}{\pi}, \quad |d| \to \infty.$$

We are now in a position to state the main theorem.

Theorem 1.1. *Let the notation be as above. Assume that*

$$f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0, \quad \hat{d} = d/|d|,$$

at energy $E_0 > 0$. Then we can take $\delta_0 > 0$ so small that the neighborhood D_d defined by (1.7) has the following property: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $|d| > d_\varepsilon$, H_d has the resonances

$$\{\zeta_{\mathrm{res},j}(d)\}_{j=1}^{N_d} \subset D_d, \qquad \operatorname{Re}\zeta_{\mathrm{res},1}(d) < \cdots < \operatorname{Re}\zeta_{\mathrm{res},N_d}(d),$$

in the neighborhood

$$\{\zeta \in \boldsymbol{C}\colon\ |\zeta - \zeta_j(d)| < \varepsilon/|d|\}$$

and the resolvent $R(\zeta; H_d)$ is analytic over $D_d \setminus \{\zeta_{\mathrm{res},1}(d), \ldots, \zeta_{\mathrm{res},N_d}(d)\}$ as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega_d)$ to $L^2_{\mathrm{loc}}(\Omega_d)$.

It does not matter whether the obstacles $\mathcal{O}_\pm$ have smooth boundaries. The idea covers the case where $\mathcal{O}_\pm$ shrink to the origin. Here we discuss the scattering by two solenoids as a special case of the above theorem. We first consider

$$P = H(\alpha\Phi) \tag{1.12}$$

with one solenoid with center at the origin as an operator acting on $L^2(\boldsymbol{R}^2)$, where $\Phi(x)$ is defined by (1.3). The energy operator P governs quantum particles moving in the field $2\pi\alpha\delta(x)$ and is often called the Aharonov–Bohm Hamiltonian in the physics literature. This model was employed by Aharonov and Bohm [4] in 1959 in order to show theoretically that a magnetic potential itself has a direct significance in quantum mechanics, and it is now called the Aharonov–Bohm effect, which is known as one of the most remarkable quantum phenomena. The operator P formally defined by (1.12) is symmetric over $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$, but it is not necessarily essentially self-adjoint in the space $L^2 = L^2(\boldsymbol{R}^2)$ because of the strong singularity at the origin of Φ. We know (see [1,9]) that it is a symmetric operator with type $(2,2)$ of deficiency indices. The self-adjoint extension is realized by imposing a boundary condition at the origin. Its Friedrichs extension denoted by the same notation P has the domain

$$\mathcal{D}(P) = \Big\{u \in L^2(\boldsymbol{R}^2)\colon\ (-i\nabla - \alpha\Phi)^2 u \in L^2(\boldsymbol{R}^2),\ \lim_{|x|\to 0}|u(x)| < \infty\Big\}, \tag{1.13}$$

where $(-i\nabla - \alpha\Phi)^2 u$ is understood in the sense of distributions. The scattering by one solenoid is known as one of the exactly solvable quantum systems. In particular, the scattering amplitude $f(\theta \to \omega; E)$ is explicitly calculated as

$$f(\omega \to \theta; E) = \left(\frac{2}{\pi}\right)^{1/2} e^{i\pi/4} E^{-1/4} \sin(\alpha\pi) e^{i\lfloor\alpha\rfloor(\theta-\omega)} \frac{e^{i(\theta-\omega)}}{1 - e^{i(\theta-\omega)}}, \tag{1.14}$$

where the Gauss notation $\lfloor\alpha\rfloor$ denotes the greatest integer not exceeding α and the coordinates θ, ω over the unit circle S^1 are identified with the azimuth angles from the positive x_1 axis (see [2,4,16] for details).

Let $d_\pm \in \boldsymbol{R}^2$ be as in (1.4). We now consider

$$\tilde{H}_d = H(\Phi_d), \qquad \Phi_d(x) = \alpha_-\Phi(x - d_-) + \alpha_+\Phi(x - d_+), \tag{1.15}$$

as an energy operator with two solenoids. This operator becomes self-adjoint under the boundary conditions $\lim_{|x-d_\pm|\to 0}|u(x)| < \infty$, and it follows from (1.14) that the backward amplitude is given by

$$f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) = (2\pi)^{-1/2} e^{i\pi/4} E_0^{-1/4} (-1)^{\lfloor\alpha_\pm\rfloor+1} \sin(\alpha_\pm \pi)$$

for scattering by each solenoid $2\pi\alpha_\pm\delta(x)$ at energy E_0. Hence $h(\zeta; d)$ defined by (1.8) takes the explicit form

$$\tilde{h}(\zeta; d) = \frac{e^{2ik|d|}}{k|d|}(2\pi)^{-1} i(-1)^{\lfloor\alpha_-\rfloor+\lfloor\alpha_+\rfloor} \sin(\alpha_-\pi)\sin(\alpha_+\pi) \tag{1.16}$$

for the operator $\tilde{H}_d$. The assumption $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0$ of Theorem 1.1 is automatically fulfilled, provided that $\alpha_\pm$ is not an integer. Thus we obtain the following theorem.

Theorem 1.2. *Let D_d be defined by (1.7) with $0 < \delta_0 \ll 1$, and let $\tilde{H}_d = H(\Phi_d)$ be defined by (1.15). Assume that $\alpha_\pm$ is not an integer. Then the resonances in D_d of $\tilde{H}_d$ are approximated by the solutions to equation $\tilde{h}(\zeta; d) = 1$ in the same sense as in Theorem 1.1.*

In the previous work [5], we have shown that there are no resonances of $\tilde{H}_d$ with $|d| \gg 1$ in the region where $|\tilde{h}(\zeta; d)| < 1$ strictly, so that a sharp lower bound on the resonance widths has been established. However the neighborhood D_d contains points at which $|h(\zeta; d)| > 1$. Thus we can improve considerably the result obtained by [5] by showing the actual existence of resonances in D_d.

The theorem above has an application to the semiclassical theory of quantum resonances in scattering by two solenoids. We now consider the self-adjoint operator

$$\hat{H}_h = (-ih\nabla - \Psi)^2, \qquad \Psi(x) = \alpha_-\Phi(x - p_-) + \alpha_+\Phi(x - p_+), \tag{1.17}$$

with $0 < h \ll 1$ under the boundary conditions $\lim_{|x-p_\pm|\to 0}|u(x)| < \infty$ at the two centers $p_\pm$, where $\Phi(x)$ is again defined by (1.3). Let $\gamma(x)$ denote the azimuth angle from the positive x_1 axis to $\hat{x} = x/|x|$. Then we have the relation

$$\nabla\gamma(x) = \Phi(x). \tag{1.18}$$

We define the two unitary operators

$$(U_1 u)(x) = h^{-1} u\big(h^{-1}x\big), \qquad (U_2 u)(x) = \exp\big(i g_h(x)\big) u(x)$$

acting on $L^2(\boldsymbol{R}^2)$, where

$$g_h = \lfloor\alpha_-/h\rfloor\gamma(x - d_-) + \lfloor\alpha_+/h\rfloor\gamma(x - d_+)$$

with $d_\pm = p_\pm/h$. The function $\exp(ig_h(x))$ is well defined as a single valued function, and $g_h(x)$ satisfies

$$\nabla g_h = \lfloor \alpha_-/h \rfloor \Phi(x - d_-) + \lfloor \alpha_+/h \rfloor \Phi(x - d_+)$$

by (1.18). Thus $\hat{H}_h$ turns out to be unitarily equivalent to

$$H(\Psi_d) = (U_1 U_2)^* \hat{H}_h (U_1 U_2),$$

where

$$\Psi_d(x) = \beta_- \Phi(x - d_-) + \beta_+ \Phi(x - d_+), \qquad \beta_\pm = \alpha_\pm/h - \lfloor \alpha_\pm/h \rfloor.$$

Hence the semiclassical resonances of $\hat{H}_h$ coincide with those in scattering by two solenoids with the centers at large separation

$$|d| = |d_+ - d_-| = |p_+ - p_-|/h = |p|/h \gg 1.$$

For $\tilde{h}(\zeta; d)$ defined by (1.16) with $d = p/h$ and $\alpha_\pm = \beta_\pm$, the equation $\tilde{h}(\zeta; d) = 1$ has the solutions $\{\zeta_j(h)\}_{1 \leqslant j \leqslant N_h}$ with properties

$$\operatorname{Im} \zeta_j(h) \sim E_0^{1/2} (h \log h)/|p|, \qquad \operatorname{Re}\big(\zeta_{j+1}(h) - \zeta_j(h)\big) \sim 2\pi E_0^{1/2} h/|p|$$

for $0 < h \ll 1$. The result below is obtained as an immediate consequence of Theorem 1.2.

Corollary 1.1. *Let $\hat{H}_h$ be defined by (1.17) and let $p = p_+ - p_-$. Assume that*

$$c \leqslant \beta_\pm = \alpha_\pm/h - \lfloor \alpha_\pm/h \rfloor \leqslant 1 - c$$

for some $0 < c < 1/2$ uniformly in h. Define D_d by (1.7) with $d = p/h$ and $\tilde{h}(\zeta; d)$ by (1.16) with $d = p/h$ and $\alpha_\pm = \beta_\pm$. Then, for $0 < h \ll 1$ small enough, the resonances in D_d of $\hat{H}_h$ is approximated by the solutions to equation $\tilde{h}(\zeta; d) = 1$ in the same sense as in Theorem 1.1.

In his work [15], Martinez has obtained the following result: For any $M \gg 1$, there exists $h_M(E)$ such that $\zeta = E + i\eta$ with $\eta > Mh \log h$ is not a resonance for $0 < h < h_M(E)$, if E is in the nontrapping energy range. Corollary 1.1 asserts that this is not the case for the trapping energy range.

As another application of Theorem 1.2, we study the location of resonances at high frequencies in scattering by two solenoids. We consider the self-adjoint operator

$$\hat{H} = H(\Psi), \qquad \Psi(x) = \alpha_- \Phi(x - p_-) + \alpha_+ \Phi(x - p_+),$$

under the boundary conditions $\lim_{|x-d_\pm|\to 0}|u(x)| < \infty$. We write $R(\zeta^2;\hat{H})$ for the resolvent $(\hat{H}-\zeta^2)^{-1}$. If we define the unitary operator U_3 by $(U_3u)(x) = \lambda u(\lambda x)$ for $\lambda \gg 1$ fixed, then we have the relation

$$U_3^* R(\zeta^2;\hat{H})U_3 = \lambda^{-2}R\big((\zeta/\lambda)^2;\tilde{H}_d\big),$$

where $\tilde{H}_d = H(\Phi_d)$ is defined by (1.15) with $d_\pm = \lambda p_\pm$. We define $\tilde{h}(\zeta;d)$ by (1.16) with ζ and d replaced by $(\zeta/\lambda)^2$ and λp, respectively. Then

$$\tilde{h}\big((\zeta/\lambda)^2;\lambda|p|\big) = \frac{e^{2i\zeta|p|}}{\zeta|p|}(2\pi)^{-1}i(-1)^{\lfloor\alpha_-\rfloor+\lfloor\alpha_+\rfloor}\sin(\alpha_-\pi)\sin(\alpha_+\pi).$$

We also define the neighborhood D as

$$D = \left\{\zeta\in \boldsymbol{C}\colon\ |\operatorname{Re}\zeta-\lambda| < \delta_0\lambda,\ |\operatorname{Im}\zeta| < \frac{1}{2}(1+\delta_0)\frac{\log\lambda}{|p|}\right\}.$$

Then the equation $\tilde{h}((\zeta/\lambda)^2;\lambda|p|) = 1$ has the solutions $\{\zeta_j(\lambda)\}_{1\leqslant j\leqslant N_\lambda}$ with properties

$$\operatorname{Im}\zeta_j(\lambda)\sim -(\log\lambda)/|p|, \qquad \operatorname{Re}\big(\zeta_{j+1}(\lambda)-\zeta_j(\lambda)\big)\sim \pi/|p|$$

in D for $\lambda \gg 1$. Thus we obtain the following corollary from Theorem 1.2. The result is similar to the location of the resonances generated by a corner in the case when a trapping trajectory exists [6].

Corollary 1.2. *Let $\hat{H} = H(\Psi)$ be as above and let $p = p_+ - p_-$. Assume that $\alpha_\pm$ is not an integer. Then, for $\lambda \gg 1$ large enough, the resonances in D of $\hat{H}$ is approximated by the solutions to the equation $\tilde{h}((\zeta/\lambda)^2;\lambda|p|) = 1$ in the same sense as in Theorem 1.1: For any $\varepsilon > 0$ small enough, there exists $\lambda_\varepsilon \gg 1$ such that $\hat{H}$ has the resonances in the neighborhood $\{\zeta\in D\colon\ |\zeta-\zeta_j(\lambda)| < \varepsilon\}$ for $\lambda > \lambda_\varepsilon$.*

We end the section by explaining by heuristic arguments how reasonable (1.11) is as an approximate relation to determine the location of the resonances. We denote by

$$\varphi_0(x;\omega,E) = \exp\big(iE^{1/2}x\cdot\omega\big) \tag{1.19}$$

the plane wave incident from the direction ω at energy $E > 0$, where the notation $\cdot$ denotes the scalar product in $\boldsymbol{R}^2$. We write $x_\pm$ for $x_\pm = x - d_\pm$. The incident plane wave $\varphi_0(x_-;-\hat{d},E)$ takes the form $f_-(-\hat{d}\to\hat{d};E)e^{iE^{1/2}|x_-|}/|x_-|^{1/2}$ after scattered into the direction $\hat{d} = d/|d|$ by the obstacle $\mathcal{O}_{-d}$, and the scattered wave hits the obstacle $\mathcal{O}_{+d}$. Since $|x_-|$ behaves like

$$|x_-| = |x-d_-| = |d+x_+| = |d| + \hat{d}\cdot x_+ + O\big(|d|^{-1}\big)$$

around $\mathcal{O}_{+d}$, the scattered wave behaves like the plane wave

$$\left(e^{iE^{1/2}|d|}/|d|^{1/2}\right) f_-(-\hat{d} \to \hat{d}; E)\varphi_0(x_+; \hat{d}, E)$$

when it arrives at $\mathcal{O}_{+d}$. We repeat a similar argument for the incident plane wave $\varphi_0(x_+; \hat{d}, E)$ after scattered into the direction $-\hat{d}$ by $\mathcal{O}_{+d}$, so that it returns to $\mathcal{O}_{-d}$, taking the approximate form $h(E; d)\varphi_0(x_-; -\hat{d}, E)$. Hence the contribution from the trapping effect between $\mathcal{O}_{-d}$ and $\mathcal{O}_{+d}$ is described by the series

$$\left(\sum_{n=1}^{\infty} h(E; d)^n\right)\varphi_0(x_-; -\hat{d}, E).$$

For example, the term with $h(E; d)^n$ describes the contribution from the trajectory oscillating n times. Thus the location of the resonances is approximately determined by relation (1.11).

2. Aharonov–Bohm Hamiltonian

We make a full use of the information about the spectral theory for the Hamiltonian with one solenoid in order to prove Theorem 1.1. The scattering by one solenoid is known as one of the exactly solvable models in quantum mechanics. In this section we give a brief review of it. We refer to [1,2,4,9,16] for more detailed expositions.

We consider the self-adjoint operator $P = H(\alpha\Phi)$ defined by (1.12) with domain (1.13). We calculate the generalized eigenfunction of the eigenvalue problem

$$P\varphi = E\varphi, \qquad \lim_{|x|\to 0}\left|\varphi(x)\right| < \infty, \tag{2.1}$$

with energy $E > 0$ as an eigenvalue. Since P is rotationally invariant, we work in the polar coordinate system (r, θ). Let U be the unitary mapping defined by

$$(Uu)(r, \theta) = r^{1/2}u(r\theta) : L^2 \to L^2\left((0, \infty); dr\right) \otimes L^2\left(S^1\right).$$

We write $\sum_l$ for the summation ranging over all integers l. Then U allows us to decompose P into the partial wave expansion

$$P \simeq UPU^* = \sum_l \oplus(P_l \otimes Id), \tag{2.2}$$

where $P_l = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}$ with $\nu = |l - \alpha|$ is self-adjoint in $L^2((0, \infty); dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2}|u(r)| < \infty$ at $r = 0$. Let $\varphi_0(x; \omega, E)$ be the plane wave defined by (1.19). We denote by $\gamma(x; \omega)$ the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. Then the outgoing eigenfunction $\varphi_{0+}(x; \omega, E)$ with ω as an incident direction is calculated as

$$\varphi_{0+}(x;\omega,E)=\sum_l \exp(-i\nu\pi/2)\exp\bigl(il\gamma(x;-\omega)\bigr)J_\nu\bigl(E^{1/2}|x|\bigr) \tag{2.3}$$

with $\nu=|l-\alpha|$, where $J_\mu(z)$ denotes the Bessel function of order μ. The eigenfunction φ_{0+} behaves like $\varphi_{0+}(x;\omega,E)\sim\varphi_0(x;\omega,E)$ as $|x|\to\infty$ in the direction $-\omega$ ($x=-|x|\omega$), and the difference $\varphi_{0+}-\varphi_0$ satisfies the outgoing radiation condition at infinity. On the other hand, the incoming eigenfunction $\varphi_{0-}(x;\omega,E)$ is given by

$$\varphi_{0-}(x;\omega,E)=\sum_l \exp(i\nu\pi/2)\exp\bigl(il\gamma(x;\omega)\bigr)J_\nu\bigl(E^{1/2}|x|\bigr), \tag{2.4}$$

which behaves like $\varphi_{0-}\sim\varphi_0(x;\omega,E)$ as $|x|\to\infty$ in the direction ω. The eigenfunctions $\varphi_{0\pm}(x;\omega,E)$ admit the analytic extension

$$\varphi_{0\pm}(x;\omega,\zeta)=\sum_l \exp(\mp i\nu\pi/2)\exp\bigl(il\gamma(x;\mp\omega)\bigr)J_\nu\bigl(k|x|\bigr),\quad k=\zeta^{1/2},$$

over the complex plane.

Remark 2. By definition (2.4), we have

$$\bar\varphi_{0-}(x;\omega,\bar\zeta)=\sum_l \exp(-i\nu\pi/2)\exp\bigl(-il\gamma(x;\omega)\bigr)J_\nu\bigl(\zeta^{1/2}|x|\bigr),$$

and hence we see that $\bar\varphi_{0-}(x;\omega,\bar\zeta)$ is analytic in ζ.

