Aharonov–Bohm Effect in Resonances of Magnetic Schrödinger Operators with Potentials with Supports at Large Separation

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Abstract

Vector potentials are known to have a direct significance to quantum particles moving in the magnetic field. This is called the Aharonov–Bohm effect and is known as one of the most remarkable quantum phenomena. Here we study this quantum effect through the resonance problem. We consider the scattering system consisting of two scalar potentials and one magnetic field with supports at large separation in two dimensions. The system has trajectories oscillating between these supports. We give a sharp lower bound on the resonance widths as the distances between the three supports go to infinity. The bound is described in terms of the backward amplitude for scattering by each of the scalar potentials and by the magnetic field, and it also depends heavily on the magnetic flux of the field.

Running Head : Aharonov–Bohm effect in resonances

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1. Introduction

In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This is called the Aharonov–Bohm effect (AB effect) and is known as one of the most remarkable quantum phenomena ([3]). In this work we study the AB effect in resonances through scattering by electrostatic and magnetic fields with compact supports in two dimensions. As a simple system in which the AB effect is expected to be observed, we consider the scattering system consisting of two scalar potentials and one magnetic field with supports at large separation, where the center of the support of the magnetic field is assumed to be located on the line segment joining the two centers of the supports of the potentials. Then the resonances are expected to be generated near the real axis by the trajectories oscillating between the three supports. We give a sharp lower bound on the resonance widths (the imaginary parts of resonances) when the distances between the three centers go to infinity. The bound is described in terms of the backward amplitudes for scattering by each of two potentials and the magnetic field. We analyze how the AB effect from quantum mechanics and the trapping effect from classical mechanics are reflected in the lower bound on the resonance widths in this simple scattering system.

We always work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$. We write

$$H(A,V) = (-i\nabla - A)^2 + V = \sum_{j=1}^{2} (-i\partial_j - a_j)^2 + V, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with the scalar potential $V : \boldsymbol{R}^2 \to \boldsymbol{R}$ and the vector potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ associated with A is defined by

$$b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1$$

and the quantity defined as the integral $\alpha = (2\pi)^{-1} \int b(x)\, dx$ is called the magnetic flux of b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation. The Hamiltonian $H(A, V)$ above is the energy operator for the quantum system of particles subjected to the electrostatic potential $V(x)$ and to the magnetic field $b(x)$.

We consider the operator

$$H_d = H(A, U_d) = (-i\nabla - A)^2 + U_d, \tag{1.1}$$

where the potential $U_d(x)$ takes the form

$$U_d = V_{1d}(x) + V_{2d}(x) = V_1(x + \kappa d) + V_2(x - (1 - \kappa)d), \quad 0 < \kappa < 1, \tag{1.2}$$

with $d \in \boldsymbol{R}^2$, $|d| \gg 1$. We write $b = \nabla \times A$ for the magnetic field of $A \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ and α for the magnetic flux of b. We assume throughout the entire discussion that

$$V_1, \quad V_2, \quad b \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$$

are smooth functions with compact support. For a given magnetic field b, the corresponding vector potential A is not uniquely determined, but it is easily seen that the Schrödinger operators with the same magnetic fields are unitarily equivalent to one another through a gauge transformation. We should also note that A is not necessarily of compact support even for $b \in C_0^\infty(\boldsymbol{R}^2)$ but it falls off slowly at infinity. This is the case when α does not vanish. In fact, we can construct $A \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ with the property

$$A(x) = \alpha \left(-x_2/|x|^2, x_1/|x|^2\right) \tag{1.3}$$

over an exterior domain of the support $\operatorname{supp} b$, and hence A has the long–range property at infinity. Throughout we fix $A(x)$ as one of such vector potentials. Then the symmetric operator H_d formally defined above has a unique self–adjoint realization H_d (denoted by the same notation) with domain $\mathcal{D}(H_d) = H^2(\boldsymbol{R}^2)$ in the space $L^2 = L^2(\boldsymbol{R}^2)$, $H^s(\boldsymbol{R}^2)$ being the Sobolev space of order s. If $|d| \gg 1$ is sufficiently large, then the supports of V_{1d}, V_{2d} and b are distant from one another. Thus H_d defines the energy operator for the particle moving in electrostatic and magnetic fields with compact supports at large separation.

We denote by $R(\zeta; T) = (T - \zeta)^{-1}$ the resolvent of the self–adjoint operator T acting on L^2. It is known ([7]) that H_d has no positive eigenvalues and the continuous spectrum occupied by $(0,\infty)$ is absolutely continuous. We further know that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2 \to L^2, \quad \zeta = E + i\eta, \quad E > 0,\ \eta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to the lower half plane across the positive real axis where the continuous spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\operatorname{Im}\zeta \le 0$ is well defined as an operator from L^2_{comp} to L^2_{loc} in the sense that $\chi R(\zeta; H_d)\chi : L^2 \to L^2$ is bounded for every $\chi \in C_0^\infty(\boldsymbol{R}^2)$, where L^2_{comp} and L^2_{loc} denote the spaces of compactly supported and locally square integrable functions over $\boldsymbol{R}^2$, respectively. This can be easily shown as an application of the analytic perturbation theory of Fredholm for compact operators. We use the same notation $R(\zeta; H_d)$ to denote this meromorphic function with values in operators from L^2_{comp} to L^2_{loc}. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane (the unphysical sheet). Our aim is to study to what extent $R(\zeta; H_d)$ can be analytically extended across the positive real axis for $|d| \gg 1$.

The obtained results are formulated in terms of the backward amplitudes for scattering by V_1, V_2 and b. We write $K_0 = -\Delta$ for the free Hamiltonian and introduce the following operators :

$$K_1 = K_0 + V_1, \quad K_2 = K_0 + V_2, \quad P_\alpha = H(A, 0) = (-i\nabla - A)^2. \tag{1.4}$$

These three operators are all self–adjoint with the same domain $H^2(\boldsymbol{R}^2)$ as H_d. We denote by $f_j(\omega \to \theta; E)$, $j = 1, 2$, the scattering amplitude from the initial direction $\omega \in S^1$ (the unit circle) to the final one θ at energy $E > 0$ for the pair (K_0, K_j) and by $g_\alpha(\omega \to \theta; E)$ the amplitude for (K_0, P_α). We make a brief comment on these amplitudes. The definition and properties of the amplitude $f_j(\omega \to \theta; E)$ for scattering by the short-range potential V_j are well known (see the book [4, chapter 10] for example). We skip its precise representation. On the other hand, the scattering by the magnetic field b requires a little explanation, because it is a long–range scattering. The definition and representation of $g_\alpha(\omega \to \theta; E)$ are given in section 4 (see Lemma 4.2 for its representation).

With the above notation, we are now in a position to formulate the three theorems obtained in the present work, which give a sharp lower bound on the resonance widths as $|d| \to \infty$. We first consider the simple case with $A(x)$ vanishing identically. The first result is mentioned as follows.

Theorem 1.1 *Let $L_d = H(0, U_d) = K_0 + U_d$ and let $E > 0$ be fixed. Assume that $f_1(-\hat{d} \to \hat{d}; E)$ and $f_2(\hat{d} \to -\hat{d}; E)$ do not vanish for $\hat{d} = d/|d| \in S^1$. Define*

$$\eta_{0d}(E) = \frac{E^{1/2}}{|d|} \left\{ \log|d| - \log\left| f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) \right| \right\}.$$

Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ large enough such that $\zeta = E + i\eta$ with $\eta > -\eta_{0d}(E) + \varepsilon/|d|$ is not a resonance of L_d for $|d| > d_\varepsilon(E)$.

We prove this theorem in section 2. Here we explain from a physical point of view how the lower bound in Theorem 1.1 is determined. We first define

$$\rho_0 = \left(e^{2ik|d|}/|d|\right) f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) \tag{1.5}$$

as the product of the two backward amplitudes for $k = \zeta^{1/2}$, $\operatorname{Im} k < 0$, with $\zeta = E + i\eta$ in the lower half plane. If ζ satisfies the assumption of the theorem, then it follows that $|\rho_0| < 1$ strictly for $|d| \gg 1$. By invariance of translation, we may assume that $U_d(x) = V_1(x) + V_2(x - d)$. Let $G_1(x, y; \zeta)$ be the Green function of the resolvent $R(\zeta; K_1) = (K_1 - \zeta)^{-1}$. Then $G_1(x, y; \zeta)$ behaves like

$$G_1(x, y; \zeta) \sim e^{ik|x-y|}|x - y|^{-1/2} + e^{ik(|y|+|x|)}(|y||x|)^{-1/2} f_1(-\hat{y} \to \hat{x}; E) \tag{1.6}$$

with $\hat{y} = y/|y|$ and $\hat{x} = x/|x|$ when $|x|$, $|y| \gg 1$ and $|x - y| \gg 1$, where some numerical factors are ignored for brevity. This is seen from the rather formal argument after Remark 2 in section 2. The first term on the right side corresponds to the free trajectory which goes from y to x directly without being scattered by the potential V_1, while the second term comes from the scattering trajectory which starts from y and arrives at x after being scattered by V_1. If we consider the scattering by U_d, then we have to take into account the contribution from the trajectory oscillating between the two supports $\operatorname{supp} V_1$ and $\operatorname{supp} V_{2d}$ at large separation. The contribution from the trajectory starting from $y \in \operatorname{supp} V_1$ coming back to $x \in \operatorname{supp} V_1$ after being scattered by V_{2d} takes the form $\left(e^{2ik|d|}/|d|\right) f_2$ with $f_2 = f_2(\hat{d} \to -\hat{d}; E)$, which is seen by setting $x = y = -d$ in the second term on the right side of the asymptotic formula (1.6). We now look at the contribution from the trajectory which starts from y, hits $\operatorname{supp} V_1$ and arrives at x from $\operatorname{supp} V_1$ after oscillating between $\operatorname{supp} V_1$ and $\operatorname{supp} V_{2d}$. Then the contribution from such a trajectory to the asymptotic form of the Green function $G_d(x, y; \zeta)$ of $R(\zeta; K_d)$ is formally given by the series

$$\begin{gathered} e^{ik|x-y|}|x - y|^{-1/2} + e^{ik(|y|+|x|)} \left(|y||x|\right)^{-1/2} f_1(-\hat{y} \to \hat{x}; E) + \\ e^{ik|y|}|y|^{-1/2} f_1(-\hat{y} \to \hat{d}; E) \left(\sum_{n=0}^{\infty} \rho_0^n\right) \Big(\frac{e^{2ik|d|}}{|d|} f_2\Big) f_1(-\hat{d} \to \hat{x}; E) e^{ik|x|}|x|^{-1/2}, \end{gathered}$$

where $f_2 = f_2(\hat{d} \to -\hat{d}; E)$ again and ρ_0 is the quantity defined above. For example, the term with ρ_0^n describes the contribution from the trajectory oscillating $n+1$ times. Thus the location of the resonances is approximately determined by the relation $\rho_0 = 1$. This intuitive argument clarifies how sharp the lower bound in Theorem 1.1 is and how trapping trajectories generate the resonances near the positive real axis. A similar argument is also used to determine the lower bounds in Theorems 1.2 and 1.3 below.