We decompose $\varphi_{0+}(x;\omega,E)$ into the sum $\varphi_{0+}=\varphi_{\mathrm{inc}}+\varphi_{\mathrm{sc}}$ of incident and scattering waves to calculate the scattering amplitude through the asymptotic behavior at infinity of the scattering wave $\varphi_{\mathrm{sc}}(x;\omega,E)$. If we set $\sigma=\sigma(x;\omega)=\gamma(x;\omega)-\pi$, then

$$\varphi_{0+}=\sum_l e^{-i\nu\pi/2}e^{il\sigma}J_\nu\bigl(E^{1/2}|x|\bigr),\quad \nu=|l-\alpha|.$$

If we further make use of the formula $e^{-i\mu\pi/2}J_\mu(iw)=I_\mu(w)$ for the Bessel function

$$I_\mu(w)=(1/\pi)\left(\int_0^\pi e^{w\cos\rho}\cos(\mu\rho)\,d\rho-\sin(\mu\pi)\int_0^\infty e^{-w\cosh p-\mu p}\,dp\right) \tag{2.5}$$

with $\operatorname{Re} w\geqslant 0$ (see [21, p. 181]), then $\varphi_{0+}(x;\omega,E)$ takes the form

$$\begin{aligned}\varphi_{0+}={}&\left(\frac{1}{\pi}\right)\sum_l e^{il\sigma}\int_0^\pi e^{-iE^{1/2}|x|\cos\rho}\cos(\nu\rho)\,d\rho\\&-\left(\frac{1}{\pi}\right)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{iE^{1/2}|x|\cosh p}e^{-\nu p}\,dp.\end{aligned} \tag{2.6}$$

We take the incident wave $\varphi_{\mathrm{inc}}(x;\omega,E)$ as

$$\varphi_{\mathrm{inc}} = e^{i\alpha\sigma}\varphi_0(x;\omega,E) = e^{i\alpha\sigma}e^{iE^{1/2}|x|\cos\gamma(x;\omega)} = e^{i\alpha\sigma}e^{-iE^{1/2}|x|\cos\sigma},$$

which is different from the usual plane wave $\varphi_0(x;\omega,E)$. Since the vector potential $\alpha\Phi(x)$ satisfies $\alpha\Phi(x) = \alpha\nabla\gamma(x;\omega)$ by (1.18), the modified factor $e^{i\alpha\sigma}$ may be interpreted as the change of the phase factor

$$\alpha\int_{l_x}\Phi(y)\cdot dy = \alpha\int_{-\infty}^{0}(d/ds)\gamma(x+s\omega;\omega)\,ds = \alpha\big(\gamma(x;\omega)-\pi\big) = \alpha\sigma(x;\omega)$$

which the vector potential $\alpha\Phi$ causes to the wave function of the quantum particle moving in the direction ω by the Aharonov–Bohm effect, where $l_x = \{y = x + s\omega\colon\ s \leqslant 0\}$.

The incident wave admits the Fourier expansion

$$\varphi_{\mathrm{inc}}(x;\omega,E) = \left(\frac{1}{\pi}\right)\sum_l\left(\int_0^{\pi} e^{-iE^{1/2}|x|\cos\rho}\cos(\nu\rho)\,d\rho\right)e^{il\sigma(x;\omega)}$$

for $|\sigma| < \pi$. This, together with (2.6), yields

$$\varphi_{\mathrm{sc}}(x;\omega,E) = -\left(\frac{1}{\pi}\right)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^{\infty} e^{iE^{1/2}|x|\cosh p}e^{-\nu p}\,dp.$$

We compute the series

$$\begin{aligned}\sum_l e^{il\sigma}e^{-\nu p}\sin(\nu\pi) &= \left\{\sum_{l\leqslant\lfloor\alpha\rfloor} + \sum_{l\geqslant\lfloor\alpha\rfloor+1}\right\}e^{il\sigma}e^{-\nu p}\sin(\nu\pi)\\ &= \sin(\alpha\pi)(-1)^{\lfloor\alpha\rfloor}\left\{\frac{e^{-\alpha p}(e^{i\sigma}e^{p})^{\lfloor\alpha\rfloor}}{1+e^{-i\sigma}e^{-p}} + \frac{e^{\alpha p}(e^{i\sigma}e^{-p})^{\lfloor\alpha\rfloor}}{1+e^{-i\sigma}e^{p}}\right\}\end{aligned}$$

for $|\sigma| < \pi$. Thus we have

$$\varphi_{\mathrm{sc}} = -\frac{\sin(\alpha\pi)}{\pi}(-1)^{\lfloor\alpha\rfloor}e^{i\lfloor\alpha\rfloor\sigma(x;\omega)}\int_{-\infty}^{\infty} e^{iE^{1/2}|x|\cosh p}\frac{e^{-\beta p}}{1+e^{-i\sigma}e^{-p}}\,dp$$

with $\beta = \alpha - \lfloor\alpha\rfloor$. We apply the stationary phase method to the integral on the right side. Since

$$e^{i\sigma(x;\omega)} = e^{i(\gamma(x;\omega)-\pi)} = -e^{i(\theta-\omega)}$$

by identifying $\theta = x/|x| = \hat{x} \in S^1$ with the azimuth angle θ, $\varphi_{\mathrm{sc}}(x;\omega,E)$ satisfies

$$\varphi_{\rm sc} = f(\omega \to \hat{x}; E) e^{iE^{1/2}|x|}|x|^{-1/2} + o\big(|x|^{-1/2}\big), \quad |x| \to \infty,$$

where $f(\omega \to \theta; E)$, $\theta \neq \omega$, is given by (1.14) with $\omega, \theta \in S^1$ identified with the azimuth angles from the positive x_1 axis. The quantity $f(\omega \to \theta; E)$ is called the amplitude for scattering from the initial direction $\omega \in S^1$ to the final one θ at energy $E > 0$. By definition, the amplitude admits the analytic extension $f(\omega \to \theta; \zeta)$ over the complex plane.

We calculate the resolvent kernel of $R(\zeta; P)$ with $\operatorname{Im} \zeta > 0$. Let P_l be as in (2.2) and let $k = \zeta^{1/2}$ with $\operatorname{Im} k > 0$. Then the equation $(P_l - \zeta)u = 0$ has $\{r^{1/2} J_\nu(kr), r^{1/2} H_\nu(kr)\}$ with Wronskian $2i/\pi$ as a pair of linearly independent solutions, where $H_\mu(z) = H_\mu^{(1)}(z)$ denotes the Hankel function of the first kind. Thus $(P_l - \zeta)^{-1}$ has the integral kernel

$$R(\zeta; P_l)(r, \rho) = (i\pi/2) r^{1/2} \rho^{1/2} J_\nu\big(k(r \wedge \rho)\big) H_\nu\big(k(r \vee \rho)\big), \quad \nu = |l - \alpha|,$$

where $r \wedge \rho = \min(r, \rho)$ and $r \vee \rho = \max(r, \rho)$. Hence the resolvent kernel of P is given by

$$R(\zeta; P)(x, y) = (i/4) \sum_l e^{il(\theta - \omega)} J_\nu\big(k\big(|x| \wedge |y|\big)\big) H_\nu\big(k\big(|x| \vee |y|\big)\big), \tag{2.7}$$

where $x = (|x| \cos\theta, |x| \sin\theta)$ and $y = (|y| \cos\omega, |y| \sin\omega)$ in the polar coordinates. This makes sense even for ζ in the lower half plane of the complex plane by analytic continuation. Then $R(\zeta; P)$ with $\operatorname{Im} \zeta \leqslant 0$ is well defined as an operator from $L^2_{\rm comp}(\boldsymbol{R}^2)$ to $L^2_{\rm loc}(\boldsymbol{R}^2)$. Thus $R(\zeta; P)$ does not have any poles as a function with values in operators from $L^2_{\rm comp}(\boldsymbol{R}^2)$ to $L^2_{\rm loc}(\boldsymbol{R}^2)$. We can say that P with one solenoid $2\pi\alpha\delta(x)$ has no resonances. Here we do not discuss the possibility of resonances at zero energy.

The next proposition is obtained as a particular case of Proposition 2.3 of [20], if we have only to apply the proposition with $\lambda = |x|$ (see also Proposition 6.2 in Section 6).

Proposition 2.1. *Let $\varphi_{0-}(x; \omega, E)$ be the incoming eigenfunction of P. If $E > 0$ and y fulfills $1/c < |y| < c$ for some $c > 1$, then*

$$R(E; P)(x, y) = c_0(E) e^{iE^{1/2}|x|}|x|^{-1/2}\big(\bar{\varphi}_{0-}(y; \hat{x}, E) + e(x, y; E)\big), \quad |x| \to \infty,$$

where

$$c_0(E) = (8\pi)^{-1/2} e^{i\pi/4} E^{-1/4} \tag{2.8}$$

and $e(x, y; E)$ obeys $|\partial_x^n \partial_y^m e| = O(|x|^{-1-|n|})$ uniformly in y.

3. Magnetic Schrödinger operators in exterior domains

In this section we consider the Schrödinger operator $H = H(\alpha\Phi)$ over the domain $\Omega = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}$ exterior to a simply connected bounded obstacle $\mathcal{O}$, where $\mathcal{O}$ is assumed to contain the origin as an interior point and to have a smooth boundary. For brevity, we also assume that $\mathcal{O} \subset \{|x| < 1\}$. Let

$$H = H(\alpha\Phi), \qquad \mathcal{D}(H) = H^2(\Omega) \cap H_0^1(\Omega), \tag{3.1}$$

be the self-adjoint realization in $L^2(\Omega)$. We denote by $f_0(\omega \to \theta; E)$ the scattering amplitude at energy E for the pair (H_0, H), where $H_0 = -\Delta$ is the free Hamiltonian acting on $L^2(\boldsymbol{R}^2)$. The aim of the present section is to establish the relations between the resolvents $R(\zeta; H)$ and $R(\zeta; P)$ and between the scattering amplitudes $f_0(\omega \to \theta; E)$ and $f(\omega \to \theta; E)$. Here we introduce a smooth non-negative cut-off function $\chi_0 \in C_0^\infty[0,\infty)$ with the properties

$$0 \leqslant \chi_0 \leqslant 1, \qquad \operatorname{supp} \chi_0 \subset [0,2], \qquad \chi_0 = 1 \quad \text{on } [0,1]. \tag{3.2}$$

This function is often used in the future discussion without further references. We also use the notation $(\ ,\ )$ to denote the L^2 scalar product in $L^2(\boldsymbol{R}^2)$ or $L^2(\Omega)$.

Lemma 3.1. *Let $H = H(\alpha\Phi)$ be defined by (3.1) and let $E > 0$ be fixed. Then there exists a complex neighborhood of E where the resolvent $R(\zeta; H)$ is analytic as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$.*

Proof. The proof is based on the complex scaling method developed by [7,19], and the lemma follows as a particular case of such a general theory. The operator H is a long-range perturbation to the free Hamiltonian H_0, but the coefficients of H are analytic in Ω. The operator P defined by (1.12) is transformed into $e^{-2\theta}P$ under the group of dilations $x \to e^\theta x$. By assumption, $\mathcal{O} \subset \{|x| < 1\}$. Let $\Sigma = \{|x| < 8\}$. Since H has no positive eigenvalues, we can show by making use of the analytic dilation which leaves Σ invariant that there exists a complex neighborhood of E in which the operator

$$j_0 R(\zeta; H) : L^2_{\mathrm{comp}}(\Sigma_0) \to L^2_{\mathrm{comp}}(\Sigma_0), \quad \Sigma_0 = \Sigma \setminus \mathcal{O},$$

is analytic as a function with values in bounded operators, where the multiplication operator by the characteristic function j_0 of Σ_0 is considered to be the restriction to $L^2_{\mathrm{comp}}(\Sigma_0)$ from $L^2_{\mathrm{loc}}(\Omega)$. We assert that $R(\zeta; H)$ is analytic over the neighborhood above as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. To see this, we set

$$u_0(x) = \chi_0\bigl(|x|/2\bigr), \qquad u_1(x) = \chi_0\bigl(|x|/4\bigr), \qquad v_0 = 1 - u_0, \qquad v_1 = 1 - u_1$$

for the cut-off function $\chi_0 \in C_0^\infty[0,\infty)$ with properties (3.2). Recall that

$$R(\zeta; P) : L^2_{\mathrm{comp}}\big(\boldsymbol{R}^2\big) \to L^2_{\mathrm{loc}}\big(\boldsymbol{R}^2\big)$$

depends analytically on ζ. If we regard the multiplication operator $f \mapsto v_1 f$ as the extension from $L^2(\Omega)$ to $L^2(\boldsymbol{R}^2)$, then $R(\zeta; P)v_1$ makes sense as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$, and similarly for $R(\zeta; P)v_0$. Since $v_0 v_1 = v_1$ and since $H = P$ over Ω, $R(\zeta; H) = R(\zeta; H)(u_1 + v_1)$ is decomposed into the sum of three terms

$$R(\zeta; H) = R(\zeta; H)u_1 + v_0 R(\zeta; P)v_1 - R(\zeta; H)[P, v_0]R(\zeta; P)v_1$$

at least for ζ with $\operatorname{Im}\zeta > 0$, where $[X, Y] = XY - YX$ denotes the commutator between two operators X and Y. The coefficients of $[P, v_0]$ have supports in Σ_0. Hence we see that

$$j_0 R(\zeta; H) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{comp}}(\Sigma_0)$$

depends analytically on ζ in the complex neighborhood of E. Similarly we obtain the relation

$$R(\zeta; H) = u_1 R(\zeta; H) + v_1 R(\zeta; P)v_0 + v_1 R(\zeta; P)[P, v_0]R(\zeta; H)$$

on $L^2_{\mathrm{comp}}(\Omega)$. This yields the analytic dependence on ζ of $R(\zeta; H) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$ and the proof is complete. □

We now define the scattering amplitude $f_0(\omega \to \theta; E)$, $\theta \neq \omega$, for the pair (H_0, H) with H defined by (3.1). Let $\varphi_+(x; \omega, E)$ be the outgoing eigenfunction of H. Then the amplitude is defined through the asymptotic form

$$\varphi_+ = e^{i\alpha(\gamma(x;\omega)-\pi)}\varphi_0(x; \omega, E) + f_0(\omega \to \theta; E)e^{iE^{1/2}|x|}|x|^{-1/2} + o\big(|x|^{-1/2}\big)$$

as $|x| \to \infty$ in the direction θ ($x = |x|\theta$).

Lemma 3.2. *Let $\varphi_{0+}(x; \omega, E)$ and $\varphi_{0-}(x; \theta, E)$ denote the outgoing and incoming eigenfunctions of $P = H(\alpha\Phi)$, respectively, and let $f(\omega \to \theta; E)$ be the scattering amplitude for the pair (H_0, P). Set $u_0 = \chi_0(|x|/2)$ and $u_1 = \chi_0(|x|/4)$. Then the amplitude $f_0(\omega \to \theta; E)$, $\theta \neq \omega$, for the pair (H_0, H) has the representation*

$$f_0 = f(\omega \to \theta; E) + c_0(E)\big(R(E; H)[P, u_0]\varphi_{0+}(\cdot; \omega, E), [P, u_1]\varphi_{0-}(\cdot; \theta, E)\big),$$

where $c_0(E)$ is defined by (2.8).

Remark 3. This lemma, together with Lemma 3.1, enables us to extend $f_0(\omega \to \theta; E)$ analytically in a complex neighborhood of E, and its extension $f_0(\omega \to \theta; \zeta)$ is given by

$$f_0 = f(\omega \to \theta; \zeta) + c_0(\zeta)\big(R(\zeta; H)[P, u_0]\varphi_{0+}(\cdot; \omega, \zeta), [P, u_1]\varphi_{0-}(\cdot; \theta, \bar{\zeta})\big).$$

Proof. Let $\varphi_+(x;\omega,E)$ be as above. Since $\mathcal{O} \subset \{|x| < 1\}$ by assumption, $P = H = H(\alpha\Phi)$ outside the support of u_0, and hence we have

$$\varphi_+ = (1-u_0)\varphi_{0+} + R(E;H)[P,u_0]\varphi_{0+}. \tag{3.3}$$

Similarly

$$\varphi_{0+} = (1-u_1)\varphi_+ + R(E;P)[P,u_1]\varphi_+,$$

and hence

$$\varphi_+ = \varphi_{0+} + u_1\varphi_+ - R(E;P)[P,u_1]\varphi_+. \tag{3.4}$$

It follows from Proposition 2.1 that the last term on the right side of (3.4) behaves like

$$c_0(E)\big(\varphi_+(\cdot;\omega,E),[P,u_1]\varphi_{0-}(\cdot;\theta,E)\big)|x|^{-1/2}e^{iE^{1/2}|x|} + o\big(|x|^{-1/2}\big)$$

as $|x| \to \infty$ in the direction θ. We insert (3.3) into φ_+ on the right side. Since

$$\big((1-u_0)\varphi_{0+},[u_1,P]\varphi_{0-}\big) = \big(\varphi_{0+},[u_1,P]\varphi_{0-}\big) = 0,$$

we obtain the desired relation. □

Proposition 3.1. *Let $E > 0$ and $c > 0$ be fixed. Set again $u_0 = \chi_0(|x|/2)$ and $u_1 = \chi_0(|x|/4)$. Assume that ζ satisfies $|\operatorname{Re}\zeta - E| < E/2$ and $|\operatorname{Im}\zeta| \leqslant c(\log|d|)/|d|$ for $|d| \gg 1$. If p and q fulfill $|p|,|q| \geqslant c_1|d|$ for some $c_1 > 0$, then*

$$w_pR(\zeta;H)w_q = w_pR(\zeta;P)w_q + w_pR(\zeta;P)[u_1,P]R(\zeta;H)[P,u_0]R(\zeta;P)w_q,$$

where $w_p(x) = \chi_0(|x-p|)$ and $w_q(x) = \chi_0(|x-q|)$.

Proof. We set $v_0 = 1-u_0$ and $v_1 = 1-u_1$. Then $w_pv_0 = w_p$ and $w_pv_1 = w_p$, and similarly for $w_q = \chi_0(|x-q|)$. The operator H coincides with P on the support of v_1. We compute

$$\begin{aligned} w_pR(\zeta;H)w_q &= w_pR(\zeta;P)w_q + w_pR(\zeta;P)(Pv_1 - v_1H)R(\zeta;H)w_q \\ &= w_pR(\zeta;P)w_q + w_pR(\zeta;P)[u_1,P]R(\zeta;H)w_q. \end{aligned}$$

Since $v_0 = 1$ on the support of ∇u_1 and since $H = P$ on the support of v_0, we repeat the above argument to get

$$[u_1,P]R(\zeta;H)w_q = [u_1,P]\big(R(\zeta;P) + R(\zeta;H)[P,u_0]R(\zeta;P)\big)w_q.$$

Note that

$$w_p R(\zeta; P)[u_1, P]R(\zeta; P)w_q = w_p R(\zeta; P)u_1 w_q - w_p u_1 R(\zeta; P)w_q = 0$$

and hence the desired relation is obtained. □

4. Proof of main theorem

In this section we complete the proof of Theorem 1.1 by reducing it to the proof of Theorem 4.1 formulated below. For notational brevity, we fix the coordinates of the two centers $d_\pm$ of the obstacles $\mathcal{O}_{\pm d}$ as

$$d_- = (-d/2, 0), \qquad d_+ = (d/2, 0),$$

so that $|d| = |d_+ - d_-| = d$ and $\hat{d} = \omega_1 = (1, 0)$. The function $h(\zeta; d)$ defined by (1.8) takes the form

$$h(\zeta; d) = \frac{e^{2ikd}}{d} f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta). \tag{4.1}$$

The next lemma is proved in almost the same way as Lemma 4.6 in [20]. We skip its proof.