Next we consider the case when $A(x)$ does not vanish identically. We study what change takes place in the lower bound on the resonance widths by the

AB effect. The obtained results are divided into the two cases according as the magnetic flux α is a half integer or not. We define

$$\eta_{1d}(E) = \frac{E^{1/2}}{|d|}\left\{\frac{\log(\kappa|d|)}{\kappa} - \log\left|f_1(-\hat{d}\to\hat{d};E)g_\alpha(\hat{d}\to-\hat{d};E)\right|\right\},$$
$$\eta_{2d}(E) = \frac{E^{1/2}}{|d|}\left\{\frac{\log((1-\kappa)|d|)}{1-\kappa} - \log\left|g_\alpha(-\hat{d}\to\hat{d};E)f_2(\hat{d}\to-\hat{d};E)\right|\right\}.$$

If $f_1(-\hat{d}\to\hat{d};E)$ or $g_\alpha(\hat{d}\to-\hat{d};E)$ vanishes, then $\eta_{1d}(E)$ is interpreted as $\eta_{1d}(E)=\infty$ and similarly for $\eta_{2d}(E)$.

Theorem 1.2 *Let $H_d = H(A,U_d)$ and let $E>0$ be fixed. Assume that the backward amplitudes $f_1(-\hat{d}\to\hat{d};E)$ and $f_2(\hat{d}\to-\hat{d};E)$ do not vanish and that the magnetic flux α of the field $b=\nabla\times A$ is not a half integer. Define*

$$\eta_d(E) = \frac{E^{1/2}}{|d|}\left\{\log|d| - \log\left|\cos^2(\alpha\pi)\, f_1(-\hat{d}\to\hat{d};E)f_2(\hat{d}\to-\hat{d};E)\right|\right\}.$$

Then, for any $\varepsilon>0$ small enough, there exists $d_\varepsilon(E)\gg 1$ such that $\zeta=E+i\eta$ with $\eta>-\eta_d(E)+\varepsilon/|d|$ is not a resonance of H_d for $|d|>d_\varepsilon(E)$.

Theorem 1.3 *Let $H_d = H(A,U_d)$ and let $E>0$ be fixed. Assume that the flux α is a half integer. Then we have the following three statements according to the values of κ.*

(1) *Let $0<\kappa<1/2$. Assume that $f_2(\hat{d}\to-\hat{d};E)$ and $g_\alpha(-\hat{d}\to\hat{d};E)$ do not vanish. Then, for any $\varepsilon>0$ small enough, there exists $d_\varepsilon(E)\gg 1$ such that $\zeta=E+i\eta$ with $\eta>-\eta_{2d}(E)+\varepsilon/|d|$ is not a resonance of H_d for $|d|>d_\varepsilon(E)$.*

(2) *Let $1/2<\kappa<1$. Assume that $f_1(-\hat{d}\to\hat{d};E)$ and $g_\alpha(\hat{d}\to-\hat{d};E)$ do not vanish. Then, for any $\varepsilon>0$ small enough, there exists $d_\varepsilon(E)\gg 1$ such that $\zeta=E+i\eta$ with $\eta>-\eta_{1d}(E)+\varepsilon/|d|$ is not a resonance of H_d for $|d|>d_\varepsilon(E)$.*

(3) *If $\kappa=1/2$, then, for any $\varepsilon>0$ small enough, there exists $d_\varepsilon(E)\gg 1$ such that $\zeta=E+i\eta$ with*

$$\eta > -\left(E^{1/2}/|d|\right)\left((2-\varepsilon)\log|d|\right)$$

is not a resonance of H_d for $|d|>d_\varepsilon(E)$.

The choice of $d_\varepsilon(E)$ in the above theorems may depend on the parameter κ and on V_1, V_2 and b (and hence α) as well as on E, but we skip its dependence on these objects. The two theorems are proved in section 5 after preparing preliminary propositions on the asymptotic properties of the Green function of magnetic Schrödinger operators with fields of compact support in sections 3 and 4. These latter results, on the other hand, are based on the asymptotic properties of the Green function of the magnetic Schrödinger operator with a single solenoidal field, which we state at the end of section 3 but prove in section 6.

We shall explain briefly and intuitively how the lower bound in Theorem 1.2 is determined from the trajectories oscillating between the supports of V_{1d} and V_{2d} and how the AB effect is reflected in the lower bound on the resonance widths. For brevity, we assume that the centers $-\kappa d$ and $(1-\kappa)d$ of the potentials V_{1d} and V_{2d} are located on the x_1 axis, respectively, and that b is given by the solenoidal field $b = 2\pi\alpha\delta(x)$ with the center at the origin. Let

$$\Phi(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = (-\partial_2 \log|x|, \partial_1 \log|x|) \tag{1.7}$$

be defined as the vector potential on the right side of (1.3). Then we have

$$\nabla \times \Phi = \Delta \log|x| = 2\pi\delta(x),$$

and hence $\alpha\Phi(x)$ turns out to be the vector potential associated with $2\pi\alpha\delta(x)$. The potential $\Phi : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is often called the Aharonov–Bohm potential in physics literatures. This potential is also represented as

$$\Phi(x) = \nabla\,\gamma(x), \tag{1.8}$$

where $\gamma(x)$ denotes the azimuth angle from the positive x_1 axis. We now consider the particle which starts from $-\kappa d$ and arrives at $(1-\kappa)d$. It passes near the center (the origin) of the field $2\pi\alpha\delta(x)$. We distinguish between these trajectories passing over $x_2 > 0$ and $x_2 < 0$ to denote the former and latter trajectories by τ_+ and τ_-. Then the AB effect causes the change in the phase of the wave function, which is given by the line integral

$$\int_{\tau_\pm} \alpha\Phi(y)\cdot dy = \mp\alpha\pi$$

along $\tau_\pm$. Thus the factor $\cos(\alpha\pi)$ comes from the sum $\exp(i\alpha\pi) + \exp(-i\alpha\pi)$. A similar argument applies to the particle starting from $(1-\kappa)d$ and arriving at $-\kappa d$, and the contribution from the trajectory oscillating between $\operatorname{supp} V_{1d}$ and $\operatorname{supp} V_{2d}$ takes the form

$$\cos^2(\alpha\pi)\left(e^{2ik|d|}/|d|\right) f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) = \cos^2(\alpha\pi)\rho_0,$$

where ρ_0 is defined by (1.5). This is the reason why the factor $\cos^2(\alpha\pi)$ appears in the lower bound on the resonance width in Theorem 1.2. If, in particular, α is a half integer, then this factor vanishes by cancellation, and Theorem 1.3 asserts that the second longest trajectory determines the lower bound.

2. Potential scattering : proof of Theorem 1.1

In this section we prove Theorem 1.1 and point out what modifications need to be made for proving Theorems 1.2 and 1.3 in which the vector potential A is added as a new perturbation. We assume throughout the discussion in the sequel that

$$\operatorname{supp} V_1, \ \operatorname{supp} V_2, \ \operatorname{supp} b \subset \Sigma_0 = \{|x| < 1\}.$$

We also use the notation

$$\Sigma_{1d} = \{|x + \kappa d| < 1\}, \qquad \Sigma_{2d} = \{|x - (1-\kappa)d| < 1\}. \tag{2.1}$$

Then we have $\operatorname{supp} V_{jd} \subset \Sigma_{jd}$ for $j = 1, 2$. We further use the notation $L^2_{\mathrm{comp}}(\Omega)$ to denote the space of functions $f \in L^2$ with support in $\Omega \subset \boldsymbol{R}^2$, and we often identify $L^2_{\mathrm{comp}}(\Omega)$ with $L^2(\Omega)$, including the topologies in these two spaces.

Proof of Theorem 1.1 We recall the notation $L_d = H(0, U_d) = K_0 + U_d$ and $\eta_{0d}(E)$ in the theorem. Throughout the proof, ζ is assumed to be in the neighborhood

$$D_{0d} = \{\zeta = E + i\mathrm{Im}\,\zeta \in \boldsymbol{C} : |E - E_0| < \delta_0, \ |\mathrm{Im}\,\zeta| < \eta_{0d}(E)\},$$

where $E_0 > 0$ is fixed and $\delta_0 > 0$ is taken small enough. Let V_{1d} and V_{2d} be as in (1.2). We write $G_0(\zeta) = R(\zeta; K_0)$ for the resolvent of the free Hamiltonian $K_0 = -\Delta$. Then we have

$$(L_d - \zeta)\, G_0(\zeta) = Id + V_{1d}G_0(\zeta) + V_{2d}G_0(\zeta), \tag{2.2}$$

where Id is the identity operator. The operator on the right side can be regarded as an operator from

$$\mathcal{X} = L^2\,(\Sigma_{1d}) \oplus L^2\,(\Sigma_{2d}) \tag{2.3}$$

into itself. If it is shown to have a bounded inverse, then the resolvent $R(\zeta; L_d)$ in question has the representation

$$R(\zeta; L_d) = G_0(\zeta) - G_0(\zeta)\,(Id + V_{1d}G_0(\zeta) + V_{2d}G_0(\zeta))^{-1}\,(V_{1d}G_0(\zeta) + V_{2d}G_0(\zeta))$$

and it follows that $R(\zeta; L_d)$ is well defined as an operator from L^2_{comp} to L^2_{loc}. Thus the desired statement is verified. We represent the operator on the right side of (2.2) in a matrix form. It takes the form

$$\begin{pmatrix} Id + V_{1d}G_0(\zeta)\chi_{1d} & V_{1d}G_0(\zeta)\chi_{2d} \\ V_{2d}G_0(\zeta)\chi_{1d} & Id + V_{2d}G_0(\zeta)\chi_{2d} \end{pmatrix}, \tag{2.4}$$

where $\chi_{jd}(x)$ is the characteristic function of Σ_{jd}.

(1) We apply the analytic perturbation theory of Fredholm to the resolvent $R(\zeta; K_1)$ of $K_1 = K_0 + V_1$. Then we see that $Id + V_1G_0(\zeta)$ has a bounded inverse as an operator from $L^2(\Sigma_0)$ into itself, so that

$$R(\zeta; K_1) = G_0(\zeta) - G_0(\zeta)\,(Id + V_1G_0(\zeta))^{-1}\,V_1G_0(\zeta)$$

is analytic as a function with values in operators from L^2_{comp} to L^2_{loc} in a neighborhood (independent of d) around E in the complex plane. We have a similar result for the resolvent $R(\zeta; K_2)$ of $K_2 = K_0 + V_2$. We introduce the auxiliary self–adjoint operators $K_{jd} = K_0 + V_{jd}$ for $j = 1, 2$. By translation, $G_{jd}(\zeta) = R(\zeta; K_{jd})$ is also well defined and we obtain the bound

$$\|V_{jd}\left(G_{jd}(\zeta) - G_{jd}(E)\right)\chi_{jd}\| = O\left((\log|d|)/|d|\right) \tag{2.5}$$

as a bounded operator acting on L^2, since

$$\zeta = E + iO\left((\log|d|)/|d|\right) \tag{2.6}$$

for $\zeta \in D_{0d}$. The operator $G_{jd}(\zeta)$ satisfies the relation

$$G_{jd}(\zeta)\chi_{jd} = G_0(\zeta)\chi_{jd}\left(Id + V_{jd}G_0(\zeta)\chi_{jd}\right)^{-1} \tag{2.7}$$

on $L^2(\Sigma_{jd})$, and hence the operator defined by (2.4) admits the decomposition

$$\begin{pmatrix} Id & V_{1d}G_{2d}(\zeta)\chi_{2d} \\ V_{2d}G_{1d}(\zeta)\chi_{1d} & Id \end{pmatrix} \begin{pmatrix} Id + V_{1d}G_0(\zeta)\chi_{1d} & 0 \\ 0 & Id + V_{2d}G_0(\zeta)\chi_{2d} \end{pmatrix}.$$

Thus it suffices to show that the first factor

$$Id + \begin{pmatrix} 0 & V_{1d}G_{2d}(\zeta)\chi_{2d} \\ V_{2d}G_{1d}(\zeta)\chi_{1d} & 0 \end{pmatrix} \tag{2.8}$$

in the above decomposition has a bounded inverse on the space $\mathcal{X}$.

(2) The operator $G_0(\zeta)$ is the integral operator with the kernel

$$G_0(x, y; \zeta) = (i/4)\, H_0(k|x-y|), \quad k = \zeta^{1/2}, \tag{2.9}$$

where $H_\nu(z)\left(= H_\nu^{(1)}(z)\right)$ is the Hankel function of the first kind and of order ν. The Hankel function $H_\nu(z)$ is known to obey the asymptotic formula

$$H_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \exp\left[i\left(z - (2\nu+1)\pi/4\right)\right]\left(1 + O(|z|^{-1})\right), \tag{2.10}$$

as $|z| \to \infty$. If $y \in \Sigma_{1d}$ and $x \in \Sigma_{2d}$, then we have

$$y = -\kappa d + O(1), \quad x = (1-\kappa)d + O(1), \quad |d| \to \infty. \tag{2.11}$$

Thus it follows from (2.6) and (2.10) that

$$G_0(x, y; \zeta) = c_0(E)e^{ik|x-y|}|x-y|^{-1/2}\left(1 + O\left((\log|d|)/|d|\right)\right) \tag{2.12}$$

with

$$c_0(E) = (8\pi)^{-1/2}\, e^{i\pi/4}E^{-1/4}. \tag{2.13}$$

We use the notation $x \cdot y$ to denote the scalar product between the two vectors x and y in $\boldsymbol{R}^2$. By (2.11), we have $|x-y| = (x-y)\cdot\hat{d} + O(|d|^{-1})$ and

$$|x-y|^{-1/2} = |d|^{-1/2}(1+O(|d|^{-1}))$$

for $(x,y) \in \Sigma_{2d} \times \Sigma_{1d}$. If $k = \zeta^{1/2}$ with $\zeta \in D_{0d}$, then $\left|e^{ik|d|}|d|^{-1/2}\right| = O(1)$ as $|d| \to \infty$. Hence it follows from (2.12) that

$$\begin{aligned} & G_0(x,y;\zeta) \sim c_0(E) e^{ik(x\cdot\hat{d})} e^{-ik(y\cdot\hat{d})} |d|^{-1/2} \\ = \; & c_0(E)\left(e^{ik|d|}|d|^{-1/2}\right) e^{ik((x-(1-\kappa)d)\cdot\hat{d})} e^{-ik((y+\kappa d)\cdot\hat{d})} \\ \sim \; & c_0(E)\left(e^{ik|d|}|d|^{-1/2}\right) \varphi_0(x-(1-\kappa)d;\hat{d},E)\varphi_0(y+\kappa d;-\hat{d},E) \end{aligned} \tag{2.14}$$

with the error bound $O\left((\log|d|)/|d|\right)$, where $\varphi_0(x;\omega,E)$ is defined by

$$\varphi_0(x;\omega,E) = \exp\left(iE^{1/2} x\cdot\omega\right) \tag{2.15}$$

with $\omega \in S^1$. If $y \in \Sigma_{2d}$ and $x \in \Sigma_{1d}$, then

$$G_0(x,y;\zeta) \sim c_0(E)\left(e^{ik|d|}|d|^{-1/2}\right)\varphi_0(x+\kappa d;-\hat{d},E)\varphi_0(y-(1-\kappa)d;\hat{d},E).$$

Thus we see that both the operators

$$\chi_{2d}G_0(\zeta)\chi_{1d}, \quad \chi_{1d}G_0(\zeta)\chi_{2d} : L^2 \to L^2 \tag{2.16}$$

are bounded uniformly in d.

(3) We analyze the behavior as $|d| \to \infty$ of $V_{2d}G_{1d}(\zeta)\chi_{1d}$ in (2.8) by making use of the resolvent identity

$$G_{1d}(\zeta) = G_0(\zeta) - G_0(\zeta)V_{1d}G_{1d}(\zeta).$$

We introduce the new notation. We denote by $f \otimes g$ the integral operator with the kernel $f(x)\overline{g}(y)$ and by $\mathrm{Op}(|d|^\rho)$ bounded operators obeying the bound $O(|d|^\rho)$. Then it follows from (2.5), (2.14) and (2.16) that

$$V_{2d}G_{1d}(\zeta)\chi_{1d} = \Lambda_{1d}(\zeta) - \Lambda_{1d}(\zeta)V_{1d}G_{1d}(E)\chi_{1d} + \mathrm{Op}\left((\log|d|)/|d|\right),$$

where

$$\Lambda_{1d}(\zeta) = c_0(E)\left(\frac{e^{ik|d|}}{|d|^{1/2}}\right)\left(V_{2d}\varphi_0(\cdot-(1-\kappa)d;\hat{d},E)\otimes\varphi_0(\cdot+\kappa d;\hat{d},E)\chi_{1d}\right).$$

The incoming eigenfunction $\psi_{1-}(x;\omega,E)$ of K_1 with $-\omega$ as a final direction is given by

$$\psi_{1-}(x;\omega,E) = \varphi_0(x;\omega,E) - \left(R(E;K_1)^* V_1\varphi_0\right)(x;\omega,E).$$

Hence $V_{2d}G_{1d}(\zeta)\chi_{1d}$ is further represented in the form

$$V_{2d}G_{1d}(\zeta)\chi_{1d} = \Lambda_{1d}^-(\zeta) + \mathrm{Op}\left((\log|d|)/|d|\right),$$

where

$$\Lambda_{1d}^-(\zeta) = c_0(E)\left(\frac{e^{ik|d|}}{|d|^{1/2}}\right)\left(V_{2d}\varphi_0(\cdot - (1-\kappa)d;\hat{d},E)\otimes\psi_{1-}(\cdot+\kappa d;\hat{d},E)\chi_{1d}\right).$$

Similarly the other component $V_{1d}G_{2d}(\zeta)\chi_{2d}$ in (2.8) takes the form

$$V_{1d}G_{2d}(\zeta)\chi_{2d} = \Lambda_{2d}^-(\zeta) + \mathrm{Op}\left((\log|d|)/|d|\right),$$

where

$$\Lambda_{2d}^-(\zeta) = c_0(E)\left(\frac{e^{ik|d|}}{|d|^{1/2}}\right)\left(V_{1d}\varphi_0(\cdot+\kappa d;-\hat{d},E)\otimes\psi_{2-}(\cdot-(1-\kappa)d;-\hat{d},E)\chi_{2d}\right)$$

with the incoming eigenfunction $\psi_{2-}(x;\omega,E)$ of K_2.

(4) The proof is completed in this step. Let $\Lambda_{1d}^-(\zeta)$ and $\Lambda_{2d}^-(\zeta)$ be defined as above. Then we set

$$\Lambda_{0d}(\zeta) = \begin{pmatrix} 0 & \Lambda_{2d}^-(\zeta) \\ \Lambda_{1d}^-(\zeta) & 0 \end{pmatrix}$$

and assert that

$$Id + \Lambda_{0d}(\zeta) : \mathcal{X} \to \mathcal{X} \tag{2.17}$$

is invertible for ζ as in the theorem. The amplitude $f_2(\hat{d}\to-\hat{d};E)$ is given by the integral

$$\begin{aligned} f_2(\hat{d}\to-\hat{d};E) &= -c_0(E)\int V_2(x)\varphi_0(x;\hat{d},E)\overline{\psi_{2-}}(x;-\hat{d},E)\,dx \\ &= -c_0(E)\int V_{2d}(x)\varphi_0(x-(1-\kappa)d;\hat{d},E)\overline{\psi_{2-}}(x-(1-\kappa)d;-\hat{d},E)\,dx. \end{aligned}$$

Similarly we have

$$c_0(E)\int V_{1d}(x)\varphi_0(x+\kappa d;-\hat{d},E)\overline{\psi_{1-}}(x+\kappa d;\hat{d},E)\,dx = -f_1(-\hat{d}\to\hat{d};E).$$

Thus the nonzero eigenvalues of $\Lambda_{0d}(\zeta)$ are calculated as the eigenvalues of the matrix

$$\begin{pmatrix} 0 & -\left(e^{ik|d|}/|d|^{1/2}\right)f_2(\hat{d}\to-\hat{d};E) \\ -\left(e^{ik|d|}/|d|^{1/2}\right)f_1(-\hat{d}\to\hat{d};E) & 0 \end{pmatrix}.$$

It is easy to see that

$$\left|\left(e^{i2k|d|}/|d|\right)f_1(-\hat{d}\to\hat{d};E)f_2(\hat{d}\to-\hat{d};E)\right| < 1-\varepsilon/2$$

for ζ as in the theorem, which implies (2.17), and the proof is now complete. □

Remark 2. In the works [8, 9], the asymptotic behaviors as $|d| \to \infty$ of the scattering amplitudes and of the spectral shift functions have been studied for the Schrödinger operator $-\Delta + V_1(x) + V_2(x-d)$ with V_j falling off rapidly at infinity.