Lemma 4.1. *Let $h(\zeta; d)$ be as above and let D_d be defined by (1.7) with $|d|$ replaced by d. Then the equation $h(\zeta; d) = 1$ has a finite number of the solutions*

$$\{\zeta_j(d)\}_{j=1}^{N_d} \subset D_d, \qquad \operatorname{Re} \zeta_1(d) < \cdots < \operatorname{Re} \zeta_{N_d}(d),$$

and each solution $\zeta_j(d)$ has the properties

$$\bigl|\operatorname{Im} \zeta_j(d) + E_0^{1/2}(\log d)/d\bigr| < \delta_0 E_0^{1/2}(\log d)/d,$$
$$\bigl|\operatorname{Re}\bigl(\zeta_{j+1}(d) - \zeta_j(d)\bigr) - 2\pi E_0^{1/2}/d\bigr| < 2\pi\delta_0 E_0^{1/2}/d$$

for $d \gg 1$.

We define the region S_0 by

$$S_0 = \bigl\{x = (x_1, x_2) \in \Omega_d\colon\ |x_1| \leqslant d,\ |x_2| \leqslant L_0\bigr\}, \quad L_0 \gg 1, \tag{4.2}$$

and denote by s_0 the characteristic function of S_0. We may assume that the two obstacles $\mathcal{O}_{\pm d}$ are included in the strip $\{x\colon\ |x_2| < L_0\}$. We further define

$$R_{0d}(\zeta) = s_0 R(\zeta; H_d)s_0 : L^2(S_0) \to L^2(S_0) \tag{4.3}$$

for $\zeta \in D_d$ with $\operatorname{Im} \zeta > 0$, where we note that the multiplication by the characteristic function s_0 has been used in two senses of the restriction $(L^2(\Omega_d) \to L^2(S_0))$ and the extension $(L^2(S_0) \to L^2(\Omega_d))$. For brevity, we often use multiplications by characteristic functions in these two different senses. We are now in a position to prove Theorem 1.1, accepting the theorem below as proved.

Theorem 4.1. *Let the notation be as above. Assume that the same assumptions as in Theorem 1.1 are fulfilled. Then we can take $\delta_0 > 0$ so small that the neighborhood D_d defined by (1.7) has the following property: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $|d| > d_\varepsilon$, $R_{0d}(\zeta)$ admits the meromorphic extension over D_d as a function with values in bounded operators over $L^2(S_0)$, and it has the poles*

$$\{\zeta_{\mathrm{res},j}(d)\}_{j=1}^{N_d} \subset D_d, \qquad \mathrm{Re}\,\zeta_{\mathrm{res},1}(d) < \cdots < \mathrm{Re}\,\zeta_{\mathrm{res},N_d}(d),$$

in the neighborhood $\{\zeta \in \boldsymbol{C}\colon\ |\zeta - \zeta_j(d)| < \varepsilon/|d|\}$. Moreover, $R_{0d}(\zeta)$ is analytic over

$$\tilde{D}_d = D_d \setminus \{\zeta_{\mathrm{res},1}(d), \zeta_{\mathrm{res},2}(d), \ldots, \zeta_{\mathrm{res},N_d}(d)\}.$$

Proof of Theorem 1.1. Let $\{\zeta_{\mathrm{res},j}(d)\}$ be as in Theorem 4.1. We claim that all the resonances in D_d of H_d coincide with $\{\zeta_{\mathrm{res},j}(d)\}$. To prove it, it suffices to show that $R(\zeta; H_d) : L^2_{\mathrm{comp}}(\Omega_d) \to L^2_{\mathrm{loc}}(\Omega_d)$ is analytic over $\tilde{D}_d$. To see this, we introduce the auxiliary operator $P_0 = H(\alpha_0\Phi)$ with $\alpha_0 = \alpha_- + \alpha_+$, and use the same notation P_0 to denote the self-adjoint extension (Friedrichs extension) realized by imposing the boundary condition $\lim_{|x|\to 0} |u(x)| < \infty$ at the origin. By the Stokes formula, the line integral

$$\int_C \big(\Phi_d(x) - \alpha_0\Phi(x)\big) \cdot dx = 0$$

along any curve C outside S_0. This makes it possible to construct a smooth real function $g(x)$ in such a way that

$$\Phi_d(x) = \alpha_0\Phi(x) + \nabla g(x) \tag{4.4}$$

outside S_0, where $\Phi_d(x)$ is defined by (1.4). In fact, it is given by the line integral

$$g(x) = -\int_1^\infty \big(\big(\Phi_d(tx) - \alpha_0\Phi(tx)\big) \cdot \hat{x}\big)\, dt, \quad \hat{x} = x/|x|,$$

for $|x| \gg 1$ and obeys $g(x) = O(|x|^{-1})$ as $|x| \to \infty$. Let $\{\psi_0, \psi_1\}$ be a smooth partition of unity over $\boldsymbol{R}^2$ such that

$$\psi_0 + \psi_1 = 1, \qquad \mathrm{supp}\,\psi_0 \subset S_0,$$

and let ψ_2 be a smooth function such that it has a slightly wider support than ψ_1 and satisfies $\psi_2\psi_1 = \psi_1$. We may assume that (4.4) remains true on $\mathrm{supp}\,\psi_2$ (and hence on

$\operatorname{supp}\psi_1$ also). If we define $\tilde{P}_0 = e^{ig}P_0e^{-ig}$, then it follows that $H_d = \tilde{P}_0$ on $\operatorname{supp}\psi_2$. This relation enables us to decompose $R(\zeta; H_d) = R(\zeta; H_d)(\psi_0 + \psi_1)$ into the sum of three terms as follows:

$$R(\zeta; H_d) = R(\zeta; H_d)\psi_0 + \psi_2 R(\zeta; \tilde{P}_0)\psi_1 - R(\zeta; H_d)[\tilde{P}_0, \psi_2]R(\zeta; \tilde{P}_0)\psi_1$$

at least for $\zeta \in D_d$ with $\operatorname{Im}\zeta > 0$. Since

$$R(\zeta; \tilde{P}_0) : L^2_{\mathrm{comp}}(\boldsymbol{R}^2) \to L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$$

depends analytically on ζ and since the commutator $[\tilde{P}_0, \psi_2]$ vanishes outside S_0, we see from Theorem 4.1 that

$$s_0 R(\zeta; H_d) : L^2_{\mathrm{comp}}(\Omega_d) \to L^2(S_0)$$

depends analytically on $\zeta \in \tilde{D}_d$. Similarly we obtain the relation

$$R(\zeta; H_d) = \psi_0 R(\zeta; H_d) + \psi_1 R(\zeta; \tilde{P}_0)\psi_2 + \psi_1 R(\zeta; \tilde{P}_0)[\tilde{P}_0, \psi_2]R(\zeta; H_d)$$

on $L^2_{\mathrm{comp}}(\Omega_d)$. This implies the desired analyticity in $\zeta \in \tilde{D}_d$ of $R(\zeta; H_d)$, and the proof of the theorem is complete. □

5. Strategy by the complex scaling method

The remaining sections are devoted to proving Theorem 4.1. It is proved by constructing the resolvent kernel $R(\zeta; H_d)(x, y)$ of $R(\zeta; H_d)$ with the spectral parameter ζ in the lower half plane. To do this, we make use of the complex scaling method to combine the Green kernel constructed for each obstacle $\mathcal{O}_\pm$, as stated in Section 1. In this section, we introduce some new notation and explain the strategy based on this method. Besides the cut-off function $\chi_0 \in C_0^\infty([0, \infty))$ with properties in (3.2), we introduce smooth cut-off functions χ_∞ and $\chi_\pm$ over the real line with the following properties: $0 \leqslant \chi_\infty, \chi_\pm \leqslant 1$ and

$$\chi_\infty(t) = 1 - \chi_0(|t|),$$
$$\chi_+(t) = 1 \quad \text{for } t \geqslant 1, \qquad \chi_+(t) = 0 \quad \text{for } t \leqslant -1, \qquad \chi_-(t) = 1 - \chi_+(t).$$

We may assume that $\chi_\infty(t)$ is increasing in $t \geqslant 0$ and decreasing in $t \leqslant 0$. We often use these functions without further references throughout the future discussion.

We define $j_d(x) : \boldsymbol{R}^2 \to \boldsymbol{R} \times \boldsymbol{C}$ by

$$j_d(x_1, x_2) = \big(x_1, x_2 + i\eta_d(x_2)x_2\big), \qquad \eta_d(t) = L_1\big((\log d)/d\big)\chi_\infty(t/d), \tag{5.1}$$

where $L_1 \gg 1$ is fixed large enough, and we consider the complex scaling mapping

$$(J_d f)(x) = \left[\det(\partial j_d/\partial x)\right]^{1/2} f\left(j_d(x)\right) \tag{5.2}$$

associated with $j_d(x)$. The Jacobian $\det(\partial j_d/\partial x)$ of $j_d(x)$ does not vanish for $d \gg 1$, and it is easily seen that J_d is a one-to-one mapping. Since the coefficients of H_d are analytic over Ω_d, we can define the operator

$$K_d = J_d H_d J_d^{-1}. \tag{5.3}$$

This becomes a closed operator in $L^2(\Omega_d)$ with the same domain as H_d, but it is not necessarily self-adjoint. We do not require the explicit form of K_d in the future discussion. We construct the resolvent kernel $R(\zeta; K_d)(x, y)$ of $R(\zeta; K_d)$ with $\zeta \in D_d$ without constructing directly the kernel $R(\zeta; H_d)(x, y)$. The mapping j_d acts as the identity mapping on the region S_0 defined by (4.2), so that

$$R(\zeta; H_d)(x, y) = R(\zeta; H_d)\left(j_d(x), j_d(y)\right) = R(\zeta; K_d)(x, y)$$

for $(x, y) \in S_0 \times S_0$. Thus the necessary information can be obtained through the kernel $R(\zeta; K_d)(x, y)$.

We introduce the auxiliary operators

$$H_{\pm d} = H(\Phi_{\pm d}), \qquad \mathcal{D}(H_{\pm d}) = H^2(\Omega_{\pm d}) \cap H_0^1(\Omega_{\pm d}), \tag{5.4}$$

where $\Phi_{\pm d}(x)$ is defined in (1.4), and $\Omega_{\pm d} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm d}$. We also define the complex scaled operator as in (5.3) for these auxiliary operators $H_{\pm d}$. Recall that $\gamma(x; \omega)$ denotes the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. The potential $\Phi(x)$ defined by (1.3) satisfies the relation $\Phi(x) = \nabla\gamma(x; \omega)$. Hence it follows that

$$\Phi_{\pm d}(x) = \alpha_\pm \nabla\gamma(x - d_\pm; \pm\omega_1), \quad \omega_1 = (1, 0).$$

The angle function $\gamma(x; \omega_1)$ is represented as

$$\gamma(x; \omega_1) = -\frac{i}{2}\left(\log(x_1 + ix_2) - \log(x_1 - ix_2)\right) + \pi,$$

so that it is well defined also for complex variables. We take $\arg z$, $0 \leqslant \arg z < 2\pi$, to be a single valued function over the complex plane slit along the direction ω_1 and define

$$\gamma\left(j_d(x); \omega_1\right) = \frac{1}{2}\left(\arg\left(b_{+d}(x)\right) - \arg\left(b_{-d}(x)\right)\right) + \pi - \frac{i}{2}\log\left|b_d(x)\right|,$$

where

$$b_{+d}(x) = x_1 - \eta_d(x_2)x_2 + ix_2, \qquad b_{-d}(x) = x_1 + \eta_d(x_2)x_2 - ix_2,$$

and $b_d(x) = b_{+d}(x)/b_{-d}(x)$. The function $\gamma(j_d(x); -\omega_1)$ is similarly defined by taking $\arg z$ to be a single valued function over the complex plane slit along the direction $-\omega_1$.

We define $g_{\pm d}(x)$ by

$$g_{\pm d}(x) = \alpha_\pm \chi_\mp\big((32x_1/d) \mp 13\big)\gamma\big(j_d(x) - d_\pm; \pm\omega_1\big) \tag{5.5}$$

and $g_{0d}(x)$ by

$$g_{0d}(x) = \chi_0\left(\frac{4|x_1|}{d}\right)\big(\alpha_-\gamma\big(j_d(x) - d_-; -\omega_1\big) + \alpha_+\gamma\big(j_d(x) - d_+; \omega_1\big)\big). \tag{5.6}$$

By definition, $\operatorname{supp} g_{-d} \subset \{x\colon\ x_1 > -7d/16\}$ and

$$g_{-d}(x) = \alpha_-\gamma\big(j_d(x) - d_-; -\omega_1\big) \quad \text{on } \Sigma_+ = \{x\colon\ x_1 > -3d/8\}.$$

Hence $\exp(ig_{-d})$ acts as

$$\exp(ig_{-d})f(x) = \big(J_d \exp\big(i\alpha_-\gamma(x - d_-; -\omega_1)\big)J_d^{-1}f\big)(x)$$

on functions $f(x)$ with support in Σ_+. On the other hand, $g_{+d}(x)$ has support in $\{x\colon\ x_1 < 7d/16\}$ and

$$g_{+d}(x) = \alpha_+\gamma\big(j_d(x) - d_+; \omega_+\big) \quad \text{on } \Sigma_- = \{x\colon\ x_1 < 3d/8\},$$

so that $\exp(ig_{+d})$ acts as

$$\exp(ig_{+d})f(x) = \big(J_d \exp\big(i\alpha_+\gamma(x - d_+; \omega_1)\big)J_d^{-1}f\big)(x)$$

on functions $f(x)$ with support in Σ_-. We take these relations into account to define the following complex scaled operator

$$K_{\pm d} = \exp(ig_{\mp d})\big(J_d H_{\pm d} J_d^{-1}\big)\exp(-ig_{\mp d}) \tag{5.7}$$

for $H_{\pm d}$ defined by (5.4), where $K_{\pm d}$ has the same domain as $H_{\pm d}$. Since

$$K_{+d} = J_d H\big(\alpha_-\nabla\gamma(x - d_-; -\omega_1) + \Phi_{+d}\big)J_d^{-1}$$

on Σ_+, we have

$$K_{+d} = K_d \quad \text{on } \Sigma_+ = \{x\colon\ x_1 > -3d/8\}. \tag{5.8}$$

Similarly we have

$$K_{-d} = K_d \quad \text{on } \Sigma_- = \{x\colon\ x_1 < 3d/8\}. \tag{5.9}$$

The function $g_{0d}(x)$ defined by (5.6) has support in $\{x\colon |x_1| < d/2\}$ and satisfies

$$g_{0d} = \alpha_- \gamma\bigl(j_d(x) - d_-; -\omega_1\bigr) + \alpha_+ \gamma\bigl(j_d(x) - d_+; \omega_1\bigr)$$

on $\Sigma_0 = \{x\colon |x_1| \leqslant d/4\}$. If we define the operator K_{0d} by

$$K_{0d} = \exp(ig_{0d})\bigl(J_d H_0 J_d^{-1}\bigr)\exp(-ig_{0d}) \tag{5.10}$$

as a closed operator with domain $\mathcal{D}(K_{0d}) = H^2(\boldsymbol{R}^2)$, then we obtain

$$K_{0d} = K_{\pm d} = K_d \quad \text{on } \Sigma_0 = \bigl\{x\colon |x_1| \leqslant d/4\bigr\}. \tag{5.11}$$

We set $\chi_{\pm d}(x) = \chi_\pm(16x_1/d)$ and take $\tilde{\chi}_{\pm d} \in C^\infty(\boldsymbol{R}^2)$ in such a way that

$$\tilde{\chi}_{\pm d} \quad \text{has a slightly larger support than } \chi_{\pm d}, \qquad \tilde{\chi}_{\pm d}\chi_{\pm d} = \chi_{\pm d}. \tag{5.12}$$

For the exterior domain $\Omega_{\pm d} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm d}$, we regard $\tilde{\chi}_{\pm d}$ as the extension from $L^2(\Omega_d)$ to $L^2(\Omega_{\pm d})$ and $\chi_{\pm d}$ as the restriction to $L^2(\Omega_d)$ from $L^2(\Omega_{\pm d})$. Then we define

$$\Lambda(\zeta; d) = \chi_{-d} R(\zeta; K_{-d})\tilde{\chi}_{-d} + \chi_{+d} R(\zeta; K_{+d})\tilde{\chi}_{+d}, \quad \zeta \in D_d, \tag{5.13}$$

as an operator acting on $L^2(\Omega_d)$. Since $K_d = K_{\pm d}$ on $\operatorname{supp}\chi_{\pm d}$ by (5.8) and (5.9), we compute

$$\begin{aligned}(K_d - \zeta)\Lambda &= (K_{-d} - \zeta)\chi_{-d} R(\zeta; K_{-d})\tilde{\chi}_{-d} + (K_{+d} - \zeta)\chi_{+d} R(\zeta; K_{+d})\tilde{\chi}_{+d} \\ &= Id + [K_{-d}, \chi_{-d}] R(\zeta; K_{-d})\tilde{\chi}_{-d} + [K_{+d}, \chi_{+d}] R(\zeta; K_{+d})\tilde{\chi}_{+d}.\end{aligned}$$

The function $\chi_{\pm d}$ depends on x_1 only, and the derivative $\chi'_{\pm d}$ has support in

$$\Pi_0 = \bigl\{x = (x_1, x_2)\colon |x_1| < d/16\bigr\}. \tag{5.14}$$

By (5.11), $K_{\pm d} = K_{0d}$ on Π_0, so that both the commutators $[K_{-d}, \chi_{-d}]$ and $[\chi_{+d}, K_{+d}]$ on the right side equal $[K_{0d}, \chi_{-d}]$. Hence we have

$$(K_d - \zeta)\Lambda(\zeta; d) = Id + \Gamma_0\bigl(R(\zeta; K_{-d})\tilde{\chi}_{-d} - R(\zeta; K_{+d})\tilde{\chi}_{+d}\bigr), \tag{5.15}$$

where

$$\Gamma_0 = [K_{0d}, \chi_{-d}], \qquad \chi_{-d} = \chi_-(16x_1/d). \tag{5.16}$$

We define

$$T(\zeta; d) = \Gamma_0\bigl(R(\zeta; K_{-d}) - R(\zeta; K_{+d})\bigr)\pi_0 \tag{5.17}$$

as an operator acting on $L^2(\Pi_0)$, where the multiplication π_0 by the characteristic function $\pi_0(x)$ of Π_0 is regarded as the extension from $L^2(\Pi_0)$ to $L^2(\Omega_{+d})$ or to $L^2(\Omega_{-d})$. Thus the resolvent $R(\zeta; K_d)$ is represented as

$$R(\zeta; K_d) = \Lambda(\zeta; d)\bigl(Id - \pi_0(Id + T)^{-1}\Gamma_0\bigl(R(\zeta; K_{-d})\tilde{\chi}_{-d} - R(\zeta; K_{+d})\tilde{\chi}_{+d}\bigr)\bigr).$$

In the previous work [5], we have shown that for $\zeta \in D_d$ with $\operatorname{Im}\zeta > 0$, $Id + T(\zeta; d)$ is invertible in $L^2(\Pi_0)$ and $R(\zeta; K_d)$ admits the above representation. This allows us to obtain the basic relation

$$R_{0d}(\zeta) = s_0\Lambda(\zeta; d)\bigl(Id - \pi_0\bigl(Id + T(\zeta; d)\bigr)^{-1}\Gamma_0\bigl(R(\zeta; K_{-d})\tilde{\chi}_{-d} - R(\zeta; K_{+d})\tilde{\chi}_{+d}\bigr)\bigr)s_0 \tag{5.18}$$

for $R_{0d}(\zeta)$ defined by (4.3).

We now have the following lemma.

Lemma 5.1. *The operators*

$$\pi_0 R(\zeta; K_{\pm d})s_0, \qquad s_0 R(\zeta; K_{\pm d})\pi_0, \qquad s_0 R(\zeta; K_{\pm d})s_0 : L^2(\Omega_d) \to L^2(\Omega_d)$$

are bounded and are analytic in $\zeta \in D_d$.

Proof. We first note that the mapping j_d acts as the identity over S_0. We prove the lemma for $\pi_0 R(\zeta; K_{+d})s_0$ only. A similar argument applies to $s_0 R(\zeta; K_{+d})\pi_0$ also, and it is easy to see that the lemma is true for $s_0 R(\zeta; K_{+d})s_0$, since

$$s_0 R(\zeta; K_{+d})s_0 = \exp(ig_{-d})\bigl(s_0 R(\zeta; H_{+d})s_0\bigr)\exp(-ig_{-d})$$

is analytic over D_d by Lemma 3.1 and (5.7). To prove the assertion, we introduce the self-adjoint operator

$$P_{+d} = H(\Phi_{+d}), \qquad \Phi_{+d} = \alpha_+\Phi(x - d_+),$$

with one solenoid with center at d_+. The operator P_{+d} acting on $L^2(\boldsymbol{R}^2)$ coincides with H_{+d} defined by (5.4) over Ω_{+d}. We take S_1 and S_2 to be

$$S_0 \subset S_1 = \bigl\{|x_1| < 2d, |x_2| \leqslant 2L_0\bigr\} \subset S_2 = \bigl\{|x_1| < 3d, |x_2| \leqslant 3L_0\bigr\}.$$

Let $s_1, s_2 \in C_0^\infty(\boldsymbol{R}^2)$ be functions such that

$$\operatorname{supp} s_1 \subset S_1, \qquad s_1 = 1 \quad \text{on } S_0, \qquad \operatorname{supp} s_2 \subset S_2, \qquad s_2 = 1 \quad \text{on } S_1.$$

If we set $\tilde{s}_1 = 1 - s_1$ and $\tilde{s}_2 = 1 - s_2$, then $\tilde{s}_1\tilde{s}_2 = \tilde{s}_2$ and we have the relation

$$R(\zeta; H_{+d})s_0 = s_2 R(\zeta; H_{+d})s_0 + \tilde{s}_2 R(\zeta; P_{+d})[P_{+d}, \tilde{s}_1]R(\zeta; H_{+d})s_0.$$

Since j_d also acts as the identity over S_1, S_2 and $\operatorname{supp}\nabla s_1 \subset S_2$, the same relation remains true for H_{+d} and P_{+d} replaced by $\tilde{H}_{+d} = J_d H_{+d} J_d^{-1}$ and $\tilde{P}_{+d} = J_d P_{+d} J_d^{-1}$, respectively. We know by [5, Lemma 3.4] that $\pi_0 R(\zeta; \tilde{P}_{+d})[P_{+d}, \tilde{s}_1]$ is analytic in $\zeta \in D_d$ as a function with values in operators on $L^2(\boldsymbol{R}^2)$. This yields that $\pi_0 R(\zeta; \tilde{H}_{+d}) s_0$ is also analytic in $\zeta \in D_d$, and the proof is complete. □

By this lemma, the problem is reduced to studying when the operator

$$Id + T(\zeta; d) : L^2(\varPi_0) \to L^2(\varPi_0)$$

is not invertible for $\zeta \in D_d$ with $\operatorname{Im}\zeta \leqslant 0$. The remaining sections are devoted to showing that the points ζ at which $Id + T(\zeta; d)$ is not invertible coincide with the poles of $R_{0d}(\zeta)$ in Theorem 4.1.

We end the section by making a brief comment on the complex scaling method developed here. The mapping J_d defined by (5.2) takes a form different from the standard mapping

$$(\tilde{J}_\theta f)(x) = \bigl[\det\bigl(1 + i\theta\, dF(x)\bigr)\bigr]^{1/2} f\bigl(x + i\theta F(x)\bigr), \quad \theta > 0,$$

used in the existing complex scaling method (see, for example, [12]), where $F : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a smooth vector field satisfying $F(x) = x$ for $|x| \gg 1$. If we define $\tilde{K}_{d\theta} = \tilde{J}_\theta H_d \tilde{J}_\theta^{-1}$, then it follows by the Weyl perturbation theorem that the essential spectrum of $\tilde{K}_{d\theta}$ is given by

$$\sigma_{\mathrm{ess}}(\tilde{K}_{d\theta}) = \bigl\{\zeta \in \boldsymbol{C}\colon\ \arg\zeta = -2\arg(1 + i\theta)\bigr\},$$

and the resonances of H_d in question are defined as eigenvalues near the positive real axis of the distorted operator $\tilde{K}_{d\theta}$. The spectrum $\sigma(\tilde{K}_{d\theta})$ is discrete in the sector

$$\varSigma_\theta = \bigl\{\zeta \in \boldsymbol{C}\colon\ \operatorname{Re}\zeta > 0,\ -2\arg(1 + i\theta) < \arg\zeta \leqslant 0\bigr\}$$

and it is known that $\sigma(\tilde{K}_{d\theta}) \cap \varSigma_\theta$ is independent of the vector field F and of θ. On the other hand, the essential spectrum of K_d expands into the region

$$\sigma_{\mathrm{ess}}(K_d) = \bigl\{\zeta \in \boldsymbol{C}\colon\ -2\arg(1 + i\eta_{0d}) \leqslant \arg\zeta \leqslant 0\bigr\}$$

with $\eta_{0d} = L_1(\log d)/d$, and K_d has no discrete eigenvalues in this sector. This follows from the Weyl perturbation theorem, if we consider K_d as a perturbation of the operator $-\partial_1^2 - (1 + i\eta_{0d})^{-2}\partial_2^2$. Hence we have to define the resonances of H_d directly as the poles of the resolvent $R(\zeta; H_d)$ meromorphically continued over the unphysical sheet but not as the eigenvalues of K_d. It seems to be difficult to apply the standard complex scaling method to the resonance problem in scattering by several magnetic fields in two dimensions. In particular, it is difficult to separate the fields from each other without introducing auxiliary operators such as $K_{\pm d}$ with one field. For this reason, we develop a new type of complex scaling method, which enables us to separate the two fields from each other by changing only the x_2 variable into the complex variable and to control the composition of the resolvent kernels of $K_{\pm d}$ in the inversion of $Id + T(\zeta; d)$.

6. Resolvent kernels of distorted operators

In this section we study the asymptotic properties of the resolvent kernels of the distorted operators $K_{\pm d}$ defined by (5.7). We introduce the new notation. We define

$$\psi_d(x,y) = \gamma\big(j_d(x); -\omega_1\big) - \gamma\big(j_d(y); -\omega_1\big), \quad \omega_1 = (1,0), \tag{6.1}$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_1 > 0$ and $y_1 > 0$, and

$$r_d(x,y) = \Big((x_1 - y_1)^2 + \big(\big(x_2 + i\eta_d(x_2)x_2\big) - \big(y_2 + i\eta_d(y_2)y_2\big)\big)^2\Big)^{1/2}, \tag{6.2}$$

where $j_d(x)$ and $\eta_d(t)$ are defined in (5.1). If, in particular, y is the origin, then we write $r_d(x)$ for $r_d(x,y)$. We further recall the notation from Sections 1 and 2. Let $P = H(\alpha\Phi)$ be define by (1.12) with domain (1.13). We have denoted by $\varphi_{0+}(x; \omega, \zeta)$ and $\varphi_{0-}(x; \omega, \zeta)$ the outgoing and incoming eigenfunctions of P and by $f(\omega \to \theta; \zeta)$ the scattering amplitude for the pair (H_0, P). We denote by $R(\zeta; P)(x,y)$ the resolvent kernel of $R(\zeta; P)$ with $\zeta \in D_d$, and we define

$$Q_d(x,y;\zeta) = R(\zeta;P)\big(j_d(x), j_d(y); \zeta\big). \tag{6.3}$$

We know that the resolvent kernel $R(\zeta; H_0)(x,y)$ of the free Hamiltonian H_0 is given by

$$R(\zeta; H_0)(x,y) = (i/4)H_0\big(k|x-y|\big), \quad k = \zeta^{1/2}, \tag{6.4}$$

where $H_0(z) = H_0^{(1)}(z)$ denotes the Hankel function of order zero. The arguments here are based on the following two propositions. We postpone proving them until the last section (Section 9).

Proposition 6.1. *Let $Q_d(x,y;\zeta)$ be as above. Assume that $d/c_1 < x_1$, $y_1 < c_1 d$ for some $c_1 > 1$. Then $Q_d(x,y;\zeta)$ admits the decomposition*

$$Q_d(x,y;\zeta) = \exp\big(i\alpha\psi_d(x,y)\big)R(\zeta; H_0)\big(j_d(x), j_d(y)\big) + Q_{\mathrm{sc}}(x,y;\zeta), \tag{6.5}$$

and the analytic function $Q_{\mathrm{sc}}(x,y;\zeta)$ over D_d satisfies the following estimates uniformly in $\zeta \in D_d$.

(1) *If $|x_2| + |y_2| > Ld$ for $L \gg 1$ fixed arbitrarily, then*

$$Q_{\mathrm{sc}}(x,y;\zeta) = O\Big(\big(|x| + |y|\big)^{-\sigma L}\Big)$$

for some $\sigma > 0$ independent of L together with the derivatives $\partial Q_{\mathrm{sc}}/\partial x_1$ and $\partial Q_{\mathrm{sc}}/\partial y_1$.

(2) *If* $|x_2| + |y_2| < 2Ld$ *for* $L \gg 1$ *fixed above, then* Q_{sc} *takes the form*

$$Q_{\mathrm{sc}} = \exp\bigl(ikr_d(x)\bigr)q_0(x,y;\zeta)\exp\bigl(ikr_d(y)\bigr) \tag{6.6}$$

and $q_0(x,y;\zeta)$ *satisfies*

$$\bigl|(\partial/\partial x_2)^j(\partial/\partial y_2)^l q_0\bigr| = O\bigl(d^{-1-j-l}\bigr).$$

Similar estimates hold true for $\partial q_0/\partial x_1$ *and* $\partial q_0/\partial y_1$*. If, in particular,* $|x_2| < d$ *and* $|y_2| < d$*, then* q_0 *takes the asymptotic form*

$$q_0(x,y;\zeta) = |x|^{-1/2}\bigl\{c_0(\zeta)f(-\hat{y}\to\hat{x};\zeta) + e(x,y;\zeta)\bigr\}|y|^{-1/2}, \tag{6.7}$$

where $c_0(\zeta)$ *is defined by* (2.8)*, and the remainder term* $e(x,y;\zeta)$ *obeys*

$$\bigl|\partial_x^m\partial_y^n e(x,y;\zeta)\bigr| = O\bigl(d^{-1-(|m|+|n|)}\bigr). \tag{6.8}$$

Proposition 6.2. *The following statements hold true for* $Q_d(x,y;\zeta)$*.*

(1) *Assume that* $|x| > d/c_1$ *and* $1/c_1 < |y| < c_1$ *for some* $c_1 > 1$*. Then* $Q_d(x,y;\zeta)$ *takes the form*

$$Q_d(x,y;\zeta) = \exp\bigl(ikr_d(x)\bigr)q_1(x,y;\zeta),$$

where q_1 *satisfies*

$$\bigl|(\partial/\partial x_2)^l q_1\bigr| = O\bigl(|x|^{-1/2-l}\bigr) \tag{6.9}$$

and similarly for $\partial q_1/\partial x_1$*. If, in particular,* $|x_2| < d$*, then* q_1 *takes the asymptotic form*

$$q_1 = |x|^{-1/2}\bigl(c_0(\zeta)\overline{\varphi}_{0-}(y;\hat{x},\bar{\zeta}) + O\bigl(d^{-1}\bigr)\bigr) \tag{6.10}$$

uniformly in y *and* $\zeta \in D_d$*.*

(2) *Assume that* $1/c_1 < |x| < c_1$ *and* $|y| > d/c_1$ *for some* $c_1 > 1$*. Then* $Q_d(x,y;\zeta)$ *takes the form*

$$Q_d(x,y;\zeta) = q_2(x,y;\zeta)\exp\bigl(ikr_d(y)\bigr),$$

where $q_2(x,y;\zeta)$ *satisfies* $(\partial/\partial y_2)^l q_2 = O(|y|^{-1/2-l})$*, and similarly for* $\partial q_1/\partial y_1$*. If, in particular,* $|y_2| < d$*, then* q_2 *takes the asymptotic form*

$$q_2 = \bigl(c_0(\zeta)\varphi_{0+}(x;-\hat{y},\zeta) + O\bigl(d^{-1}\bigr)\bigr)|y|^{-1/2}$$

uniformly in x *and* $\zeta \in D_d$*.*

Remark 6.

(1) In decomposition (6.5), the first term describes the free trajectory which goes from y to x directly without being scattered at the origin, while the second term comes from the scattering trajectory which starts from y and arrives at x after being scattered at the origin.
(2) If x and y satisfy $-c_1 d < x_1, y_1 < -d/c_1$ for some $c_1 > 1$, then we have a similar decomposition

$$Q_d(x, y; \zeta) = \exp\bigl(i\alpha\tilde{\psi}_d(x, y)\bigr)R(\zeta; H_0)\bigl(j_d(x), j_d(y)\bigr) + Q_{\mathrm{sc}}(x, y; \zeta)$$

with $\tilde{\psi}_d(x, y) = \gamma(j_d(x); \omega_1) - \gamma(j_d(y); \omega_1)$.

Let $H = H(\alpha\Phi)$ be the self-adjoint operator defined by (3.1). Lemma 3.1 and Proposition 3.1 allow us to define

$$G_d(x, y; \zeta) = R(\zeta; H)\bigl(j_d(x), j_d(y)\bigr)$$

for the resolvent kernel $R(\zeta; H)(x, y)$ of $R(\zeta; H)$ with $\zeta \in D_d$, provided that $d \gg 1$. Then the two propositions above, together with Lemma 3.2 (see Remark 3 also), imply the following proposition.

Proposition 6.3. *Assume that $d/c_1 < x_1, y_1 < c_1 d$ for some $c_1 > 1$. Then $G_d(x, y; \zeta)$ admits the decomposition*

$$G_d(x, y; \zeta) = \exp\bigl(i\alpha\psi_d(x, y)\bigr)R(\zeta; H_0)\bigl(j_d(x), j_d(y)\bigr) + G_{\mathrm{sc}}(x, y; \zeta)$$

and the analytic function $G_{\mathrm{sc}}(x, y; \zeta)$ over D_d satisfies the following estimates uniformly in $\zeta \in D_d$.

(1) *If $|x_2| + |y_2| > Ld$ for $L \gg 1$ fixed arbitrarily, then*

$$G_{\mathrm{sc}}(x, y; \zeta) = O\bigl(\bigl(|x| + |y|\bigr)^{-\sigma L}\bigr)$$

for some $\sigma > 0$ independent of L together with the derivatives $\partial G_{\mathrm{sc}}/\partial x_1$ and $\partial G_{\mathrm{sc}}/\partial y_1$.

(2) *If $|x_2| + |y_2| < 2Ld$ for $L \gg 1$ fixed, then G_{sc} takes the form*

$$G_{\mathrm{sc}}(x, y; \zeta) = \exp\bigl(ikr_d(x)\bigr)p_0(x, y; \zeta)\exp\bigl(ikr_d(y)\bigr)$$

and $p_0(x, y; \zeta)$ satisfies

$$\bigl|(\partial/\partial x_2)^j(\partial/\partial y_2)^l p_0\bigr| = O\bigl(d^{-1-j-l}\bigr).$$

Similar estimates hold true for $\partial p_0/\partial x_1$ and $\partial p_0/\partial y_1$. If, in particular, $|x_2| < d$ and $|y_2| < d$, then p_0 takes the asymptotic form

$$p_0(x, y; \zeta) = |x|^{-1/2}\{c_0(\zeta) f_0(-\hat{y} \to \hat{x}; \zeta) + e_0(x, y; \zeta)\}|y|^{-1/2},$$

where $f_0(\omega \to \theta; E)$ is the scattering amplitude for the pair (H_0, H), and the remainder term $e_0(x, y; \zeta)$ obeys the same bound as (6.8).

The resolvent kernel $R(\zeta; \tilde{K})(x, y)$ of the distorted operator $\tilde{K} = J_d H J_d^{-1}$ with the same domain as H is given by

$$R(\zeta; \tilde{K})(x, y) = \left[\det(\partial j_d/\partial x)\right]^{1/2} G_d(x, y; \zeta)\left[\det(\partial j_d/\partial y)\right]^{1/2}.$$

If we denote by $R(\zeta; H_\pm)(x, y)$ the resolvent kernel of the self-adjoint operator

$$H_\pm = H(\alpha_\pm \Phi), \qquad \mathcal{D}(H_\pm) = H^2(\Omega_\pm) \cap H_0^1(\Omega_\pm), \qquad \Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm,$$

then the resolvent kernel $R(\zeta; K_{\pm d})(x, y)$ of $K_{\pm d}$ defined by (5.7) is given by

$$\begin{aligned} R(\zeta; K_{\pm d})(x, y) = \exp\bigl(i g_{\mp d}(x)\bigr)\left[\det(\partial j_d/\partial x)\right]^{1/2} R(\zeta; H_\pm)\bigl(j_d(x) - d_\pm, j_d(y) - d_\pm\bigr) \\ \times \left[\det(\partial j_d/\partial y)\right]^{1/2} \exp\bigl(-i g_{\mp d}(y)\bigr), \end{aligned}$$

and also it follows from (5.10) and (6.4) that the resolvent kernel $R(\zeta; K_{0d})(x, y)$ of K_{0d} takes the form

$$\begin{aligned} R(\zeta; K_{0d})(x, y) = (i/4)\exp\bigl(i g_{0d}(x)\bigr)\left[\det(\partial j_d/\partial x)\right]^{1/2} \\ \times H_0\bigl(k r_d(x, y)\bigr)\left[\det(\partial j_d/\partial y)\right]^{1/2} \exp\bigl(-i g_{0d}(y)\bigr). \end{aligned}$$

We now recall the definitions of $g_{\pm d}(x)$ and $g_{0d}(x)$ from (5.5) and (5.6), respectively. Then we have the relation

$$\begin{aligned} \alpha_\pm\bigl(\gamma\bigl(j_d(x) - d_\pm; \pm\omega_1\bigr) - \gamma\bigl(j_d(y) - d_\pm; \pm\omega_1\bigr)\bigr) \\ = g_{0d}(x) - g_{0d}(y) - \bigl(g_{\mp d}(x) - g_{\mp d}(y)\bigr) \end{aligned}$$

on $\Pi_0 \times \Pi_0$, Π_0 being as in (5.14). Thus the next proposition is obtained as an immediate consequence of Proposition 6.3.