We shall derive the asymptotic form (1.6) of the Green function $G_1(x, y; \zeta)$ of $G_1(\zeta) = R(\zeta; K_1)$ in a rather formal way, ignoring some numerical factors. If $|x| \gg 1$ and $|z| < c$ for some $c > 0$, then $|x - z| \sim |x| - \hat{x} \cdot z$, and hence

$$G_0(x, z; \zeta) \sim e^{ik|x|}|x|^{-1/2}\varphi_0(z; -\hat{x}; E)$$

by (2.9) and (2.10). Similarly $G_0(z, y; \zeta) \sim \varphi_0(z; -\hat{y}; E)e^{ik|y|}|y|^{-1/2}$ for $|y| \gg 1$. By the resolvent identity, it follows that

$$G_1(\zeta) = G_0(\zeta) - G_0(\zeta)\left(Id - V_1 G_1(\zeta)\right) V_1 G_0(\zeta).$$

Since

$$\varphi_0(z; \hat{x}; E) - \left(G_1(E)^* V_1 \varphi_0\right)(z; \hat{x}; E) = \psi_{1-}(z; \hat{x}, E)$$

and since $\int V_1(z)\varphi_0(z; -\hat{y}; E)\overline{\psi}_{1-}(z; \hat{x}, E)\, dz \sim f_1(-\hat{y} \to \hat{x}; E)$, $G_1(x, y; \zeta)$ takes the asymptotic form (1.6) when $|x|,\ |y| \gg 1$ and $|x - y| \gg 1$. The argument above is repeated and is justified in the course of the proof of Theorem 1.2.

We end the section by pointing out the modifications towards the proofs of Theorems 1.2 and 1.3. We have to overcome two new difficulties. Let $P_\alpha = H(A, 0)$ be defined in (1.4). Then the resolvent $R_\alpha(\zeta) = R(\zeta; P_\alpha)$ plays the same role as $G_0(\zeta) = R(\zeta; K_0)$ in proving these theorems. We introduce the auxiliary self–adjoint operators

$$Q_{1d} = H(A, V_{1d}) = P_\alpha + V_{1d}, \quad Q_{2d} = H(A, V_{2d}) = P_\alpha + V_{2d} \tag{2.18}$$

according to the notation in section 1, and we write $R_{jd}(\zeta) = R(\zeta; Q_{jd})$ for $j = 1, 2$. The first difficulty is to establish the relation

$$R_{jd}(\zeta)\chi_{jd} = R_\alpha(\zeta)\chi_{jd}\left(Id + V_{jd}R_\alpha(\zeta)\chi_{jd}\right)^{-1}, \tag{2.19}$$

which corresponds to (2.7) of step (1) in the proof above. Since

$$\left(Id + V_{jd}R_\alpha(\zeta)\chi_{jd}\right)^{-1} = Id - V_{jd}R_{jd}(\zeta)\chi_{jd}$$

by the resolvent identity, this is equivalent to studying the analytic extension near the real axis of $R_{jd}(\zeta) : L^2_{\text{comp}} \to L^2_{\text{loc}}$. However, there are trajectories oscillating between $\operatorname{supp} b$ and $\operatorname{supp} V_{jd}$, so that the resonances of Q_{jd} are not expected to be away from the real axis uniformly in d. Thus (2.19) is not obtained as an immediate application of the analytic perturbation theory of Fredholm.

The second difficulty is to analyze the asymptotic behavior of the Green function $R_\alpha(x, y; \zeta)$ of $R_\alpha(\zeta)$ along the forward direction

$$(x, y) \sim ((1-\kappa)d, \ -\kappa d) \ \text{ or } \ (x, y) \sim (-\kappa d, \ (1-\kappa)d)\,.$$

This importance is seen from (2.12) or (2.14) of step (2). The operator P_α is considered to be a perturbation of long–range class of K_0, so that the forward scattering amplitude $g_\alpha(\omega \to \omega; E)$ is divergent. In fact, the modified factor $\cos(\alpha\pi)$ comes from such a singular behavior. Thus the asymptotic analysis of the behavior $R_\alpha(x, y; \zeta)$ with $|x - y| \gg 1$ plays an important role in proving Theorems 1.2 and 1.3.

3. Hamiltonians with one solenoidal field

In this section we make a brief review of the scattering by one solenoidal field as the first step towards the proofs of Theorems 1.2 and 1.3. Such a system is known as one of the exactly solvable models in quantum mechanics. We refer to [1, 2, 3, 5, 13] for more detailed expositions.

Let $\Phi : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the Aharonov–Bohm potential defined by (1.7). We consider the energy operator

$$P_0 = H(\alpha\Phi, 0) = (-i\nabla - \alpha\Phi)^2 \tag{3.1}$$

which governs the movement in the solenoidal field $2\pi\alpha\delta(x)$. This operator is symmetric over $C_0^\infty\left(\boldsymbol{R}^2 \setminus \{0\}\right)$, but it is not necessarily essentially self–adjoint in L^2 because of the strong singularity at the origin of Φ. We know ([1, 5]) that it is a symmetric operator with type $(2, 2)$ of deficiency indices. The self–adjoint extension is realized by imposing a boundary condition at the origin. Its Friedrichs extension denoted by the same notation P_0 is obtained by imposing the boundary condition $\lim_{|x|\to 0} |u(x)| < \infty$ at the center of the solenoidal field.

We calculate the generalized eigenfunction of the eigenvalue problem

$$P_0\,\varphi = E\,\varphi, \qquad \lim_{|x|\to 0} |\varphi(x)| < \infty, \tag{3.2}$$

with energy $E > 0$ as an eigenvalue. Since P_0 is rotationally invariant, we work in the polar coordinate system (r, θ). Let U be the unitary mapping defined by

$$(Uu)(r, \theta) = r^{1/2} u(r\theta) : L^2 \to L^2((0, \infty); dr) \otimes L^2(S^1).$$

We write $\sum_l$ for the summation ranging over all integers l. Then U enables us to decompose P_0 into the partial wave expansion

$$P_0 \simeq U P_0 U^* = \sum_l \oplus \, (P_{0l} \otimes Id)\,, \tag{3.3}$$

where $P_{0l} = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}$ with $\nu = |l-\alpha|$ is self–adjoint in $L^2((0,\infty);dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2}|u(r)| < \infty$ at $r=0$. We again define $\varphi_0(x;\omega,E)$ by (2.15). We denote by $\gamma(x;\omega)$ the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. Then the outgoing eigenfunction $\varphi_+(x;\omega,E)$ of (3.2) with ω as an incident direction is calculated as

$$\varphi_+(x;\omega,E) = \sum_l \exp(-i\nu\pi/2)\exp(il\gamma(x;-\omega))J_\nu(E^{1/2}|x|) \tag{3.4}$$

with $\nu = |l-\alpha|$, where $J_\mu(z)$ denotes the Bessel function of order μ. The eigenfunction φ_+ behaves like $\varphi_+(x;\omega,E) \sim \varphi_0(x;\omega,E)$ as $|x| \to \infty$ in the direction $-\omega$ ($x = -|x|\omega$), and the difference $\varphi_+ - \varphi_0$ satisfies the outgoing radiation condition at infinity. On the other hand, the incoming eigenfunction $\varphi_-(x;\omega,E)$ is given by

$$\varphi_-(x;\omega,E) = \sum_l \exp(i\nu\pi/2)\exp(il\gamma(x;\omega))J_\nu(E^{1/2}|x|), \tag{3.5}$$

which behaves like $\varphi_- \sim \varphi_0(x;\omega,E)$ as $|x| \to \infty$ in the direction ω.

We decompose $\varphi_+(x;\omega,E)$ into the sum $\varphi_+ = \varphi_{\rm in} + \varphi_{\rm sc}$ of incident and scattering waves to calculate the scattering amplitude through the asymptotic behavior at infinity of the scattering wave $\varphi_{\rm sc}(x;\omega,E)$. The idea is due to Takabayashi ([11]). If we set $\sigma = \sigma(x;\omega) = \gamma(x;\omega) - \pi$, then

$$\varphi_+ = \sum_l e^{-i\nu\pi/2} e^{il\sigma} J_\nu(E^{1/2}|x|), \quad \nu = |l-\alpha|.$$

If we further make use of the formula $e^{-i\mu\pi/2}J_\mu(iw) = I_\mu(w)$ for the Bessel function

$$I_\mu(w) = (1/\pi)\left(\int_0^\pi e^{w\cos\rho}\cos(\mu\rho)\,d\rho - \sin(\mu\pi)\int_0^\infty e^{-w\cosh p - \mu p}\,dp\right) \tag{3.6}$$

with $\mathrm{Re}\, w \ge 0$ ([15, p.181]), then $\varphi_+(x;\omega,E)$ takes the form

$$\begin{aligned}\varphi_+ = {} & (1/\pi)\sum_l e^{il\sigma}\int_0^\pi e^{-i\sqrt{E}|x|\cos\rho}\cos(\nu\rho)\,d\rho \\ & - \ (1/\pi)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{i\sqrt{E}|x|\cosh p} e^{-\nu p}\,dp. \end{aligned}\tag{3.7}$$

We take the incident wave $\varphi_{\rm in}(x;\omega,E)$ as

$$\varphi_{\rm in} = e^{i\alpha\sigma}\varphi_0(x;\omega,E) = e^{i\alpha\sigma}e^{i\sqrt{E}|x|\cos\gamma(x;\omega)} = e^{i\alpha\sigma}e^{-i\sqrt{E}|x|\cos\sigma},$$

which is different from the usual plane wave $\varphi_0(x;\omega,E)$. The modified factor $e^{i\alpha\sigma}$ appears because of the long–range property of the potential $\Phi(x)$. By (1.8), we have

$$\int_{l_x}\alpha\Phi(y)\cdot dy = \alpha\int_{-\infty}^0 (d/ds)\,\gamma(x+s\omega;\omega)\,ds = \alpha\left(\gamma(x;\omega)-\pi\right) = \alpha\sigma(x;\omega),$$

where $l_x = \{y = x + s\omega\}$. Thus we may interpret $e^{i\alpha\sigma}$ as the change of the phase which the potential $\alpha\Phi$ causes to the wave function of the quantum particle moving in the direction ω due to the AB effect.

The incident wave admits the Fourier expansion

$$\varphi_{\rm in}(x;\omega,E) = (1/\pi)\sum_l \left(\int_0^\pi e^{-i\sqrt{E}|x|\cos\rho}\cos(\nu\rho)\,d\rho\right) e^{il\sigma(x;\omega)}$$

for $|\sigma| < \pi$. This, together with (3.7), yields

$$\varphi_{\rm sc}(x;\omega,E) = -\,(1/\pi)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{i\sqrt{E}|x|\cosh p}e^{-\nu p}\,dp.$$

We compute the series

$$\begin{aligned}\sum_l e^{il\sigma}e^{-\nu p}\sin(\nu\pi) &= \left\{\sum_{l\le[\alpha]} + \sum_{l\ge[\alpha]+1}\right\} e^{il\sigma}e^{-\nu p}\sin(\nu\pi)\\ &= \sin(\alpha\pi)\,(-1)^{[\alpha]}\left\{\frac{e^{-\alpha p}\left(e^{i\sigma}e^{p}\right)^{[\alpha]}}{1+e^{-i\sigma}e^{-p}} + \frac{e^{\alpha p}\left(e^{i\sigma}e^{-p}\right)^{[\alpha]}}{1+e^{-i\sigma}e^{p}}\right\}\end{aligned}$$

for $|\sigma| < \pi$. Thus we have

$$\varphi_{\rm sc} = -\frac{\sin(\alpha\pi)}{\pi}\,(-1)^{[\alpha]}\,e^{i[\alpha]\sigma(x;\omega)}\int_{-\infty}^{\infty} e^{i\sqrt{E}|x|\cosh p}\frac{e^{-\beta p}}{1+e^{-i\sigma}e^{-p}}\,dp$$

with $\beta = \alpha - [\alpha]$, where the Gauss notation $[\alpha]$ denotes the greatest integer not exceeding α. We apply the stationary phase method to the integral on the right side. Since $e^{i\sigma(x;\omega)} = e^{i(\gamma(x;\omega)-\pi)} = -e^{i(\theta-\omega)}$ by identifying $\theta = x/|x| = \hat{x} \in S^1$ with the azimuth angle θ, $\varphi_{\rm sc}(x;\omega,E)$ obeys

$$\varphi_{\rm sc} = g_0(\omega\to\hat{x};E)e^{i\sqrt{E}|x|}|x|^{-1/2} + o(|x|^{-1/2}), \qquad |x|\to\infty,$$

where $g_0(\omega\to\theta;E)$ is defined as

$$g_0(\omega\to\theta;E) = \left(\frac{2}{\pi}\right)^{1/2} e^{i\pi/4}E^{-1/4}\sin(\alpha\pi)e^{i[\alpha](\theta-\omega)}\frac{e^{i(\theta-\omega)}}{1-e^{i(\theta-\omega)}} \tag{3.8}$$

by identifying $\omega,\ \theta \in S^1$ with the azimuth angles from the positive x_1 axis. This quantity $g_0(\omega\to\theta;E)$ is called the amplitude for scattering from the initial direction $\omega \in S^1$ to the final one θ at energy $E > 0$. It should be noted that the forward amplitude $g_0(\omega\to\omega;E)$ is divergent, as stated in the previous section.

We calculate the Green function of the resolvent $R_0(\zeta) = R(\zeta;P_0)$ with $\operatorname{Im}\zeta > 0$. Let P_{0l} be as in (3.3) and let $k = \zeta^{1/2}$ with $\operatorname{Im}k > 0$. Then the equation $(P_{0l}-\zeta)\,u = 0$ has $\left\{r^{1/2}J_\nu(kr), r^{1/2}H_\nu(kr)\right\}$ with Wronskian $2i/\pi$ as a pair of linearly independent solutions, where $H_\mu(z) = H_\mu^{(1)}(z)$ again denotes the Hankel function of the first kind. Thus $(P_{0l} - \zeta)^{-1}$ has the integral kernel

$$R_{0l}(r, \rho; \zeta) = (i\pi/2)\, r^{1/2} \rho^{1/2} J_\nu\left(k(r \wedge \rho)\right) H_\nu\left(k(r \vee \rho)\right), \quad \nu = |l - \alpha|,$$

where $r \wedge \rho = \min(r, \rho)$ and $r \vee \rho = \max(r, \rho)$. Hence the Green function $R_0(x, y; \zeta)$ of $R_0(\zeta)$ is given by

$$R_0(x, y; \zeta) = (i/4) \sum_l e^{il(\theta - \omega)} J_\nu\left(k(|x| \wedge |y|)\right) H_\nu\left(k(|x| \vee |y|)\right), \tag{3.9}$$

where $x = (|x| \cos\theta, |x| \sin\theta)$ and $y = (|y| \cos\omega, |y| \sin\omega)$ in the polar coordinates. This makes sense even for ζ in the lower–half plane of the complex plane by analytic continuation. Then $R_0(\zeta)$ with $\operatorname{Im}\zeta \le 0$ is well defined as an operator from L^2_{comp} to L^2_{loc}. Thus $R_0(\zeta)$ does not have any poles as a function with values in operators from L^2_{comp} to L^2_{loc}. We can say that P_0 with one solenoidal field $2\pi\alpha\delta(x)$ has no resonances. We do not discuss the possibility of resonances at zero energy.

We summarize the asymptotic properties of the Green function $R_0(x, y; \zeta)$ in the three propositions below. These propositions are proved in section 6 after completing the proofs of Theorems 1.2 and 1.3 in section 5. The propositions are used in the next section to derive the corresponding asymptotic properties of the Green function of the operator P_α.

Proposition 3.1 *Let $E > 0$ and $c_1 > 0$ be fixed. Let $\lambda \gg 1$ be large enough. Assume that $\zeta = E + i\eta$ satisfies $|\eta| \le c_1 (\log \lambda)/\lambda$. If x and y fulfill*

$$\lambda/c \le |x|, \ |y|, \ |x - y| \le c\lambda$$

for some $c > 1$ and if $\hat{x}$ and $\hat{y}$ satisfy $|\hat{x} \cdot \hat{y} + 1| < c\lambda^{2(\mu-1)}$ for some $0 \le \mu < 1/2$, then

$$\begin{aligned} R_0(x, y; \zeta) &= (i/4) \cos(\alpha\pi) e^{i\alpha(\gamma(\hat{x};\hat{y}) - \pi)} H_0(k|x - y|) \\ &\quad + \ e^{ik(|x|+|y|)} \left(|x| + |y|\right)^{-1/2} e_{1N}(x, y; \zeta, \lambda) + O(\lambda^{-N}) \end{aligned}$$

for any $N \gg 1$, where $c_0(E)$ is defined by (2.13), and e_{1N} obeys

$$\left|\partial_x^n \partial_y^m e_{1N}\right| = O\left(\lambda^{\mu - 1/2 - |n|/2 - |m|/2}\right) \tag{3.10}$$

uniformly in x, y and ζ.

Proposition 3.2 *Let $E > 0$, $c_1 > 0$ and $\lambda \gg 1$ be as in Proposition 3.1. Assume that $\zeta = E + i\eta$ satisfies $|\eta| \le c_1 (\log \lambda)/\lambda$. If x and y fulfill*

$$\lambda/c \le |x|,\ |y|,\ |x-y| \le c\lambda$$

for some $c > 1$ and if $\hat{x}$ and $\hat{y}$ satisfy $|\hat{x}\cdot\hat{y}+1| > 1/c$, then

$$R_0(x,y;\zeta) = (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) + c_0(E)e^{ik(|x|+|y|)}(|x||y|)^{-1/2}\Big(g_0(-\hat{y}\to\hat{x};E) + e_{2N}(x,y;\zeta,\lambda)\Big) + O(\lambda^{-N})$$

for any $N \gg 1$, where $c_0(E)$ is defined by (2.13), and e_{2N} obeys

$$\left|\partial_x^n \partial_y^m e_{2N}\right| = O\left((\log\lambda)^2\lambda^{-1-|n|-|m|}\right) \tag{3.11}$$

uniformly in x, y and ζ.

Remark 3. The proof of this proposition is based on the stationary phase method. If we use instead the method of steepest descent in the complex plane, then we may be able to expand asymptotically the second term on the right side of the relation in $k(|x|+|y|)$, as the first term $H_0(k|x-y|)$ is asymptotically expanded in $k|x-y|$. However, the proofs of Theorems 1.2 and 1.3 do not require the precise remainder estimate.

Proposition 3.3 *Let $\lambda \gg 1$. Let $\varphi_+(x;\omega,E)$ and $\varphi_-(x;\omega,E)$ be the outgoing and incoming eigenfunctions of P_0, respectively. Then we have the following statements:*

(1) *If x and y fulfill $\lambda/c \le |x| \le c\lambda$ and $1/c \le |y| \le c$ for some $c > 1$, then*

$$R_0(x,y;\zeta) = c_0(E)e^{ik|x|}|x|^{-1/2}\left(\overline{\varphi}_-(y;\hat{x},E) + e_{3N}(x,y;\zeta,\lambda)\right) + O(\lambda^{-N}),$$

where e_{3N} obeys $\left|\partial_x^n\partial_y^m e_{3N}\right| = O\left((\log\lambda)^2\lambda^{-1-|n|}\right)$.

(2) *If x and y fulfill $1/c \le |x| \le c$ and $\lambda/c \le |y| \le c\lambda$, then*

$$R_0(x,y;\zeta) = c_0(E)e^{ik|y|}|y|^{-1/2}\left(\varphi_+(x;-\hat{y},E) + e_{4N}(x,y;\zeta,\lambda)\right) + O(\lambda^{-N}),$$

where e_{4N} obeys $\left|\partial_x^n\partial_y^m e_{4N}\right| = O\left((\log\lambda)^2\lambda^{-1-|m|}\right)$.

4. Magnetic Schrödinger operators with fields of compact support

The aim of this section is to analyze the asymptotic behavior of the Green function for magnetic Schrödinger operators with compactly supported fields. We prove that the Green function has asymptotic properties similar to those in

Propositions 3.1 and 3.2. The main results here are formulated as Propositions 4.1 and 4.2.