Proposition 6.4. *Let $(x, y) \in \Pi_0 \times \Pi_0$, so that $|x_1| < d/16$ and $|y_1| < d/16$. Then $R(\zeta; K_{\pm d})(x, y)$ admits the decomposition*

$$R(\zeta; K_{\pm d})(x, y) = R(\zeta; K_{0d})(x, y) + R_{\mathrm{sc}}(\zeta; K_{\pm d})(x, y)$$

over $\Pi_0 \times \Pi_0$, and the analytic function $R_{\mathrm{sc}}(\zeta; K_{\pm d})(x, y)$ over D_d satisfies the following estimates uniformly in $\zeta \in D_d$.

(1) *If* $|x_2| + |y_2| > Ld$ *for* $L \gg 1$ *fixed arbitrarily, then*

$$R_{\mathrm{sc}}(\zeta; K_{\pm d})(x, y) = O\big(\big(|x_\pm| + |y_\pm|\big)^{-\sigma L}\big)$$

for some $\sigma > 0$ *independent of* L *together with the derivatives* $\partial R_{\mathrm{sc}}(\zeta; K_{\pm d})/\partial x_1$ *and* $\partial R_{\mathrm{sc}}(\zeta; K_{\pm d})/\partial y_1$*, where* $x_\pm = x - d_\pm$ *and* $y_\pm = y - d_\pm$*.*

(2) *We write* $r_{\pm d}(x)$ *for* $r_d(x, y)$ *with* $y = d_\pm$*. If* $|x_2| + |y_2| < 2Ld$ *for* $L \gg 1$ *fixed, then* $R_{\mathrm{sc}}(\zeta; K_{\pm d})(x, y)$ *takes the form*

$$R_{\mathrm{sc}}(\zeta; K_{\pm d})(x, y) = \exp\big(ikr_{\pm d}(x)\big)p_\pm(x, y; \zeta)\exp\big(ikr_{\pm d}(y)\big)$$

and $p_\pm(x, y; \zeta)$ *satisfies*

$$\big|(\partial/\partial x_2)^j(\partial/\partial y_2)^l p_\pm\big| = O\big(d^{-1-j-l}\big).$$

Similar estimates hold true for $\partial p_\pm/\partial x_1$ *and* $\partial p_\pm/\partial y_1$*. If, in particular,* $|x_2| < d$ *and* $|y_2| < d$*, then* $p_\pm(x, y; \zeta)$ *takes the asymptotic form*

$$p_\pm = |x_\pm|^{-1/2}\big\{c_0(\zeta)e^{ig_{\mp d}(x)}f_\pm(-\hat{y}_\pm \to \hat{x}_\pm; \zeta)e^{-ig_{\mp d}(y)} + e_\pm(x, y; \zeta)\big\}|y_\pm|^{-1/2},$$

where $\hat{x}_\pm = x_\pm/|x_\pm|$ *and* $e_\pm(x, y; \zeta)$ *obeys the same bound as* (6.8)*.*

Let $R_{\mathrm{sc}}(\zeta; K_{\pm d})(x, y)$ be as in Proposition 6.4. We denote by $R_{\mathrm{sc}}(\zeta; K_{\pm d})$ the integral operator with this kernel. Then we have

$$R(\zeta; K_{\pm d}) = R(\zeta; K_{0d}) + R_{\mathrm{sc}}(\zeta; K_{\pm d}), \quad \zeta \in D_d, \tag{6.11}$$

as an operator from $L^2_{\mathrm{comp}}(\Pi_0)$ to $L^2_{\mathrm{loc}}(\Pi_0)$, and it also follows that the operator $T(\zeta; d)$ defined by (5.17) takes the form

$$T(\zeta; d) = \Gamma_0\big(R_{\mathrm{sc}}(\zeta; K_{-d}) - R_{\mathrm{sc}}(\zeta; K_{+d})\big)\pi_0, \quad \zeta \in D_d. \tag{6.12}$$

By Proposition 6.4, we see that $T(\zeta; d)$ is analytic in $\zeta \in D_d$ as a function with values in bounded operators on $L^2(\Pi_0)$ and obeys the bound $\|T(\zeta; d)\| = O(d^\mu)$ for some $\mu > 0$ uniformly in D_d.

7. Proof of Theorem 4.1

In this section we prove Theorem 4.1 on the basis of the three lemmas (Lemmas 7.1–7.3) formulated in the course of the proof. These lemmas are proved in the next section.

Proof of Theorem 4.1. The theorem is verified through five steps. We first note that $R_{0d}(\zeta)$ is analytic over the region where

$$Id + T(\zeta; d) : L^2(\Pi_0) \to L^2(\Pi_0)$$

is invertible, which is an immediate consequence of (5.18) and Lemma 5.1.

(1) We specify a point $\zeta \in D_d$ at which $Id + T(\zeta; d)$ is not invertible. We fix ρ as

$$0 < \rho - 1/2 \ll 1 \tag{7.1}$$

and we define

$$v_0(x_2) = \chi_0\big(2|x_2|/d^\rho\big), \qquad \tilde{v}_0 = \chi_0\big(|x_2|/d^\rho\big), \qquad v_1(x_2) = 1 - v_0(x_2)$$

and $\tilde{v}_1(x_2) = 1 - \chi_0(4|x_2|/d^\rho)$. Then $v_j \tilde{v}_j = v_j$ for $0 \leqslant j \leqslant 1$. We further define

$$T_{jk} = T_{jk}(\zeta; d) = v_j T(\zeta; d) \tilde{v}_k, \quad 0 \leqslant j,\ k \leqslant 1.$$

Then all the operators $T_{jk}(\zeta; d)$ fulfill $\|T_{jk}\| = O(d^\mu)$ for some $\mu > 0$ uniformly in D_d, and $Id + T(\zeta; d)$ has the matrix representation

$$X = X(\zeta; d) = \begin{pmatrix} Id + T_{00} & T_{01} \\ T_{10} & Id + T_{11} \end{pmatrix}$$

as an operator acting on $L^2(\Pi_0) \oplus L^2(\Pi_0)$.

(2) We denote by $\mathrm{Op}(d^{-N})$ the class of bounded operators on $L^2(\Pi_0)$ with bound $O(d^{-N})$ for any $N \gg 1$.

Lemma 7.1. *The operators*

$$T_{11}T_{11}, T_{11}T_{10}, T_{01}T_{11}, T_{01}T_{10} : L^2(\Pi_0) \to L^2(\Pi_0)$$

are all of class $\mathrm{Op}(d^{-N})$.

By the lemma above, we have

$$(Id + T_{11})^{-1} = \big(Id - T_{11}^2\big)^{-1}(Id - T_{11}) = Id - T_{11} + \mathrm{Op}\big(d^{-N}\big) \tag{7.2}$$

and X admits the decomposition

$$X = \begin{pmatrix} Id & 0 \\ 0 & Id + T_{11} \end{pmatrix} \begin{pmatrix} Id & T_{01} \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id + T_{00} + T_N & 0 \\ (Id + T_{11})^{-1} T_{10} & Id \end{pmatrix},$$

where

$$T_N = T_N(\zeta; d) = -T_{01}(Id + T_{11})^{-1} T_{10}$$

is of class $\mathrm{Op}(d^{-N})$. We now consider the operator

$$Y_0 = Y_0(\zeta; d) = Id + T_{00}(\zeta; d) + T_N(\zeta; d) : L^2(\Pi_0) \to L^2(\Pi_0).$$

If Y_0 is invertible, then it follows that X is also invertible, so that $Id + T(\zeta; d)$ becomes invertible. We can easily see that the converse is also true. The inverse X^{-1} is calculated as

$$X^{-1} = \begin{pmatrix} Y_0^{-1} & X_{01} \\ X_{10} & X_{11} \end{pmatrix},$$

where

$$X_{01} = -Y_0^{-1} T_{01} (Id + T_{11})^{-1}, \qquad X_{10} = -(Id + T_{11})^{-1} T_{10} Y_0^{-1}$$

and

$$X_{11} = (Id + T_{11})^{-1} T_{10} Y_0^{-1} T_{01} (Id + T_{11})^{-1} + (Id + T_{11})^{-1}.$$

If we take (7.2) into account, then it follows by Lemma 7.1 that $(Id + T)^{-1}$ behaves like

$$(Id + T)^{-1} \sim \bigl(Id - T_{10} + \mathrm{Op}\bigl(d^{-N}\bigr)\bigr) Y_0^{-1} \bigl(v_0 - T_{01} v_1 + \mathrm{Op}\bigl(d^{-N}\bigr)\bigr) \tag{7.3}$$

with remainders analytic over D_d.

(3) We analyze the asymptotic properties as $d \to \infty$ of $T_{00}(\zeta; d)$. We write $u \otimes w$ for the integral operator with the kernel $u(x)\overline{w}(y)$. We define $u_\pm = u_\pm(x; \zeta, d)$ and $w_\pm = w_\pm(x; \zeta, d)$ as

$$\begin{aligned} u_\pm &= -2ikc_0(\zeta) f_\pm(\pm\omega_1 \to \mp\omega_1; \zeta) \chi'_{-d}(x_1) v_0(x_2) e^{ik|x_\pm|} |x_\pm|^{-1/2}, \\ w_\pm &= \pi_0(x_1) \tilde{v}_0(x_2) \overline{e^{ik|x_\pm|}} |x_\pm|^{-1/2} = \pi_0(x_1) \tilde{v}_0(x_2) e^{-i\bar{k}|x_\pm|} |x_\pm|^{-1/2}, \end{aligned} \tag{7.4}$$

where $c_0(\zeta)$ is the constant defined by (2.8) and $\pi_0(x_1)$ $(= \pi_0(x))$ is the characteristic function of $\Pi_0 = \{|x_1| < d/16\}$.

Lemma 7.2. *The operator $T_{00}(\zeta; d)$ admits the decomposition*

$$T_{00}(\zeta; d) = Z_0(\zeta; d) + Z_1(\zeta; d).$$

Here the two operators on the right side have the following properties.

(i) *The operator $Z_0(\zeta; d)$ is defined by*

$$Z_0(\zeta; d) = u_- \otimes w_- + u_+ \otimes w_+$$

with $u_\pm$ and $w_\pm$ defined as above.

(ii) *We can take* $\delta_0 > 0$ *in* (1.7) *and* $\rho - 1/2 > 0$ *in* (7.1) *so small that*

$$\big\|Z_1(\zeta; d)\big\| = O\big(d^{-3/8}\big)$$

uniformly in $\zeta \in D_d$ *as a bounded operator acting on* $L^2(\Pi_0)$.

We denote by $\langle\ ,\ \rangle$ and $\|\quad\|_0$ the L^2 scalar product and the norm in $L^2(\Pi_0)$, respectively. We evaluate the norms $\|u_\pm\|_0$ and $\|w_\pm\|_0$. If $x \in \Pi_0 \cap \operatorname{supp} \tilde{v}_0$, then $7d/16 < |x_1 - d/2| < 9d/16$ and $|x_2| \leqslant 2d^\rho$. Hence we have

$$|x_+| = |x - d_+| = |x_1 - d/2|\big(1 + O\big(d^{-2+2\rho}\big)\big) \leqslant (9d/16)\big(1 + O\big(d^{-2+2\rho}\big)\big).$$

By (1.9) and (1.10), we have

$$|\operatorname{Im} k| \leqslant (1/2 + 3\delta_0/2)\big((\log d)/d\big),$$

so that $|e^{ik|x_+|}/|x_+|^{1/2}| = O(d^{-7/32+27\delta_0/32})$ uniformly in $\zeta \in D_d$, and similarly for $e^{ik|x_-|}/|x_-|^{1/2}$ with $x_- = x - d_-$. Note that $|\chi'_{-d}(x_1)| = O(d^{-1})$. Hence we can take $\delta_0 > 0$ and $\rho - 1/2 > 0$ so small that

$$-7/32 + 27\delta_0/32 - 1/2 + \rho/2 < -7/16.$$

Thus we obtain the bounds

$$\|u_\pm\|_0 = O\big(d^{-7/16}\big), \qquad \|w_\pm\|_0 = O\big(d^{9/16}\big). \tag{7.5}$$

Lemma 7.2 allows us to write

$$Y_0(\zeta; d) = Id + Z_0(\zeta; d) + Z_{\mathrm{rem}}(\zeta; d) = \big(Id + Z_{\mathrm{rem}}(\zeta; d)\big)\big(Id + \tilde{Z}_0(\zeta; d)\big),$$

where $Z_{\mathrm{rem}}(\zeta; d) = Z_1(\zeta; d) + T_N(\zeta; d)$ and

$$\tilde{Z}_0(\zeta; d) = \big(Id + Z_{\mathrm{rem}}(\zeta; d)\big)^{-1} Z_0(\zeta; d) = \tilde{u}_- \otimes w_- + \tilde{u}_+ \otimes w_+$$

with $\tilde{u}_\pm = \tilde{u}_\pm(x; \zeta, d)$ defined by

$$\tilde{u}_\pm = \big(Id + Z_{\mathrm{rem}}(\zeta; d)\big)^{-1} u_\pm = u_\pm - \big(Id + Z_{\mathrm{rem}}(\zeta; d)\big)^{-1} Z_{\mathrm{rem}}(\zeta; d) u_\pm. \tag{7.6}$$

If we set

$$h_{\pm\pm}(\zeta; d) = \langle \tilde{u}_\pm, w_\pm \rangle, \qquad h_{\pm\mp}(\zeta; d) = \langle \tilde{u}_\pm, w_\mp \rangle$$

and define $h_0(\zeta; d)$ by

$$h_0(\zeta;d) = \bigl(1 + h_{--}(\zeta;d)\bigr)\bigl(1 + h_{++}(\zeta;d)\bigr) - h_{-+}(\zeta;d)h_{+-}(\zeta;d) \tag{7.7}$$

for $\zeta \in D_d$, then a direct computation yields

$$(Id + \tilde{Z}_0)^{-1} = Id - h_0^{-1}\bigl\{(1 + h_{++})Z_{--} - h_{+-}Z_{-+} - h_{-+}Z_{+-} + (1 + h_{--})Z_{++}\bigr\},$$

where $Z_{\pm\pm} = \tilde{u}_\pm \otimes w_\pm$ and $Z_{\mp\pm} = \tilde{u}_\mp \otimes w_\pm$. Hence it follows that

$$Y_0^{-1} \sim -h_0^{-1}\bigl\{(1 + h_{++})\tilde{Z}_{--} - h_{+-}\tilde{Z}_{-+} - h_{-+}\tilde{Z}_{+-} + (1 + h_{--})\tilde{Z}_{++}\bigr\} \tag{7.8}$$

with remainders analytic over D_d, where $\tilde{Z}_{\pm\pm} = \tilde{u}_\pm \otimes \tilde{w}_\pm$ and $\tilde{Z}_{\mp\pm} = \tilde{u}_\mp \otimes \tilde{w}_\pm$ with

$$\tilde{w}_\pm = \tilde{w}_\pm(x;\zeta,d) = \bigl(Id + Z_{\mathrm{rem}}(\zeta;d)^*\bigr)^{-1} w_\pm. \tag{7.9}$$

Thus we have shown that the operator $R_{0d}(\zeta)$ in question is analytic over the region where $h_0(\zeta;d)$ does not vanish.

(4) We consider the equation $h_0(\zeta;d) = 0$ over D_d for $h_0(\zeta;d)$ defined by (7.7). We claim that the solutions to this equation coincide with the poles in D_d of $R_{0d}(\zeta)$. To see this, we analyze the asymptotic behaviors of $h_{\pm\pm}(\zeta;d)$ and $h_{\mp\pm}(\zeta;d)$. If we note that $|\partial_1|x_\pm|| > c > 0$ for $x \in \Pi_0 \cap \operatorname{supp} v_0$, then we can easily show by repeated use of integration by parts that $\langle u_\pm, w_\pm\rangle = O(d^{-N})$ for any $N \gg 1$. We make use of the stationary phase method (or the method of steepest descent) to see the behavior of $\langle u_\pm, w_\mp\rangle$. For x_1 fixed, the phase function $|x_-| + |x_+|$ attains its minimum at $x_2 = 0$ as a function of x_2. Since $c_0(\zeta) = (8\pi)^{-1/2}e^{i\pi/4}k^{-1/2}$ by (2.8) and since $\int \chi'_{-d}(x_1)\,dx_1 = -1$ for $\chi_{-d}(x_1)$ in (5.16), we have

$$\begin{aligned}\langle u_\pm, w_\mp\rangle &= (k/2\pi i)^{-1/2}2ikc_0(\zeta)f_\pm(\pm\omega_1 \to \mp\omega_1;\zeta)e^{ikd}d^{-1/2}\bigl(1 + O\bigl(d^{-1}\bigr)\bigr)\\ &= -f_\pm(\pm\omega_1 \to \mp\omega_1;\zeta)e^{ikd}d^{-1/2}\bigl(1 + O\bigl(d^{-1}\bigr)\bigr)\end{aligned}$$

by (7.4). We further obtain

$$\bigl|\langle \tilde{u}_\pm - u_\pm, w_\pm\rangle\bigr| = O(1)\|Z_{\mathrm{rem}}\|\|u_\pm\|_0\|w_\pm\|_0 = O\bigl(d^{-1/4}\bigr)$$

by (7.5), (7.6) and Lemma 7.2. Thus we have $h_{\pm\pm}(\zeta;d) = O(d^{-1/4})$ and

$$h_{\mp\pm}(\zeta;d) = -f_\mp(\mp\omega_1 \to \pm\omega_1;\zeta)e^{ikd}d^{-1/2} + O\bigl(d^{-1/4}\bigr)$$

uniformly in $\zeta \in D_d$, because

$$\bigl|e^{ikd}/d^{1/2}\bigr| = O\bigl(d^{3\delta_0/2}\bigr) \tag{7.10}$$

for $\zeta \in D_d$ (see the bound below (1.10) in Section 1).

We now write

$$h_0(\zeta;d) = \bigl(1 - h(\zeta;d)\bigr) + h_{\mathrm{rem}}(\zeta;d),$$

where $h(\zeta;d)$ is defined by (4.1), and $h_{\mathrm{rem}}(\zeta;d)$ obeys $|h_{\mathrm{rem}}(\zeta;d)| = O(d^{-1/5})$ uniformly in $\zeta \in D_d$. We apply Rouché's theorem to the equation $h_0(\zeta;d) = 0$ over D_d. Let $\{\zeta_j(d)\}$, $1 \leqslant j \leqslant N_d$, be as in Lemma 4.1 and let

$$C_{j\varepsilon} = \bigl\{\bigl|\zeta - \zeta_j(d)\bigr| = \varepsilon/|d|\bigr\}, \qquad D_{j\varepsilon} = \bigl\{\bigl|\zeta - \zeta_j(d)\bigr| < \varepsilon/|d|\bigr\},$$

for $\varepsilon > 0$ fixed arbitrarily but sufficiently small. We may assume $D_{j\varepsilon} \subset D_d$ for $|d| \gg 1$ by expanding D_d slightly, if necessary. Since $h(\zeta_j(d);d) = 1$, we have

$$h'\bigl(\zeta_j(d);d\bigr) = i\zeta_j(d)^{-1/2}|d|\bigl(1 + O\bigl(|d|^{-1}\bigr)\bigr),$$

so that $|h'(\zeta_j(d);d)| \geqslant c_1|d|$ for some $c_1 > 0$. Hence it follows that $|h(\zeta;d) - 1| \geqslant c_2\varepsilon$ on $C_{j\varepsilon}$ for some $c_2 > 0$. Thus the equation $h_0(\zeta;d) = 0$ has the unique solutions

$$\bigl\{\zeta_{\mathrm{res},j}(d)\bigr\}_{1\leqslant j\leqslant N_d}, \qquad \operatorname{Re}\zeta_{\mathrm{res},1}(d) < \cdots < \operatorname{Re}\zeta_{\mathrm{res},N_d}(d),$$

in $D_{j\varepsilon}$ for $|d| \gg 1$.

(5) In this step we complete the proof by showing that $\zeta_{\mathrm{res},j}(d)$ become the poles of $R_{0d}(\zeta)$. For brevity, we fix one of $\zeta_{\mathrm{res},j}(d)$, $1 \leqslant j \leqslant N_d$, and denote it by $\zeta_0 = \zeta_0(d)$. We restrict ourselves to the neighborhood

$$D_{0\delta} = \bigl\{\bigl|\zeta - \zeta_0(d)\bigr| < \delta/|d|\bigr\} \subset D_d, \quad 0 < \delta \ll 1,$$

of ζ_0. Then we have

$$h_0(\zeta;d) = \bigl(\zeta - \zeta_0(d)\bigr)h_1(\zeta;d)$$

with $h_1(\zeta;d)$ analytic and not vanishing over $D_{0\delta}$. Since $f_\mp(\mp\omega_1 \to \pm\omega_1;\zeta) \neq 0$ over D_d by assumption ($f_\mp(\mp\omega_1 \to \pm\omega_1;E_0) \neq 0$), the relation $h_0(\zeta_0;d) = 0$ implies

$$1 + h_{--}(\zeta_0;d) = \lambda_d h_{+-}(\zeta_0;d), \qquad 1 + h_{++}(\zeta_0;d) = h_{-+}(\zeta_0;d)/\lambda_d \tag{7.11}$$

for some nonzero constant $\lambda_d \neq 0$. We now define $u_0(x;d)$ and $w_0(x;d)$ by

$$\begin{aligned} u_0 &= \bigl(Id - T_{10}(\zeta_0;d)\bigr)\bigl(\tilde{u}_-(x;\zeta_0,d) - \lambda_d\tilde{u}_+(x;\zeta_0,d)\bigr),\\ w_0 &= \bigl(v_0 - v_1T_{01}(\zeta_0;d)^*\bigr)\bigl((\bar{h}_{-d}/\overline{\lambda_d})\tilde{w}_-(x;\zeta_0,d) - \bar{h}_{+d}\tilde{w}_+(x;\zeta_0,d)\bigr) \end{aligned} \tag{7.12}$$

with $h_{\pm d} = h_{\pm\mp}(\zeta_0;d)$. Then it follows from (7.3) and (7.8) that

$$\big(Id + T(\zeta; d)\big)^{-1} \sim -\frac{1}{(\zeta - \zeta_0(d))h_1(\zeta; d)}\big((u_0 \otimes w_0) + \mathrm{Op}\big(d^{-N}\big)\big)$$

with remainders analytic over $D_{0\delta}$. We further recall the representation for $\Lambda(\zeta_0; d)$ from (5.13), and we define $\tilde{u}_0(x; d)$ and $\tilde{w}_0(x; d)$ by

$$\begin{aligned} \tilde{u}_0 &= s_0 \Lambda(\zeta_0; d) u_0(x) = s_0\big(\chi_{-d} R(\zeta_0; K_{-d}) + \chi_{+d} R(\zeta_0; K_{+d})\big)\pi_0 u_0, \\ \tilde{w}_0 &= s_0\big(\tilde{\chi}_{-d} R(\zeta_0; K_{-d})^* - \tilde{\chi}_{+d} R(\zeta_0; K_{+d})^*\big)\Gamma_0^* w_0, \end{aligned} \tag{7.13}$$

where $s_0(x)$ is the characteristic function of S_0 defined by (4.2), and we have used the relation $\tilde{\chi}_{\pm d}\pi_0 = \pi_0$ (see (5.12)). Then it follows from (5.18) that

$$R_{0d}(\zeta) \sim \frac{1}{(\zeta - \zeta_0(d))h_1(\zeta; d)}\big((\tilde{u}_0 \otimes \tilde{w}_0) + \mathrm{Op}\big(d^{-N}\big)\big).$$

Lemma 7.3. *Let $\tilde{u}_0$ and $\tilde{w}_0$ be as above. Then $\tilde{u}_0$ and $\tilde{w}_0$ never vanish identically.*

This lemma shows that $\zeta_0(d)$ is really the pole in D_d of $R_{0d}(\zeta)$, and the proof of the theorem is now complete. □

8. Proofs of Lemmas 7.1, 7.2 and 7.3

This section is devoted to proving Lemmas 7.1, 7.2 and 7.3 which have remained unproved.

Proof of Lemma 7.1. We prove the lemma for the operator $T_{11}T_{11}$ only. The idea is based on the fact that a particle which starts from $\operatorname{supp}\tilde{v}_1 \cap \Pi_0$ and passes over $\operatorname{supp} v_1 \cap \Pi_0$ again after being scattered by the obstacles $\mathcal{O}_{\pm d}$ never returns to $\operatorname{supp} v_1 \cap \Pi_0$. The other three operators are dealt with on the basis of the same idea. By (6.12), $T_{11}T_{11}$ is represented as the sum of four operators. Moreover, we consider only the operator

$$B(\zeta; d) = v_1 \Gamma_0 R_{\mathrm{sc}}(\zeta; K_{+d}) v_1 \Gamma_0 R_{\mathrm{sc}}(\zeta; K_{-d}) \tilde{v}_1 \pi_0$$

among these operators. A similar argument applies to the other operators.

Let $N \gg 1$ be fixed arbitrarily and let

$$v_N(x_2) = \chi_0\big(|x_2|/Nd\big), \qquad v_\infty(x_2) = 1 - v_N(x_2).$$

Then Proposition 6.4 (1) shows that

$$\pi_0 v_\infty R_{\mathrm{sc}}(\zeta; K_{\pm d})\pi_0, \qquad \pi_0 R_{\mathrm{sc}}(\zeta; K_{\pm d}) v_\infty \pi_0 : L^2(\Pi_0) \to L^2(\Pi_0)$$

obey the bound $O(d^{-\sigma N})$ for some $\sigma > 0$ independent of N. Thus it suffices to show that

$$B_N(\zeta;d) = \upsilon_N v_1 \Gamma_0 R_{\mathrm{sc}}(\zeta;K_{+d})\upsilon_N v_1 \Gamma_0 R_{\mathrm{sc}}(\zeta;K_{-d})\upsilon_N \tilde{v}_1 \pi_0$$

obeys the bound $O(d^{-L})$ for any $L \gg 1$. By Proposition 6.4 (2), the kernel $B_N(x,y;\zeta,d)$ of $B_N(\zeta;d)$ takes the form

$$B_N(x,y;\zeta,d) = e^{ikr_{+d}(x)}\upsilon_N(x)I(x,y;\zeta,d)\upsilon_N(y)e^{ikr_{-d}(y)}, \quad (x,y) \in \Pi_0 \times \Pi_0$$

with $r_{\pm d}(x) = r_d(x,d_\pm)$, and $I = I(x,y;\zeta,d)$ is defined by the oscillatory integral

$$I = \int_{\Pi_0} e^{ik\varphi(z;d)}\upsilon_N(z)v_1(z)a(x,z,y;\zeta,d)\,dz,$$

where the phase function φ is defined by $\varphi = r_{-d}(z) + r_{+d}(z)$ and the amplitude a satisfies $|\partial_z^m a| = O(d^{-4-|m|\rho})$ uniformly in $x,y \in \operatorname{supp}\upsilon_N$ and in $\zeta \in D_d$. If $z \in \operatorname{supp} v_1 \cap \Pi_0$, then $|z_2| > d^\rho$ and

$$\left|(\partial/\partial z_2)\varphi\right| \geqslant cd^{-1+\rho}$$

for some $c > 0$. Since $\rho > 1-\rho$ by (7.1), we repeat the integration by parts for the integral I to obtain

$$I(x,y;\zeta,d) = \int_{\Pi_0 \cap \operatorname{supp}\upsilon_N} e^{ik\varphi(z;d)}O\left(d^{-L}\right)dz$$

for any $L \gg 1$. Thus we see that $B_N(\zeta;d)$ is of class $\mathrm{Op}(d^{-N})$, and the proof is complete. □

Proof of Lemma 7.2. We first note that $r_{\pm d}(x)$ equals $|x_\pm| = |x \mp d/2|$ for $x \in \Pi_0 \cap \operatorname{supp} v_0$ and

$$\partial_1 r_{\pm d} = \mp(x_1 \mp d/2)/|x_\pm| = \mp 1 + O\left(d^{-2(1-\rho)}\right), \quad \partial_1 = \partial/\partial x_1.$$

By definition (see (5.16)), the commutator Γ_0 takes the form

$$\Gamma_0 = [K_{0d}, \chi_{-d}] = -2\chi'_{-d}(x_1)\partial_1 + O\left(d^{-2}\right).$$

We prove statement (1). For $R_{\mathrm{sc}}(\zeta;K_{\pm d})$, we recall the representation for the kernel $R_{\mathrm{sc}}(\zeta;K_{\pm d})(x,y)$ from Proposition 6.4 (2). If $x \in \Pi_0 \cap \operatorname{supp} v_0$ and $y \in \Pi_0 \cap \operatorname{supp}\tilde{v}_0$, we have that:

$$f_\pm(-\hat{y}_\pm \to \hat{x}_\pm;\zeta) = f_\pm(\pm\omega_1 \to \mp\omega_1;\zeta) + O\left(d^{-1+\rho}\right),$$
$$g_{\pm d}(x) = \alpha_\pm \pi + O\left(d^{-1+\rho}\right), \qquad g_{\pm d}(y) = \alpha_\pm \pi + O\left(d^{-1+\rho}\right).$$

These relations imply the desired leading term $Z_0(\zeta;d)$.

Next we prove statement (2) by evaluating the norm of the remainder terms. The kernel $Z_1(x,y;\zeta,d)$ of the operator $Z_1(\zeta,d)$ takes the form

$$\begin{aligned}Z_1(x,y;\zeta,d) &= v_0(x_2)\pi_0(x_1)e^{ik|x_-|}|x_-|^{-1/2}z_-(x,y;\zeta,d)|y_-|^{-1/2}e^{ik|y_-|}\pi_0(y_1)\tilde{v}_0(y_2)\\ &\quad + v_0(x_2)\pi_0(x_1)e^{ik|x_+|}|x_+|^{-1/2}z_+(x,y;\zeta,d)|y_+|^{-1/2}e^{ik|y_+|}\pi_0(y_1)\tilde{v}_0(y_2),\end{aligned} \tag{8.1}$$

where $z_\pm(x,y;\zeta,d)$ satisfies

$$\left|\partial_x^m\partial_y^n z_\pm(x,y;\zeta,d)\right| = O\left(d^{-(2-\rho)-(|m|+|n|)\rho}\right)$$

uniformly in $x \in \Pi_0 \cap \operatorname{supp} v_0$ and $y \in \Pi_0 \cap \operatorname{supp}\tilde{v}_0$ and in $\zeta \in D_d$. We evaluate the Hilbert–Schmidt norm of the integral operator. We have shown in the previous section that

$$\left|e^{ik|x_\pm|}/|x_\pm|^{1/2}\right| + \left|e^{ik|y_\pm|}/|y_\pm|^{1/2}\right| = O\left(d^{-7/32+27\delta_0/32}\right).$$

Hence the Hilbert–Schmidt norm of $Z_1(\zeta,d)$ obeys

$$\left\|Z_1(\zeta,d)\right\|_{\mathrm{HS}} = O\left(d^{-7/16+27\delta_0/16}\right)O\left(d^{-2+\rho}\right)O\left(d^{1+\rho}\right).$$

We can take $\delta_0 > 0$ and $\rho - 1/2 > 0$ so small that

$$-7/16 + 27\delta_0/16 - 1 + 2\rho < -3/8.$$

This yields the desired bound on $\|Z_1(\zeta,d)\|$. The second statement is proved and the proof is complete. □

Proof of Lemma 7.3. Throughout the proof, the notation k is used with the meaning $k = \zeta_0^{1/2}$. Let $s_-(x)$ be the characteristic function of

$$S_- = \left\{x \in S_0\colon\ -d/4 < x_1 < -d/8,\ |x_2| < L_0/2\right\}.$$

To prove the lemma, it suffices to show that $s_-\tilde{u}_0$ and $s_-\tilde{w}_0$ never vanish identically.

We first consider $s_-\tilde{u}_0$. Since $s_-\chi_{+d} = 0$ and $s_-s_0\chi_{-d} = s_-$ (and similarly for $\tilde{\chi}_{\pm d}$ by (5.12)), we have $s_-\tilde{u}_0 = s_-R(\zeta_0;K_{-d})\pi_0u_0$ by (7.13). For the same reason as in the proof of Lemma 7.1, we further have that the operator

$$s_-R(\zeta_0;K_{-d})T_{10}(\zeta_0;d) = s_-R(\zeta_0;K_{-d})v_1T(\zeta_0;d)\tilde{v}_0$$

is of class $\mathrm{Op}(d^{-N})$. Thus it follows from (7.12) that

$$s_-\tilde{u}_0 \sim s_-R(\zeta_0;K_{-d})(\tilde{u}_- - \lambda_d\tilde{u}_+)$$

with negligible error estimates $O(d^{-N})$, where $\lambda_d \neq 0$ is as in (7.11). We recall from (7.6) that $\tilde{u}_\pm$ is expanded as

$$\tilde{u}_\pm = \big(Id + Z_{\mathrm{rem}}(\zeta_0; d)\big)^{-1} u_\pm \sim u_\pm - Z_1(\zeta_0; d)u_\pm + (-1)^2 Z_1(\zeta_0; d)^2 u_\pm + \cdots.$$

By definition (7.4), $u_\pm$ has support in $\Pi_0 \cap \operatorname{supp} v_0$ and $e^{ik|x_\pm|}$ as a phase factor. The wave function $u_\pm$ describes the state outgoing from the center $d_\pm$ to the region $\operatorname{supp} v_0 \cap \Pi_0$. Intuitively, the particle described by u_- never passes over S_-, and hence we have

$$s_- R(\zeta_0; K_{-d})u_- = s_-\big(R(\zeta_0; K_{0d}) + R_{\mathrm{sc}}(\zeta_0; K_{-d})\big)u_- = O\big(d^{-N}\big).$$

This is rigorously verified by repeated use of the integration by parts, since the resolvent kernel $R(\zeta_0; K_{0d})(x, y)$ behaves like

$$R(\zeta_0; K_{0d})(x, y) = c_0(\zeta_0)e^{ik|x-y|}|x-y|^{-1/2}\big(e^{i(g_{0d}(x)-g_{0d}(y))} + O\big(|x-y|^{-1}\big)\big)$$

by the asymptotic formula

$$H_0(z) = (2/\pi z)^{1/2} \exp\big[i(z - \pi/4)\big]\big(1 + O\big(|z|^{-1}\big)\big), \quad |z| \to \infty,$$

of the Hankel function $H_0(z)$ and since the kernel $R_{\mathrm{sc}}(\zeta_0; K_{-d})(x, y)$ takes the asymptotic form as in Proposition 6.4 (2). Thus the leading term of $s_-\tilde{u}_0$ comes from

$$s_- R(\zeta_0; K_{-d})u_+ = s_- R(\zeta_0; K_{0d})u_+ + s_- R_{\mathrm{sc}}(\zeta_0; K_{-d})u_+.$$

The first term on the right side describes the particle going directly from d_+ to $x \in S_-$, while the second one describes the particle going from d_+ to $x \in S_-$ after being scattered by the obstacle $\mathcal{O}_{-d}$. We apply the method of steepest descent to see the behaviors over S_- of the two terms. The first term behaves like

$$\big(s_- R(\zeta_0; K_{0d})u_+\big)(x) = -e^{ik|x_+|}|x_+|^{-1/2}\big(c_0(\zeta_0)f_+(\omega_1 \to -\omega_1; \zeta_0) + O\big(d^{-1}\big)\big).$$

Since $h(\zeta_0; d) = 1$, we have

$$\big(e^{ikd}/d^{1/2}\big) f_+(\omega_1 \to -\omega_1; \zeta) f_-(-\omega_1 \to \omega_1; \zeta)\big|_{\zeta=\zeta_0} = e^{-ikd}d^{1/2} \sim 1,$$

and hence

$$\big(s_- R_{\mathrm{sc}}(\zeta_0; K_{-d})u_+\big)(x) = -e^{ik|x_-|}|x_-|^{-1/2}\big(c_0(\zeta_0)e^{-ikd}d^{1/2} + O\big(d^{-1}\big)\big)$$

for the second term.