We recall that the given field $b \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ has the magnetic flux α and is supported in $\Sigma_0 = \{|x| < 1\}$. Then the potential $A \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ associated with b can be constructed in such a way that

$$A(x) = \alpha\Phi(x) = \alpha(-x_2/|x|^2, x_1/|x|^2) \tag{4.1}$$

over $\{|x| > 2\}$. See, for example, [14, section 2], for construction of such a potential. As already stated, magnetic potentials are not uniquely determined for a given field b, but Schrödinger operators with the same magnetic field become unitarily equivalent to one another through gauge transformations. Hence it does not matter to the location of resonances which potentials are chosen. We know ([7, 10, 12]) that the self–adjoint operator $P_\alpha = H(A, 0)$ has the following spectral properties : (1) P_α has no bound states ; (2) The spectrum of P_α is absolutely continuous and the principle of limiting absorption holds true. We denote by $\varphi_{\alpha+}(x; \omega, E)$ the outgoing eigenfunction of P_α with $\omega \in S^1$ as an incident direction at energy $E > 0$. The amplitude $g_\alpha(\omega \to \theta; E)$ for the scattering from the initial direction ω to the final one θ at energy E is defined through the asymptotic form

$$\varphi_{\alpha+} = e^{i\alpha(\gamma(x;\omega)-\pi)}\varphi_0(x; \omega, E) + g_\alpha(\omega \to \theta; E)e^{i\sqrt{E}|x|}|x|^{-1/2} + o(|x|^{-1/2})$$

as $|x| \to \infty$ in the direction θ ($x = |x|\theta$), where $\gamma(x; \omega)$ again denotes the azimuth angle from ω to $\hat{x} = x/|x|$. Here we introduce a smooth non–negative cut–off function $\chi \in C_0^\infty[0, \infty)$ with the properties

$$0 \le \chi \le 1, \qquad \operatorname{supp} \chi \subset [0, 2], \qquad \chi = 1 \text{ on } [0, 1]. \tag{4.2}$$

This function is often used in the future discussion without further references.

Lemma 4.1 *Let $E > 0$ be fixed. Then there exists a neighborhood of E in the complex plane where the resolvent $R_\alpha(\zeta) = R(\zeta; P_\alpha)$ is analytic as a function with values in operators from L^2_{comp} to L^2_{loc}.*

Proof. The lemma is obtained as an application of the analytic perturbation theory of Fredholm. Let $\chi \in C_0^\infty[0, \infty)$ be as above. We set $u_1(x) = \chi(|x|/4)$ and define $P = H(A_1, 0)$ with $A_1 = u_1 A$. Then the coefficients of the differential operator $P - K_0$ have support in $\Sigma = \{|x| < 8\}$, $K_0 = -\Delta$ being the free Hamiltonian. Hence the analytic perturbation theory implies that there exists a neighborhood of E in the complex plane where the resolvent $R(\zeta; P)$ is analytic as a function with values in operators from L^2_{comp} to L^2_{loc}. In fact,

$$(P - K_0)G_0(\zeta) = (P - K_0)R(\zeta; K_0) : L^2_{\mathrm{comp}}(\Sigma) \to L^2_{\mathrm{comp}}(\Sigma)$$

acts as a compact operator, and $R(\zeta; P)$ is represented as

$$R(\zeta; P) = G_0(\zeta) - G_0(\zeta)\left(Id + (P - K_0)G_0(\zeta)\right)^{-1}(P - K_0)G_0(\zeta).$$

If we set $u_0 = \chi(|x|/2)$, then $P_\alpha = P$ over the support of u_0, and also it follows from (4.1) that $P_\alpha = P_0 = H(\alpha\Phi, 0)$ over the support of $v_0 = 1 - u_0$. We recall the notation $R_0(\zeta) = R(\zeta; P_0)$ and calculate

$$(P_\alpha - \zeta)\left(u_0 R(\zeta; P) + v_0 R_0(\zeta)\right) = Id + \Lambda(\zeta),$$

where

$$\Lambda(\zeta) = [P_\alpha, u_0]R(\zeta; P) + [P_0, v_0]R_0(\zeta)$$

and $[X, Y] = XY - YX$ denotes the commutator between two operators X and Y. The operator $\Lambda(\zeta)$ acts as a compact operator on $L^2_{\mathrm{comp}}(\Sigma_1)$ with $\Sigma_1 = \{|x| < 4\}$ and is analytic as a function with values in bounded operators acting on $L^2_{\mathrm{comp}}(\Sigma_1)$. Hence there exists an inverse

$$(Id + \Lambda(\zeta))^{-1} : L^2_{\mathrm{comp}}(\Sigma_1) \to L^2_{\mathrm{comp}}(\Sigma_1)$$

for ζ in a neighborhood of E by the analytic perturbation theory. Thus we see that

$$R_\alpha(\zeta) = \left(u_0 R(\zeta; P) + v_0 R_0(\zeta)\right)(Id + \Lambda(\zeta))^{-1} : L^2_{\mathrm{comp}}(\Sigma_1) \to L^2_{\mathrm{loc}}$$

is well defined for ζ as above. As stated above, $P_\alpha = P_0$ on the support of v_0, and hence a simple manipulation yields

$$\begin{aligned} R_\alpha(\zeta) &= R_\alpha(\zeta)u_0 + (R_\alpha(\zeta)v_0 - v_0 R_0(\zeta)) + v_0 R_0(\zeta) \\ &= R_\alpha(\zeta)u_0 + R_\alpha(\zeta)[v_0, P_\alpha]R_0(\zeta) + v_0 R_0(\zeta). \end{aligned}$$

Thus it follows that $R_\alpha(\zeta) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ is well defined in a neighborhood of E in the complex plane. $\square$

Lemma 4.2 *We use the notation* $(\ ,\ )$ *to denote the* L^2 *scalar product. Let* $u_0 = \chi(|x|/2)$ *and* $u_1 = \chi(|x|/4)$, *and let* $\varphi_+(x; \omega, E)$ *and* $\varphi_-(x; \theta, E)$ *be the outgoing and incoming eigenfunctions of* $P_0 = H(\alpha\Phi, 0)$. *Then the amplitude* $g_\alpha(\omega \to \theta; E)$ *has the representation*

$$g_\alpha = g_0 + c_0(E)\left(R_\alpha(E)[P_0, u_0]\varphi_+(\cdot; \omega, E), [P_0, u_1]\varphi_-(\cdot; \theta, E)\right),$$

where $g_0 = g_0(\omega \to \theta; E)$ *is the scattering amplitude for* P_0 *and* $c_0(E)$ *is defined by* (2.13).

Proof. Note that $P_\alpha = P_0$ outside the support of u_0. Hence we have

$$\varphi_{\alpha+} = (1 - u_0)\,\varphi_+ + R_\alpha(E)[P_0, u_0]\varphi_+. \tag{4.3}$$

Similarly

$$\varphi_+ = (1 - u_1)\,\varphi_{\alpha+} + R_0(E)[P_0, u_1]\varphi_{\alpha+},$$

and hence

$$\varphi_{\alpha+} = \varphi_+ + u_1\varphi_{\alpha+} - R_0(E)[P_0, u_1]\varphi_{\alpha+}. \tag{4.4}$$

It follows from Proposition 3.3 with $\lambda = r = |x|$ that the last term on the right side of (4.4) behaves like

$$c_0(E)\,(\varphi_{\alpha+}(\cdot;\omega,E), [P_0,u_1]\varphi_-(\cdot;\theta,E))\,|x|^{-1/2}e^{i\sqrt{E}|x|} + o(|x|^{-1/2})$$

as $|x| \to \infty$ in the direction θ. We insert (4.3) into $\varphi_{\alpha+}$ on the right side. Since

$$((1-u_0)\varphi_+, [u_1, P_0]\varphi_-) = (\varphi_+, [u_1, P_0]\varphi_-) = 0,$$

we obtain the desired relation. $\square$

We write $R_\alpha(x, y; \zeta)$ for the Green function of $R_\alpha(\zeta) = R(\zeta; P_\alpha)$. The following two propositions correspond to Propositions 3.1 and 3.2. We keep the notation with the same meaning as ascribed there to formulate the propositions.

Proposition 4.1 *Assume $\zeta = E + i\eta$ with $|\eta| \le c_1\,(\log\lambda)\,/\lambda$ for $\lambda \gg 1$. If x and y fulfill*

$$\lambda/c \le |x|,\ |y|,\ |x-y| \le c\lambda$$

for some $c > 1$ and if $\hat{x}$ and $\hat{y}$ satisfy $|\hat{x}\cdot\hat{y} + 1| < c\lambda^{2(\mu-1)}$ for some $0 \le \mu < 1/2$, then

$$\begin{aligned} R_\alpha(x, y; \zeta) &= (i/4)\cos(\alpha\pi)e^{i\alpha(\gamma(\hat{x};\hat{y})-\pi)}H_0(k|x-y|) \\ &\quad +\ e^{ik(|x|+|y|)}\,(|x|+|y|)^{-1/2}\,\rho_{1N}(x,y;\zeta,\lambda) + O(\lambda^{-N}) \end{aligned}$$

for any $N \gg 1$, where ρ_{1N} obeys the same bound as in (3.10).

Proposition 4.2 *Assume $\zeta = E + i\eta$ with $|\eta| \le c_1\,(\log\lambda)\,/\lambda$ for $\lambda \gg 1$. If x and y fulfill*

$$\lambda/c \le |x|,\ |y|,\ |x-y| \le c\lambda, \quad |\hat{x}\cdot\hat{y} + 1| > 1/c$$

for some $c > 1$, then

$$\begin{aligned} R_\alpha(x, y; \zeta) &= (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &+\ c_0(E)e^{ik(|x|+|y|)}(|x||y|)^{-1/2}\Big(g_\alpha(-\hat{y}\to\hat{x};E) + \rho_{2N}(x,y;\zeta,\lambda)\Big) + O(\lambda^{-N}) \end{aligned}$$

for any $N \gg 1$, where ρ_{2N} obeys the same bound as in (3.11).