We have to deal with remainder terms such as $s_- R(\zeta_0; K_{-d})Z_1(\zeta_0; d)u_\pm$. We recall the kernel of $Z_1(\zeta_0; d)$ from (8.1). Then we obtain

$$\begin{aligned}\big(Z_1(\zeta_0;d)u_+\big)(x) &= O\big(d^{-2+\rho}\big)\pi_0(x_1)v_0(x_2)e^{ik|x_-|}|x_-|^{-1/2},\\ \big(Z_1(\zeta_0;d)u_-\big)(x) &= O\big(d^{-2+\rho}\big)\pi_0(x_1)v_0(x_2)e^{ik|x_+|}|x_+|^{-1/2}\end{aligned}$$

as a simple application of the method of steepest descent, where ρ is the constant fixed in (7.1). Hence it follows that

$$\big(s_-R(\zeta_0;K_{-d})Z_1(\zeta_0;d)u_+\big)(x) = O\big(d^{-N}\big)$$

and also we have

$$\big(s_-R(\zeta_0;K_{-d})Z_1(\zeta_0;d)u_-\big)(x) = O\big(d^{-1+\rho}\big)\big(e^{ik|x_+|}|x_+|^{-1/2} + e^{ik|x_-|}|x_-|^{-1/2}\big)$$

by the same argument as used to derive the leading term above. A similar argument applies to the other remainder terms associated with $Z_1(\zeta_0;d)^k u_\pm$. We have

$$\big(s_-R(\zeta_0;K_{-d})Z_1(\zeta_0;d)^k u_\pm\big)(x) = O\big(d^{-k(1-\rho)}\big)\big(e^{ik|x_+|}|x_+|^{-1/2} + e^{ik|x_-|}|x_-|^{-1/2}\big).$$

Thus we combine the results obtained above to see that $\tilde{u}_0$ defined by (7.13) behaves like

$$\begin{aligned}\tilde{u}_0 = \lambda_d c_0(\zeta_0)\big\{&e^{ik|x_-|}|x_-|^{-1/2}\big(e^{-ikd}d^{1/2} + O\big(d^{-1+\rho}\big)\big)\\ &+ e^{ik|x_+|}|x_+|^{-1/2}\big(f_+(\omega_1\to-\omega_1;\zeta_0) + O\big(d^{-1+\rho}\big)\big)\big\}\end{aligned}$$

over S_-. Hence $\tilde{u}_0$ never vanishes identically.

Next we shall show that $s_-\tilde{w}_0 = s_-R(\zeta_0;K_{-d})^*\Gamma_0^* w_0$ does not vanish, either. This is verified in a way similar to the argument used for $\tilde{u}_0$. We give only a sketchy explanation. We recall the representation for w_0 from (7.12). Then the operator

$$s_-R(\zeta_0;K_{-d})^* v_1 T_{01}(\zeta_0;d)^* = \big(T_{01}v_1R(\zeta_0;K_{-d})s_-\big)^* = \mathrm{Op}\big(d^{-N}\big)$$

becomes negligible, and $\tilde{w}_\pm$ is expanded as

$$\tilde{w}_\pm = \big(Id + Z_{\mathrm{rem}}(\zeta_0;d)^*\big)^{-1}w_\pm \sim w_\pm - Z_1(\zeta_0;d)^*w_\pm + \cdots$$

by (7.9). As is seen from (7.4), the wave function

$$\Gamma_0^* v_0 w_\pm = [\chi_{-d}, K_{0d}]v_0w_\pm = -v_0\Gamma_0 w_\pm$$

has support in $\Pi_0 \cap \operatorname{supp} v_0$, and it follows that

$$s_-R(\zeta_0;K_{-d})^*\Gamma_0^* v_0 w_- = O\big(d^{-N}\big).$$

In fact, the wave function w_- describes the state incoming to the center d_- from the region $\operatorname{supp}\tilde{v}_0 \cap \Pi_0$. Such a particle never passes over S_-. Thus the leading term comes from

$$s_- R(\zeta_0; K_{-d})^* \Gamma_0^* v_0 w_+ = -s_-\big(R(\zeta_0; K_{0d})^* + R_{sc}(\zeta_0; K_{-d})^*\big)\Gamma_0 v_0 w_+, \tag{8.2}$$

where $\Gamma_0 v_0 w_+$ behaves like

$$\Gamma_0 v_0 w_+ \sim -2i\bar{k}\chi'_{-d}(x_1)v_0(x_2)e^{-i\bar{k}|x_+|}|x_+|^{-1/2},$$

because $\partial|x_+|/\partial x_1 = -1 + O(d^{-1})$. The two terms on the right side of (8.2) behave as follows:

$$\begin{aligned}
&\big(s_- R(\zeta_0; K_{0d})^* \Gamma_0 v_0 w_+\big)(x) = e^{-i\bar{k}|x_+|}|x_+|^{-1/2}\big(1 + O\big(d^{-1}\big)\big),\\
&\big(s_- R_{sc}(\zeta_0; K_{-d})^* \Gamma_0 v_0 w_+\big)(x)\\
&\quad = e^{-i\bar{k}|x_-|}|x_-|^{-1/2}\big(\bar{f}_-(-\omega_1 \to \omega_1; \zeta_0)e^{-i\bar{k}d}d^{-1/2} + O\big(d^{-1}\big)\big).
\end{aligned}$$

As a result, we have

$$\begin{aligned}
\tilde{w}_0 = \bar{h}_{+d}\big\{&e^{-i\bar{k}|x_+|}|x_+|^{-1/2}\big(1 + O\big(d^{-1+\rho}\big)\big)\\
&+ e^{-i\bar{k}|x_-|}|x_-|^{-1/2}\big(\bar{f}_-(-\omega_1 \to \omega_1; \zeta_0)e^{-i\bar{k}d}d^{-1/2} + O\big(d^{-1+\rho}\big)\big)\big\}
\end{aligned}$$

over S_-, where $h_{+d} \neq 0$ is as in (7.12). This proves that $\tilde{w}_0$ never vanishes identically and the proof of the lemma is complete. □

9. Complex scaling method for point-like fields

This section is devoted to proving Propositions 6.1 and 6.2 which have remained unproved in Section 6. Proposition 6.2 is much easier to prove than Proposition 6.1, although the proof of both the propositions is based on the same idea. We give only a sketch for the proof of Proposition 6.2.

Proposition 6.1 is verified through a series of lemmas. As the first step toward the proof, we establish the representation for the resolvent kernel $R(\zeta; P)(x, y)$ of $P = H(\alpha\Phi)$ in terms of a contour integral in the complex plane. The derivation is based on the following formula

$$H_\mu(Z)J_\mu(z) = \frac{1}{i\pi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{Z^2 + z^2}{2t}\right)I_\mu\left(\frac{Zz}{t}\right)\frac{dt}{t}, \quad |z| \leqslant |Z|,$$

for the product of Bessel functions (see [21, p. 439]), where the contour is taken to be rectilinear with corner at $\kappa + i0$, $\kappa > 0$ being fixed arbitrarily. We apply to (2.7) this formula with $Z = k(|x| \vee |y|)$ and $z = k(|x| \wedge |y|)$, $k = \zeta^{1/2}$. If we write $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates, then $R(\zeta; P)(x, y)$ is represented as

$$R(\zeta; P)(x,y) = \frac{1}{4\pi} \sum_l e^{il\psi} \int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right) I_\nu\left(\frac{\zeta|x||y|}{t}\right) \frac{dt}{t} \tag{9.1}$$

with $\nu = |l - \alpha|$, where $\psi = \theta - \omega$. If, in particular, $\alpha = 0$, then the resolvent kernel $R(\zeta; H_0)(x, y) = (i/4)H_0(k|x - y|)$ of H_0 is given by

$$R(\zeta; H_0)(x,y) = \frac{1}{4\pi} \sum_l e^{il\psi} \int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right) I_l\left(\frac{\zeta|x||y|}{t}\right) \frac{dt}{t},$$

where $I_l(w) = I_{|l|}(w)$ is defined by

$$I_l(\omega) = (1/\pi) \int_0^{\pi} e^{w\cos\rho} \cos(l\rho)\, d\rho$$

according to (2.5). Since the series $\sum_l e^{il\psi} I_l(w)$ converges to $e^{w\cos\psi}$ by the Fourier expansion and since

$$|x-y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\psi,$$

the kernel has the integral representation

$$R(\zeta; H_0)(x,y) = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right) \frac{dt}{t}. \tag{9.2}$$

We now use formula (2.5) for $I_\nu(w)$ to calculate the series

$$I(w,\psi) = \sum_l e^{il\psi} I_\nu(w), \quad \nu = |l-\alpha|,$$

in the integrand above, where $w = \zeta|x||y|/t$. Then $I(w, \psi)$ is decomposed into the sum

$$I(w,\psi) = I_{\mathrm{fr}}(w,\psi) + e^{-w} I_{\mathrm{sc}}(w,\psi),$$

where

$$I_{\mathrm{fr}}(w,\psi) = (1/\pi) \sum_l e^{il\psi} \int_0^{\pi} e^{w\cos\rho} \cos(\nu\rho)\, d\rho,$$

$$I_{\rm sc}(w,\psi) = -(1/\pi)\sum_l e^{il\psi}\sin(\nu\pi)\int_0^\infty e^{-w(\cosh p-1)-\nu p}\,dp.$$

We have $I_{\rm fr}(w,\psi) = e^{i\alpha\psi}e^{w\cos\psi}$ for $|\psi| < \pi$ by the Fourier expansion and

$$I_{\rm sc}(w,\psi) = C_\alpha e^{i\lfloor\alpha\rfloor\psi}\int_{-\infty}^{\infty} e^{-w(\cosh p-1)}\frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\,dp \tag{9.3}$$

by the same argument as used in calculating the scattering wave $\varphi_{\rm sc}$ in Section 2, where

$$C_\alpha = (-1)^{\lfloor\alpha\rfloor+1}\sin(\alpha\pi)/\pi \tag{9.4}$$

and $\beta = \alpha - \lfloor\alpha\rfloor$, $0 \leqslant \beta < 1$. Thus the resolvent kernel $R(\zeta;P)(x,y)$ admits the decomposition

$$R(\zeta;P)(x,y) = R_{\rm fr}(x,y;\zeta) + R_{\rm sc}(x,y;\zeta), \tag{9.5}$$

where

$$R_{\rm fr}(x,y;\zeta) = \frac{1}{4\pi}e^{i\alpha\psi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right)\frac{dt}{t}, \tag{9.6}$$

$$R_{\rm sc}(x,y;\zeta) = \frac{1}{4\pi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2} - \frac{\zeta(|x|+|y|)^2}{2t}\right)I_{\rm sc}\left(\frac{\zeta|x||y|}{t},\psi\right)\frac{dt}{t}. \tag{9.7}$$

We should note that (9.3) is true only for $|\psi| < \pi$. If $\psi = \pm\pi$, then the denominator $e^p + e^{-i\psi}$ in (9.3) vanishes at $p = 0$. If α is an integer, then $I_{\rm sc}(\zeta|x||y|/t,\psi)$ vanishes, and hence so does $R_{\rm sc}(x,y;\zeta)$.

Proof of Proposition 6.1 (1). We recall the notation from (6.2). We set

$$\sigma_d(x,y) = r_d(x) + r_d(y), \qquad \rho_d(x,y) = r_d(x)r_d(y). \tag{9.8}$$

We assume that $|x_2| + |y_2| > Ld$ for $L \gg 1$ fixed. The three lemmas below are verified after completing the proof of statement (1).

Lemma 9.1. *We have* $\mathrm{Re}(\zeta\rho_d(x,y)/t) > 0$ *for* $\zeta = E + i\eta \in D_d$.

Lemma 9.2. *Let* $\psi_d(x,y)$ *be defined by* (6.1). *Then we have*

$$\left|\mathrm{Im}\, e^{-i\psi_d(x,y)}\right| \geqslant c\big(|x| + |y|\big)^{-1}$$

for some $c > 0$.

Lemma 9.3. *If* $0 < t < \kappa$, *then*

$$\operatorname{Re}\bigl(\zeta\sigma_d(x,y)^2/t\bigr) \geqslant c\bigl(|x|+|y|\bigr)^2/t$$

for some $c > 0$, *and if* $0 < s < M(|x|+|y|)$ *for* $t = \kappa + is$, $M \gg 1$ *being fixed arbitrarily, then there exists* $\sigma > 0$ *independent of* L *(but depending on* M*) such that*

$$\operatorname{Re}\bigl(\zeta\sigma_d(x,y)^2/t\bigr) \geqslant \sigma L \log\bigl(|x|+|y|\bigr).$$

By Lemmas 9.1 and 9.2, the integral $I_{\mathrm{sc}}(w,\psi)$ defined by (9.3) converges for $w = \zeta\rho_d(x,y)/t$ and $\psi = \psi_d(x,y)$, and hence it follows from (9.5) that $Q_d(x,y;\zeta)$ defined by (6.3) admits the decomposition

$$Q_d(x,y;\zeta) = R_{\mathrm{fr}}\bigl(j_d(x), j_d(y);\zeta\bigr) + R_{\mathrm{sc}}\bigl(j_d(x), j_d(y);\zeta\bigr).$$

By (9.2) and (9.6), we have

$$R_{\mathrm{fr}}\bigl(j_d(x), j_d(y);\zeta\bigr) = \exp\bigl(i\alpha\psi_d(x,y)\bigr)R(\zeta;H_0)\bigl(j_d(x), j_d(y)\bigr).$$

Thus the first term in the decomposition is obtained. We show that the second term

$$Q_{\mathrm{sc}}(x,y;\zeta) = R_{\mathrm{sc}}\bigl(j_d(x), j_d(y);\zeta\bigr)$$

satisfies $O((|x|+|y|)^{-\sigma L})$ as in the proposition. To see this, we consider the integral

$$\int_0^{\kappa+i\infty} \chi_\infty\left(\frac{\operatorname{Im} t}{M(|x|+|y|)}\right)\exp\left(\frac{t}{2} - \frac{\zeta\sigma_d(x,y)^2}{2t}\right) I_{\mathrm{sc}}\left(\frac{\zeta\rho_d(x,y)}{t}, \psi_d(x,y)\right)\frac{dt}{t}$$

and show that it obeys $O((|x|+|y|)^{-N})$ for any $N \gg 1$. This, together with Lemma 9.3, implies the desired bound on $Q_{\mathrm{sc}}(x,y;\zeta)$. We evaluate the integral by repeated use of integration by parts. Assume that $\operatorname{Im} t \geqslant M(|x|+|y|)$ with $M \gg 1$. Then we make use of the following relations: If $|p| < 2$, then

$$\bigl|\partial_t\bigl(t/2 - \zeta\sigma_d(x,y)^2/2t - \zeta\rho_d(x,y)(\cosh p - 1)/t\bigr)\bigr| > c > 0$$

and if $|p| > 1$, $|\partial_t(t/2 - \zeta\sigma_d(x,y)^2/2t)| > c > 0$ and

$$\partial_t e^{-(\zeta\rho_d(x,y)/t)(\cosh p - 1)} = -t^{-1}\bigl((\cosh p - 1)/\sinh p\bigr)\partial_p e^{-(\zeta\rho_d(x,y)/t)(\cosh p - 1)}.$$

A similar argument applies to the derivatives $\partial Q_{\mathrm{sc}}/\partial x_1$ and $\partial Q_{\mathrm{sc}}/\partial y_1$. Thus the proof of statement (1) is complete. □

Proof of Lemma 9.1. We set $w = \zeta\rho_d(x,y)/t$ with $\rho_d(x,y)$ as in (9.8). Recall $\eta_d(t) = O((\log d)/d)$ from (5.1). We compute

$$r_d(x)^2 = x_1^2 + \bigl(1 + 2i\eta_d(x_2) + O\bigl(\bigl((\log d)/d\bigr)^2\bigr)\bigr)x_2^2,$$

so that

$$r_d(x)^2 = |x|\bigl(1 + i\eta_d(x_2)\bigl(x_2/|x|\bigr)^2 + O\bigl(\bigl((\log d)/d\bigr)^2\bigr)\bigr) \tag{9.9}$$

and similarly for $r_d(y)$. Hence

$$\rho_d(x,y) = |x||y|\bigl(1 + i\bigl(\eta_d(x_2)\bigl(x_2/|x|\bigr)^2 + \eta_d(y_2)\bigl(y_2/|y|\bigr)^2\bigr) + O\bigl(\bigl((\log d)/d\bigr)^2\bigr)\bigr).$$

If t, $0 < t < \kappa$, is positive, then it is easy to see that $\operatorname{Re} w > 0$. If $t = \kappa + is$ with $s > 0$, then we have

$$\zeta/t = \bigl(\kappa^2 + s^2\bigr)^{-1}\bigl((E\kappa + \eta s) - i(Es - \eta\kappa)\bigr), \tag{9.10}$$

and hence $\operatorname{Re} w$ behaves like

$$\operatorname{Re} w \sim \frac{|x||y|}{\kappa^2 + s^2}\left(\left(E\eta_d(x_2)\left(\frac{x_2}{|x|}\right)^2 + E\eta_d(y_2)\left(\frac{y_2}{|y|}\right)^2 + \eta\right)s + E\kappa\right) \tag{9.11}$$

for $d \gg 1$. By definition, $\eta_d(t)$ satisfies

$$\eta_d(x_2) = L_1(\log d)/d \quad \text{or} \quad \eta_d(y_2) = L_1(\log d)/d \tag{9.12}$$

with $L_1 \gg 1$ for $|x_2| + |y_2| > Ld$. Since $|\eta| \leqslant 2E_0^{1/2}(\log d)/d$ for $\zeta = E + i\eta \in D_d$, we have that $\operatorname{Re} w > 0$ for $t = \kappa + is$ also. □

Proof of Lemma 9.2. As already mentioned, the denominator $e^p + e^{-i\psi_d(x,y)}$ of the integrand in (9.3) never vanishes but takes values close to 0 around $p = 0$, provided that $\psi_d(x,y) \sim \pm\pi$. This is the case when $x_2/x_1 \gg 1$ and $-y_2/y_1 \gg 1$ or when $-x_2/x_1 \gg 1$ and $y_2/y_1 \gg 1$. We consider only the former case. We set

$$\theta_d(x) = 3\pi/2 - \gamma\bigl(j_d(x); -\omega_1\bigr), \qquad \omega_d(y) = \gamma\bigl(j_d(y); -\omega_1\bigr) - \pi/2.$$

Then

$$\pi - \psi_d(x,y) = \pi - \bigl(\gamma\bigl(j_d(x); -\omega_1\bigr) - \gamma\bigl(j_d(y); -\omega_1\bigr)\bigr) = \theta_d(x) + \omega_d(y)$$

and $e^{-i\psi_d} = -e^{i\theta_d}e^{i\omega_d}$. We note that $|\theta_d(x)| \ll 1$. Since

$$\tan\theta_d = x_1/\bigl(\bigl(1 + i\eta_d(x_2)\bigr)x_2\bigr)$$

with $\eta_d(x_2) = O((\log d)/d)$, the function $\theta_d(x)$ behaves like

$$\theta_d = (x_1/x_2)\bigl(1 + O\bigl((\log d)/d\bigr) + O\bigl((x_1/x_2)^2\bigr)\bigr).$$

Similarly we have

$$\omega_d = -(y_1/y_2)\bigl(1 + O\bigl((\log d)/d\bigr) + O\bigl((y_1/y_2)^2\bigr)\bigr).$$

Thus the desired bound can be obtained in the following way:

$$\bigl|\operatorname{Im} e^{-i\psi_d(x,y)}\bigr| \geqslant c\bigl(x_1/|x| + y_1/|y|\bigr) \geqslant c\bigl(|x| + |y|\bigr)^{-1}.$$

This proves the lemma. □

Proof of Lemma 9.3. We set $w = \zeta\sigma_d(x,y)^2/t$ with $\sigma_d(x,y)$ as in (9.8). If $0 < t < \kappa$, then it is easy to see that $\operatorname{Re} w > c(|x| + |y|)^2/t$ for some $c > 0$. Assume that $t = \kappa + is$ with $0 < s < M(|x| + |y|)$. If we take (9.9), (9.10) and (9.12) into account, then a simple computation yields

$$\operatorname{Re} w > c\bigl((\log d)/d\bigr)\bigl(|x| + |y|\bigr)$$

for another $c > 0$. Since $(\log p)/p$ is decreasing for $p \gg 1$, we have

$$\bigl(\log\bigl(|x| + |y|\bigr)\bigr)/\bigl(|x| + |y|\bigr) \leqslant \bigl(\log(Ld)\bigr)/(Ld) \leqslant (2/L) \times \bigl((\log d)/d\bigr)$$

for $|x_2| + |y_2| > Ld$. This implies that $\operatorname{Re} w \geqslant \sigma L \log(|x| + |y|)$ for some $\sigma > 0$, and hence the lemma follows. □

Proof of Proposition 6.1 (2). The proof is divided into several steps. Throughout the proof, we assume that $|x_2| + |y_2| < 2Ld$ for $L \gg 1$ fixed.

(1) We accept the two lemmas below as proved.

Lemma 9.4. *Let κ be fixed as*

$$\kappa = M^2 \log d. \tag{9.13}$$

Then we can take $M \gg 1$ (M being dependent on L) so large that

$$\operatorname{Re}\bigl(\zeta\rho_d(x,y)/t\bigr) \geqslant 0,$$

if $0 < t < \kappa$ or $0 < s < 2M(|x| + |y|)$ for $t = \kappa + is$.

Lemma 9.5. *Let $M \gg 1$ be as in Lemma 9.4. Assume that $s > M(|x|+|y|)$ and define $Q_{\mathrm{rem}}(x,y;\zeta)$ by*

$$Q_{\mathrm{rem}} = \sum_l e^{il\psi_d(x,y)}\tilde{Q}_l(x,y;\zeta), \tag{9.14}$$

where

$$\tilde{Q}_l = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \chi_\infty\left(\frac{\operatorname{Im} t}{M(|x|+|y|)}\right)\exp\left(\frac{t}{2}-\frac{\zeta\sigma_d(x,y)^2}{2t}\right)I_\nu\left(\frac{\zeta\rho_d(x,y)}{t}\right)\frac{dt}{t}$$

with $\nu = |l-\alpha|$. Then we have $Q_{\mathrm{rem}}(x,y;\zeta) = O(d^{-N})$ for any $N \gg 1$ uniformly in $\zeta \in D_d$. A similar result remains true for $\partial Q_{\mathrm{rem}}/\partial x_1$ and $\partial Q_{\mathrm{rem}}/\partial y_1$.

(2) It should be noted that $\operatorname{Re}(\zeta\rho_d(x,y)/t)$ is not necessarily nonnegative for $t = \kappa+is$ with $s \gg 1$, so that the integral $I_{\mathrm{sc}}(w,\psi)$ defined by (9.3) is not convergent. However, Lemma 9.5 shows that the contribution from this interval is negligible. By (9.1) and (9.5)–(9.7), the kernel $Q_d(x,y;\zeta)$ under consideration admits the decomposition

$$Q_d(x,y;\zeta) = Q_{0d}(x,y;\zeta) + Q_{1d}(x,y;\zeta) + Q_{\mathrm{rem}}(x,y;\zeta),$$

where $Q_{0d}(x,y;\zeta)$ and $Q_{1d}(x,y;\zeta)$ are defined by

$$Q_{0d} = \frac{1}{4\pi}e^{i\alpha\psi_d(x,y)}\int_0^{\kappa+i\infty} \chi_0\left(\frac{\operatorname{Im} t}{M(|x|+|y|)}\right)\exp\left(\frac{t}{2}-\frac{\zeta r_d(x,y)^2}{2t}\right)\frac{dt}{t},$$

$$Q_{1d} = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \chi_0\left(\frac{\operatorname{Im} t}{M(|x|+|y|)}\right)\exp\left(\frac{t}{2}-\frac{\zeta\sigma_d^2}{2t}\right)I_{\mathrm{sc}}\left(\frac{\zeta\rho_d}{t},\psi_d\right)\frac{dt}{t}$$

with $\sigma_d = \sigma_d(x,y)$, $\rho_d = \rho_d(x,y)$ and $\psi_d = \psi_d(x,y)$. Lemma 9.4 makes the integral $I_{\mathrm{sc}}(\zeta\rho_d/t,\psi_d)$ convergent. Since

$$\left|\partial_t\left(t/2-\zeta r_d(x,y)^2/2t\right)\right| > c > 0$$

for $\operatorname{Im} t > M(|x|+|y|)$, it is easy to see from (9.2) that $Q_{0d}(x,y;\zeta)$ takes the form

$$Q_{0d} = \exp\left(i\alpha\psi_d(x,y)\right)R(\zeta;H_0)\left(j_d(x),j_d(y)\right) + O\left(d^{-N}\right)$$

for any $N \gg 1$. Thus the first term is obtained from $Q_{0d}(x,y;\zeta)$.

(3) We consider the integral $Q_{1d}(x,y;\zeta)$. If $|x|+|y| < 2Ld$, then the denominator $e^p + e^{-i\psi_d(x,y)}$ is away from zero uniformly in x, y and p, and the integral $I_{\mathrm{sc}}(\zeta\rho_d/t,\psi_d)$

is uniformly bounded. The main contribution to $Q_{1d}(x, y; \zeta)$ comes from the integral over the interval in $t = \kappa + is$ with

$$\big(|x| + |y|\big)/2M < s < 2M\big(|x| + |y|\big).$$

In fact, we have

$$\big|t^{-1} \exp\big(t/2 - \zeta\sigma_d(x, y)^2/2t\big)\big| = O\big(d^{-N}\big),$$

provided that $0 < t < \kappa$ or $0 < s < 2(|x|+|y|)^{1-\delta}$, δ being fixed small enough ($0 < \delta \ll 1$). If $(|x| + |y|)^{1-\delta} < s < (|x| + |y|)/M$, then

$$\big|\partial_t\big(t/2 - \zeta\sigma_d(x, y)^2/2t\big)\big| > c > 0.$$

This enables us to repeat the same argument as used in proving statement (1) for the case $s > M(|x| + |y|)$. Thus $Q_{\mathrm{sc}}(x, y; \zeta)$ takes the asymptotic form

$$Q_{\mathrm{sc}} = \exp\big(ikr_d(x)\big)\tilde{q}_0(x, y; \zeta) \exp\big(ikr_d(y)\big) + O\big(d^{-N}\big),$$

where $\tilde{q}_0$ is defined by the integral

$$\tilde{q}_0 = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \chi_M\left(\frac{\operatorname{Im} t}{|x| + |y|}\right) \exp\left(\frac{t}{2}\left(1 - \frac{ik\sigma_d}{t}\right)^2\right) I_{\mathrm{sc}}\left(\frac{\zeta\rho_d}{t}, \psi_d\right) \frac{dt}{t}$$

with $\chi_M(r) = \chi_0(r/M)\chi_\infty(Mr)$.

(4) We assert that the function $\tilde{q}_0(x, y; \zeta)$ defined above has the properties in the proposition. We recall the integral representation for $I_{\mathrm{sc}}(\zeta\rho_d/t, \psi_d)$ from (9.3). We note that $\operatorname{Re}(\zeta\rho_d/t) > 0$ by Lemma 9.4 and $|\rho_d/t| \sim d$. This allows us to apply the stationary phase method [13, Theorem 7.7.5] to this integral. Then it satisfies

$$\big|(\partial/\partial x_2)^m(\partial/\partial y_2)^n I_{\mathrm{sc}}\big| = \big(\rho_d(x, y)/t\big)^{-1/2} O\big(d^{-m-n}\big).$$

Next we see the asymptotic behavior of the integral with respect to t. The phase function

$$(t/2)\big(1 - ik\sigma_d(x, y)/t\big)^2 \longrightarrow ik\sigma_d(x, y)\tau^2/\big(2(1 - \tau)\big)$$

is transformed under the change of the variable $\tau = 1 - ik\sigma_d(x, y)/t$. The stationary point lies at $\tau = 0$. We deform the contour of the integral suitably in the complex domain by analyticity and use the method of steepest decent in a neighborhood of $\tau = 0$. Then we have

$$\big|(\partial/\partial x_2)^m(\partial/\partial y_2)^n \tilde{q}_0\big| = O\big(d^{-1-m-n}\big).$$

As is easily seen, $|\exp(-ikr_d(x))| = O(d^\mu)$ for some $\mu > 0$, and similarly for $\exp(-ikr_d(y))$. Hence we may write

$$O(d^{-N}) = \exp(ikr_d(x))O(d^{-N+2\mu})\exp(ikr_d(y)).$$

This implies that $Q_{\mathrm{sc}}(x,y;\zeta)$ takes the desired form (6.6).

(5) We shall show that $q_0(x,y;\zeta)$ takes the asymptotic form (6.7). Assume that $|x_2| < d$ and $|y_2| < d$. Then the mapping j_d defined by (5.1) acts as $j_d(x) = x$ and $j_d(y) = y$, so that

$$\sigma_d(x,y) = |x| + |y|, \qquad \rho_d(x,y) = |x||y|, \qquad \psi_d(x,y) = \psi = \theta - \omega$$

for $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |x|\sin\omega)$. We calculate only the leading term. It is not difficult to see that the remainder term satisfies (6.8). By the stationary phase method, the integral $I_{\mathrm{sc}}(\zeta|x||y|/t,\psi)$ behaves like

$$I_{\mathrm{sc}}(\zeta|x||y|/t,\psi) = C_\alpha e^{i\lfloor\alpha\rfloor\psi}(2\pi)^{1/2}(\zeta|x||y|/t)^{-1/2}\left(\frac{e^{i\psi}}{1+e^{i\psi}} + O(d^{-1})\right),$$

where C_α is the constant defined by (9.4). We further apply the method of steepest decent to the integral in t as in step (4). Then we have

$$q_0(x,y;\zeta) = \left(\frac{1}{4\pi}\right)C_\alpha e^{i\lfloor\alpha\rfloor\psi}(2\pi)e^{i\pi/2}(\zeta|x||y|)^{-1/2}\left(\frac{e^{i\psi}}{1+e^{i\psi}} + O(d^{-1})\right).$$

We use (2.8) and (9.4) to compute

$$\left(\frac{1}{4\pi}\right)C_\alpha e^{i\lfloor\alpha\rfloor\psi}(2\pi)e^{i\pi/2}\zeta^{-1/2} = c_0(\zeta)\left(-\left(\frac{2}{\pi}\right)^{1/2}e^{i\pi/4}\zeta^{-1/4}\sin(\alpha\pi)e^{i\lfloor\alpha\rfloor(\theta-(\omega+\pi))}\right).$$

We further write

$$e^{i\psi}/(1+e^{i\psi}) = -e^{i(\theta-(\omega+\pi))}/(1-e^{i(\theta-(\omega+\pi))})$$

and recall the definition of the scattering amplitude from (1.14). Then we see that $q_0(x,y;\zeta)$ takes the desired asymptotic form (6.7). This proves statement (2). □

It remains to prove Lemmas 9.4 and 9.5.

Proof of Lemma 9.4. Note that $|\eta| \leqslant 2E_0^{1/2}((\log d)/d)$ for $\zeta \in D_d$. By assumption, $|x| + |y| < 2Ld$, and hence $0 < s < 2M(|x| + |y|) < 4MLd$. Thus relation (9.11) enables us to take $M \gg 1$ in (9.13) so large that $\mathrm{Re}(\zeta\rho_d(x,y)/t) \geqslant 0$ for $t = \kappa + is$ as in the lemma. This completes the proof. □

Proof of Lemma 9.5. To prove the lemma, we employ the formula

$$I_\mu(w) = \frac{e^{-i\mu\pi/2}}{\pi}\left\{\int_0^\pi \cos(\mu\rho - iw\sin\rho)\,d\rho - \sin(\mu\pi)\int_0^\infty e^{-iw\sinh p - \mu p}\,dp\right\}$$

for $\operatorname{Im} w \leqslant 0$, which follows as an immediate consequence of the relation $I_\mu(w) = e^{-i\mu\pi/2}J_\mu(iw)$ (see [21, p. 176]). As in the proof of Lemma 9.1, we have

$$\operatorname{Im}\bigl(\zeta\rho_d(x,y)/t\bigr) \sim -|x||y|\bigl(Es/\bigl(\kappa^2+s^2\bigr)\bigr) < 0$$

for $t = \kappa + is$ with $s > M(|x|+|y|) > cd$. We insert $I_\nu(\zeta\rho_d(x,y)/t)$ into the integral representation for $\tilde{Q}_l(x,y;\zeta)$ and evaluate the resulting integral by partial integration for each l with $|l| < d$. To do this, we use the relations

$$\bigl|\partial_t\bigl(t - \zeta\sigma_d(x,y)^2/t \pm 2\bigl(\zeta\rho_d(x,y)/t\bigr)\sin\rho\bigr)\bigr| > c > 0,$$
$$\bigl|\partial_t\bigl(t - \zeta\sigma_d(x,y)^2/t - 2i\bigl(\zeta\rho_d(x,y)/t\bigr)\sinh p\bigr)\bigr| > c > 0$$

for $t = \kappa + is$ with $s > M(|x|+|y|)$ uniformly in ρ, $0 < \rho < \pi$, and in p, $0 < p < 2$. If $p > 1$, then we use $|\partial_t(t - \zeta\sigma_d(x,y)^2/t)| > c > 0$ and

$$e^{-\nu p}\bigl(\partial_t e^{-i(\zeta\rho_d(x,y)/t)\sinh p}\bigr)$$
$$= -t^{-1}\frac{i(\zeta\rho_d(x,y)/t)\sinh p}{i(\zeta\rho_d(x,y)/t)\cosh p + \nu}\bigl(\partial_p e^{-i(\zeta\rho_d(x,y)/t)\sinh p - \nu p}\bigr).$$

We take into account these relations to repeat the integration by parts. Since $\operatorname{Im}\psi_d(x,y) = O((\log d)/d)$, each $\tilde{Q}_l(x,y;\zeta)$ with $|l| < d$ obeys $O(d^{-N})$ uniformly in l, and hence so does the sum of $\tilde{Q}_l(x,y;\zeta)$ over l with $|l| < d$. To see that the sum over l with $|l| > d$ is of order $O(d^{-N})$, we make use of the other representation formula

$$I_\mu(w) = \frac{(w/2)^\mu}{\Gamma(\mu+1/2)\Gamma(1/2)}\int_{-1}^{1} e^{-w\rho}\bigl(1-\rho^2\bigr)^{\mu-1/2}\,d\rho \tag{9.15}$$

for $I_\mu(w)$ with $\mu \geqslant 0$ (see [21, p. 172]). Since $|x|+|y| = O(d)$, we have $|\zeta\rho_d(x,y)/t| = M^{-1}O(d)$ for $s = \operatorname{Im} t > M(|x|+|y|)$ and

$$\bigl|e^{-w\rho}\bigr| = O\bigl(e^{|\operatorname{Re}(\zeta\rho_d(x,y)/t)|}\bigr) = O\bigl(e^d\bigr), \quad |\rho| < 1,$$

for $w = \zeta\rho_d(x,y)/t$. Since $\Gamma(\mu)$ behaves like $\Gamma(\mu) \sim (2\pi)^{1/2}e^{-\mu}\mu^{\mu-(1/2)}$ for $\mu \gg 1$ by Stirling's formula, we have

$$\bigl|e^{il\psi_d(x,y)}w^\nu/\Gamma(\nu)\bigr| \leqslant (1/2)^{|l|}, \quad \nu = |l-\alpha|,$$

for $M \gg 1$. Hence the sum of $\tilde{Q}_l(x, y; \zeta)$ over l with $|l| > d$ also obeys $O(d^{-N})$, and the lemma is proved. □

Proof of Proposition 6.2. We give only a sketch for the proof of statement (1). The kernel $Q_d(x, y; \zeta)$ takes the form

$$Q_d = \exp\bigl(ikr_d(x)\bigr)q_1(x, y; \zeta),$$

where

$$q_1 = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2}\left(1 - \frac{ikr_d(x)}{t}\right)^2\right)\exp\left(-\frac{\zeta|y|^2}{2t}\right)I\left(\frac{\zeta\rho_d}{t}, \psi_d\right)\frac{dt}{t}$$

and $I(\zeta\rho_d/t, \psi_d) = \sum_l e^{il\psi_d} I_\nu(\zeta\rho_d/t)$ with $\rho_d = r_d(x)|y|$ and $\psi_d = \psi_d(x, y)$. By (9.15), the series converges absolutely. We can show in a way similar to that in the proof of Proposition 6.1 that $q_1(x, y; \zeta)$ behaves like

$$q_1 \sim \frac{1}{4\pi}\int_0^{\kappa+i\infty} \chi_M\left(\frac{\operatorname{Im} t}{|x|}\right)\exp\left(\frac{t}{2}\tau^2\right)\exp\left(-\frac{\zeta|y|^2}{2t}\right)I\left(\frac{\zeta\rho_d}{t}, \psi_d\right)\frac{dt}{t}$$

where

$$\tau = 1 - ikr_d(x)/t, \qquad \chi_M(r) = \chi_0(r/M)\chi_\infty(Mr), \quad M \gg 1,$$

and the error term obeys the bound $O(|x|^{-N})$ for any $N \gg 1$. We can show (6.9) by applying the method of steepest descent to the integral on the right side. If $|x_2| < d$, then $r_d(x) = |x|$ and $\psi_d(x, y) = \psi = \theta - \omega$ for $x = (|x| \cos\theta, |x| \sin\theta)$ and $y = (|y| \cos\omega, |y| \sin\omega)$. We see that $q_1(x, y; \zeta)$ takes the asymptotic form

$$q_1(x, y; \zeta) = c_0(\zeta)|x|^{-1/2}\exp\bigl(ik|y|^2/2|x|\bigr)\bigl(I\bigl(k|y|/i, \psi\bigr) + O\bigl(d^{-1}\bigr)\bigr).$$

We note that $\exp(ik|y|^2/2|x|) = 1 + O(d^{-1})$. Since $I_\nu(z/i) = e^{-i\nu\pi/2}J_\nu(z)$ by formula and since

$$e^{il\psi} = e^{il(\theta-\omega)} = e^{il\gamma(\hat{x};\hat{y})} = e^{-il\gamma(\hat{y};\hat{x})},$$

we have by (2.4) (see also Remark 2) that

$$I\bigl(k|y|/i, \psi\bigr) = \sum_l e^{il\psi}I_\nu\bigl(k|y|/i\bigr) = \sum_l e^{-il\gamma(\hat{y};\hat{x})}e^{-i\nu\pi/2}J_\nu\bigl(k|y|\bigr) = \bar{\varphi}_{0-}(y; \hat{x}, \bar{\zeta}).$$

Thus we get the desired asymptotic form (6.10) and the proof of statement (1) is complete. □

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