Proof of Proposition 4.1. We set

$$u_0(x) = \chi(|x|/2), \quad u_1(x) = \chi(|x|/4), \quad v_0 = 1 - u_0, \quad v_1 = 1 - u_1$$

and fix $p,\ q \in \boldsymbol{R}^2$ ($|p|,\ |q| \gg 1$) as points having the properties in the proposition. If we further set $w_p(x) = \chi(|x-p|)$, then $w_p v_0 = w_p$ and $w_p v_1 = w_p$, and similarly for $w_q = \chi(|x-q|)$. The operator P_α coincides with P_0 on the support of v_1. We compute

$$\begin{aligned} w_p R_\alpha(\zeta) w_q &= w_p R_0(\zeta) w_q + w_p R_0(\zeta)\left(P_0 v_1 - v_1 P_\alpha\right) R_\alpha(\zeta) w_q \\ &= w_p R_0(\zeta) w_q + w_p R_0(\zeta)[u_1, P_0] R_\alpha(\zeta) w_q. \end{aligned}$$

Since $v_0 = 1$ on the support of ∇u_1 and since $P_\alpha = P_0$ on the support of v_0, we repeat the above argument to get

$$w_p R_\alpha(\zeta) w_q = w_p R_0(\zeta) w_q + w_p R_0(\zeta)[u_1, P_0]\Big(R_0(\zeta) + R_\alpha(\zeta)[P_0, u_0] R_0(\zeta)\Big) w_q.$$

Note that

$$w_p R_0(\zeta)[u_1, P_0] R_0(\zeta) w_q = w_p R_0(\zeta) u_1 w_q - w_p u_1 R_0(\zeta) w_q = 0$$

and hence we have

$$w_p R_\alpha(\zeta) w_q = w_p R_0(\zeta) w_q + w_p R_0(\zeta)[u_1, P_0] R_\alpha(\zeta)[P_0, u_0] R_0(\zeta) w_q.$$

We apply Proposition 3.3 to the second operator on the right side. Since

$$e^{ik(|p|+|q|)}(|p||q|)^{-1/2} = e^{ik(|p|+|q|)}(|p| + |q|)^{-1/2} O(\lambda^{-1/2}),$$

Proposition 3.3 enables us to deal with the kernel of the second operator as a remainder term. Thus the proposition follows from Proposition 3.1. □

Proof of Proposition 4.2. We use the same notation and repeat the same argument as in the proof of Proposition 4.1. Then we obtain

$$\begin{aligned} & w_p R_\alpha(\zeta) w_q = w_p R_0(\zeta) w_q + w_p R_0(\zeta)[u_1, P_0] R_\alpha(\zeta)[P_0, u_0] R_0(\zeta) w_q \\ = \ & w_p R_0(\zeta) w_q + w_p R_0(\zeta)[u_1, P_0]\Big(R_\alpha(E) + (R_\alpha(\zeta) - R_\alpha(E)\Big)[P_0, u_0] R_0(\zeta) w_q. \end{aligned}$$

If we apply Propositions 3.2 and 3.3 to the operator

$$w_p R_0(\zeta) w_q + w_p R_0(\zeta)[u_1, P_0] R_\alpha(E)[P_0, u_0] R_0(\zeta) w_q,$$

then it follows from Lemma 4.2 that the kernel of this operator has the desired asymptotic form at points p and q fixed arbitrarily. Note that $|\zeta - E| = |\eta| = O\left((\log \lambda)/\lambda\right)$ by assumption. Since

$$[u_1, P_0]\left(R_\alpha(\zeta) - R_\alpha(E)\right)[P_0, u_0] : L^2 \to L^2$$

is bounded by elliptic estimates and its norm obeys $O((\log\lambda)/\lambda)$ by continuity (Lemma 4.1), the kernel of the operator

$$w_p R_0(\zeta)[u_1, P_0]\left(R_\alpha(\zeta) - R_\alpha(E)\right)[P_0, u_0]R_0(\zeta)w_q$$

can be dealt with as a remainder term. Thus the proof is complete. $\square$

5. AB effect in resonances : proof of Theorems 1.2 and 1.3

This section is devoted to proving Theorems 1.2 and 1.3. For brevity, we restrict ourselves to the generic case that none of the four backward amplitudes $f_j(-\hat{d} \to \hat{d}; E)$ $(j = 1, 2)$ and $g_\alpha(\pm\hat{d} \to \mp\hat{d}; E)$ vanishes. The proofs of the theorems are based on the two lemmas (Lemmas 5.1 and 5.2) below, and these lemmas are proved after completing the proof of Theorem 1.2. We use the notation with the same meaning as ascribed in section 2. In particular, we recall that $\chi_{jd}(x)$ denotes the characteristic function of Σ_{jd} defined in (2.1) and that the space $\mathcal{X}$ is defined by (2.3).

Lemma 5.1 *Let Q_{1d} and Q_{2d} be as in* (2.18)*, and write $R_{jd}(\zeta)$ for the resolvent $R(\zeta; Q_{jd})$. Let $\eta_{1d}(E)$ and $\eta_{2d}(E)$ be as in Theorem 1.3. Define*

$$D_{jd} = \left\{\zeta = E + i\mathrm{Im}\,\zeta \in \boldsymbol{C} : |E - E_0| < \delta_0,\ \ |\mathrm{Im}\,\zeta| < \eta_{jd}(E)\right\}$$

for $j = 1, 2$. Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ such that $R_{jd}(\zeta) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ is analytic as a function of ζ over

$$\left\{\zeta \in D_{jd} : \mathrm{Im}\,\zeta > -\eta_{jd}(E) + \varepsilon/|d|\right\}$$

for $|d| > d_\varepsilon(E)$.

Lemma 5.2 *Assume that the flux α is not a half integer. Let $\eta_d(E)$ be as in Theorem 1.2. Define*

$$D_d = \left\{\zeta = E + i\mathrm{Im}\,\zeta \in \boldsymbol{C} : |E - E_0| < \delta_0,\ \ |\mathrm{Im}\,\zeta| < \eta_d(E)\right\}.$$

Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ such that

$$\begin{pmatrix} Id & V_{1d}R_{2d}(\zeta)\chi_{2d} \\ V_{2d}R_{1d}(\zeta)\chi_{1d} & Id \end{pmatrix} : \mathcal{X} \to \mathcal{X}$$

is invertible for $|d| > d_\varepsilon(E)$, provided that $\zeta \in D_d$ fulfills $\mathrm{Im}\,\zeta > -\eta_d(E) + \varepsilon/|d|$.

5.1. Proof of Theorem 1.2 Before going into the proof, we note that

$$\eta_d(E) < \min\left(\eta_{1d}(E), \eta_{2d}(E)\right) \tag{5.1}$$

for $|d| \gg 1$, if α is not a half integer.

Proof of Theorem 1.2. The resolvent $R_\alpha(\zeta) = R(\zeta; P_\alpha)$ satisfies the relation

$$(H_d - \zeta)\, R_\alpha(\zeta) = Id + V_{1d}R_\alpha(\zeta) + V_{2d}R_\alpha(\zeta).$$

The operator on the right side is written in the matrix form

$$\begin{pmatrix} Id + V_{1d}R_\alpha(\zeta)\chi_{1d} & V_{1d}R_\alpha(\zeta)\chi_{2d} \\ V_{2d}R_\alpha(\zeta)\chi_{1d} & Id + V_{2d}R_\alpha(\zeta)\chi_{2d} \end{pmatrix} : \mathcal{X} \to \mathcal{X} \tag{5.2}$$

and is regarded as an operator from $\mathcal{X}$ into itself. If this operator is shown to have a bounded inverse, then

$$R(\zeta; H_d) = R_\alpha(\zeta) - R_\alpha(\zeta)\left(Id + V_{1d}R_\alpha(\zeta) + V_{2d}R_\alpha(\zeta)\right)^{-1}\left(V_{1d}R_\alpha(\zeta) + V_{2d}R_\alpha(\zeta)\right).$$

By Lemma 4.1, $R_\alpha(\zeta)$ is well defined as a meromorphic function with values in operators from L^2_{comp} to L^2_{loc}. Thus it suffices to show that the operator defined by (5.2) is invertible for $\zeta \in D_d$ as in the theorem. If $\zeta \in D_d$, then it follows from (5.1) and Lemma 5.1 that $V_{jd}R_{jd}(\zeta)\chi_{jd} : L^2 \to L^2$ is well defined for $\zeta \in D_d$, and hence we have

$$\left(Id + V_{jd}R_\alpha(\zeta)\chi_{jd}\right)^{-1} = Id - V_{jd}R_{jd}(\zeta)\chi_{jd}$$

as an operator acting on $L^2(\Sigma_{jd})$. This implies the relation

$$R_{jd}(\zeta)\chi_{jd} = R_\alpha(\zeta)\chi_{jd}\left(Id + V_{jd}R_\alpha(\zeta)\chi_{jd}\right)^{-1}$$

and enables us to decompose the operator defined by (5.2) into the product

$$\begin{pmatrix} Id & V_{1d}R_{2d}(\zeta)\chi_{2d} \\ V_{2d}R_{1d}(\zeta)\chi_{1d} & Id \end{pmatrix}\begin{pmatrix} Id + V_{1d}R_\alpha(\zeta)\chi_{1d} & 0 \\ 0 & Id + V_{2d}R_\alpha(\zeta)\chi_{2d} \end{pmatrix}.$$

By Lemma 5.2, we can take $d_\varepsilon(E)$ so large that the first factor is invertible for $|d| \gg d_\varepsilon(E)$, provided that $\operatorname{Im}\zeta > -\eta_d(E) + \varepsilon/|d|$. Thus the proof is complete. □

5.2. Proof of Lemma 5.1 To prove Lemma 5.1, we have to take into account the effect from trajectories oscillating between $\operatorname{supp} b$ and $\operatorname{supp} V_{1d}$ and between $\operatorname{supp} b$ and $\operatorname{supp} V_{2d}$. On the other hand, the trapping phenomenon between $\operatorname{supp} V_{1d}$ and $\operatorname{supp} V_{2d}$ is important in proving Lemma 5.2.

We prove Lemma 5.1 for $R_{1d}(\zeta)$ only. For notational brevity, we write

$$d_- = -\kappa d, \quad |x_-| = |x - d_-| = |x + \kappa d|, \quad \hat{x}_- = x_-/|x_-|$$

throughout the section. We set

$$u_0(x) = \chi(|x - d_-|), \quad u_1(x) = \chi(|x - d_-|/2), \quad u_2(x) = \chi(|x - d_-|/4)$$

and $v_j(x) = 1 - u_j(x)$ for the cut–off function $\chi \in C_0^\infty[0, \infty)$ with properties in (4.2). By definition, we have $u_1 u_2 = u_1$ and $v_1 v_0 = v_1$. The magnetic field $b \in C_0^\infty$ vanishes around the center d_- of $\operatorname{supp} V_{1d}$. We take a real bounded function $g \in C^\infty(\boldsymbol{R}^2)$ such that

$$g(x) = \alpha\gamma(x; \hat{d}) \quad \text{on} \quad \{|x - d_-| < |d_-|/2\}, \tag{5.3}$$

where $\gamma(x; \omega) = \gamma(\hat{x}; \omega)$ denotes the azimuth angle from $\omega \in S^1$ to $\hat{x}$. By (1.8) and (4.1), we have

$$\nabla g = \alpha\Phi(x) = A(x) \tag{5.4}$$

on $\{|x - d_-| < |d_-|/2\}$.

Proof of Lemma 5.1. The lemma is proved by reducing its proof to that of the new lemma (Lemma 5.3) in the course of the proof. We establish several relations and introduce auxiliary operators which are required to derive the representation for $R_{1d}(\zeta)$. We define the self–adjoint operator

$$\tilde{K}_{1d} = e^{ig} K_{1d} e^{-ig} = e^{ig} \left(K_0 + V_{1d}\right) e^{-ig} = H(\nabla g, V_{1d}),$$

where $K_0 = -\Delta$ again denotes the free Hamiltonian. By (5.4), $\tilde{K}_{1d}$ coincides with $Q_{1d} = P_\alpha + V_{1d}$ on $\{|x - d_-| < |d_-|/2\}$. We further define

$$\Gamma_d(\zeta) = u_1 \tilde{G}_{1d}(\zeta) u_2 + v_1 R_\alpha(\zeta) v_0 \tag{5.5}$$

for $\zeta \in D_{1d}$, where $\tilde{G}_{1d}(\zeta) = R(\zeta; \tilde{K}_{1d})$. Then we have the relation

$$(Q_{1d} - \zeta)\, \Gamma_d(\zeta) = Id + Y_d(\zeta), \tag{5.6}$$

where

$$Y_d(\zeta) = [Q_{1d}, u_1] \tilde{G}_{1d}(\zeta) u_2 + [Q_{1d}, v_1] R_\alpha(\zeta) v_0.$$

Since ∇u_1 and ∇v_1 have support in $\Omega = \{|x - d_-| < 4\}$, the two commutators $[Q_{1d}, u_1]$ and $[Q_{1d}, v_1]\Big(= [u_1, Q_{1d}]\Big)$ vanish over the outside of Ω. Thus we may consider $Y_d(\zeta)$ to be an operator from $L^2(\Omega)\Big(\simeq L^2_{\mathrm{comp}}(\Omega)\Big)$ into itself.

We continue the reduction. We represent $Y_d(\zeta)$ as a sum of two operators. Let $g \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ be defined by (5.3). We introduce the auxiliary operator

$$\tilde{K}_0 = e^{ig} K_0 e^{-ig} = H(\nabla g, 0) \tag{5.7}$$

and denote by $\tilde{G}_0(\zeta)$ the resolvent $R(\zeta; \tilde{K}_0)$ of $\tilde{K}_0$. By (5.4), $\tilde{K}_0$ equals

$$\tilde{K}_0 = H(A, 0) = P_\alpha \tag{5.8}$$

over $\{|x-d_-|<|d_-|/2\}$. If we set $w_d(x)=\chi(4|x-d_-|/|d_-|)$, then $w_d=1$ on Ω and we have

$$\begin{aligned}[Q_{1d},v_1]R_\alpha(\zeta)v_0 &= [Q_{1d},v_1]\left(\tilde{G}_0(\zeta)v_0+R_\alpha(\zeta)(w_d\tilde{K}_0-P_\alpha w_d)\tilde{G}_0(\zeta)v_0\right)\\ &= [Q_{1d},v_1]\tilde{G}_0(\zeta)v_0+[Q_{1d},v_1]R_\alpha(\zeta)[w_d,\tilde{K}_0]\tilde{G}_0(\zeta)v_0\end{aligned}$$

as an operator acting on $L^2(\Omega)$. Hence $Y_d(\zeta)$ is decomposed into the sum $Y_d(\zeta)=Y_{d0}(\zeta)+Y_{d1}(\zeta)$, and we have

$$(Q_{1d}-\zeta)\,\Gamma_d(\zeta)=Id+Y_{d0}(\zeta)+Y_{d1}(\zeta),$$

where the two operators

$$Y_{d0}(\zeta) = [Q_{1d},u_1]\tilde{G}_{1d}(\zeta)u_2+[Q_{1d},v_1]\tilde{G}_0(\zeta)v_0 \tag{5.9}$$
$$Y_{d1}(\zeta) = [Q_{1d},v_1]R_\alpha(\zeta)[w_d,\tilde{K}_0]\tilde{G}_0(\zeta)v_0 \tag{5.10}$$

act on $L^2(\Omega)$.

We shall show that $Id+Y_{d0}(\zeta)$ is invertible on $L^2(\Omega)$. By (5.8), we have

$$[Q_{1d},u_1]=[P_\alpha,u_1]=e^{ig}[K_0,u_1]e^{-ig}$$

and $[Q_{1d},v_1]=[u_1,Q_{1d}]=e^{ig}[K_0,v_1]e^{-ig}$. By definition, it also follows that

$$\tilde{G}_0(\zeta)=e^{ig}G_0(\zeta)e^{-ig},\quad \tilde{G}_{1d}(\zeta)=e^{ig}G_{1d}(\zeta)e^{-ig}.$$

Thus the operator $Y_{d0}(\zeta)$ under consideration equals

$$Y_{d0}(\zeta)=e^{ig}\Big([K_0,u_1]G_{1d}(\zeta)u_2+[K_0,v_1]G_0(\zeta)v_0\Big)e^{-ig}.$$

The operator $K_{1d}=K_0+V_{1d}$ coincides with K_0 on $\operatorname{supp} v_1$, and hence

$$\begin{gathered}(K_{1d}-\zeta)\,(u_1G_{1d}(\zeta)u_2+v_1G_0(\zeta)v_0)=\\ Id+[K_0,u_1]G_{1d}(\zeta)u_2+[K_0,v_1]G_0(\zeta)v_0=e^{-ig}\,(Id+Y_{d0}(\zeta))\,e^{ig}.\end{gathered} \tag{5.11}$$

The resonances of K_{1d} do not depend on d by translation. If $\zeta=E+i\eta\in D_{1d}$, then ζ is not a resonance of K_{1d} for $|d|\gg 1$. Hence the relation above implies that $Id+Y_{d0}(\zeta)$ is invertible on $L^2(\Omega)$ and the inverse $(Id+Y_{d0}(\zeta))^{-1}$ takes the form

$$(Id+Y_{d0}(\zeta))^{-1}=e^{ig}\,(Id+[K_0,u_1]G_{1d}(\zeta)u_2+[K_0,v_1]G_0(\zeta)v_0)^{-1}\,e^{-ig}.$$

Thus we have

$$Id+Y_d(\zeta)=\left(Id+Y_{d1}(\zeta)(Id+Y_{d0})^{-1}\right)(Id+Y_{d0}(\zeta)) \tag{5.12}$$

as an operator acting on $L^2(\Omega)$. We also note that $(Id + Y_{d0}(\zeta))^{-1} : L^2(\Omega) \to L^2(\Omega)$ is bounded uniformly in $\zeta \in D_{1d}$ and d, $|d| \gg 1$. We further obtain from (5.11) that $G_{1d}(\zeta) = R(\zeta; K_{1d})$ is represented as

$$G_{1d}(\zeta) = (u_1 G_{1d}(\zeta) u_2 + v_1 G_0(\zeta) v_0)\, e^{-ig}\, (Id + Y_{d0}(\zeta))^{-1}\, e^{ig}, \tag{5.13}$$

when considered as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to L^2_{loc}.

We recall the representation (5.10) for $Y_{d1}(\zeta)$ to calculate

$$Y_{d1}(\zeta)\, (Id + Y_{d0}(\zeta))^{-1} = [Q_{1d}, v_1] R_\alpha(\zeta) [w_d, \tilde{K}_0] \tilde{G}_0(\zeta) v_0\, (Id + Y_{d0}(\zeta))^{-1}$$

in (5.12). If we decompose $\tilde{G}_0(\zeta)$ into the sum

$$\tilde{G}_0(\zeta) = e^{ig} G_0(\zeta) e^{-ig} = e^{ig}\, (u_1 + v_1)\, G_0(\zeta) e^{-ig},$$

then $[w_d, \tilde{K}_0] u_1 = 0$ and $[w_d, \tilde{K}_0] v_1 = e^{ig} [w_d, K_0] e^{-ig}$. Hence it follows from (5.13) that

$$[w_d, \tilde{K}_0] \tilde{G}_0(\zeta) v_0\, (Id + Y_{d0}(\zeta))^{-1} = e^{ig} [w_d, K_0] G_{1d}(\zeta) e^{-ig}.$$

Thus we have

$$Y_{d1}(\zeta)\, (Id + Y_{d0}(\zeta))^{-1} = [Q_{1d}, v_1] R_\alpha(\zeta) e^{ig} [w_d, K_0] G_{1d}(\zeta) e^{-ig}.$$

By (5.8), $[Q_{1d}, v_1]$ equals $[u_1, Q_{1d}] = e^{ig} [u_1, K_0] e^{-ig}$, so that

$$Id + Y_{d1}(\zeta)\, (Id + Y_{d0}(\zeta))^{-1} = e^{ig}\, (Id + T_d(\zeta))\, e^{-ig}, \tag{5.14}$$

where

$$T_d(\zeta) = [u_1, K_0] e^{-ig} R_\alpha(\zeta) e^{ig} [w_d, K_0] G_{1d}(\zeta). \tag{5.15}$$

We complete the proof of Lemma 5.1, accepting the lemma below as proved.

Lemma 5.3 *Let the notation be as above. Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ such that*

$$Id + T_d(\zeta) : L^2(\Omega) \to L^2(\Omega)$$

has a bounded inverse for $|d| \geq d_\varepsilon(E)$, provided that $\zeta \in D_{1d}$ fulfills $\operatorname{Im} \zeta > -\eta_{1d}(E) + \varepsilon/|d|$.

Completion of Proof of Lemma 5.1. Let ζ be as in Lemma 5.3. Then it follows from (5.12) and (5.14) that

$$(Id + Y_d(\zeta))^{-1} = (Id + Y_{d0}(\zeta))^{-1}\, e^{-ig}\, (Id + T_d(\zeta))^{-1}\, e^{ig}$$

as an operator acting on $L^2(\Omega)$. This, together with (5.6), implies that

$$R_{1d}(\zeta) = \Gamma_d(\zeta)\, (Id + Y_d(\zeta))^{-1}$$

is well defined as an operator from $L^2_{\text{comp}}(\Omega)$ to L^2_{loc}. Once this is established, it is easy to see that $R_{1d}(\zeta) : L^2_{\text{comp}} \to L^2_{\text{loc}}$ is well defined. In fact, this is verified in the same way as in the proof of Lemma 4.1. Thus the proof of the lemma is complete. □

5.3. Proof of Lemma 5.3 We shall prove Lemma 5.3. The lemma is proved by analyzing the behavior as $|d| \to \infty$ of the kernel $T_d(x, y; \zeta)$ of $T_d(\zeta)$ defined by (5.15).

Proof of Lemma 5.3. The proof is long and is divided into four steps. We assume throughout the proof that $\zeta = E + i\eta$ fulfills the assumption in the lemma. Then it follows that

$$\left| \left(e^{i2k|d_-|}/|d_-| \right) f_1(-\hat{d} \to \hat{d}; E) g_\alpha(\hat{d} \to -\hat{d}; E) \right| < 1 - \varepsilon/2 \tag{5.16}$$

for $|d| \gg 1$, where $\operatorname{Im} k = \operatorname{Im} \zeta^{1/2} \le 0$.

(1) Let $w_d(x)$ again be defined by $w_d = \chi(4|x - d_-|/|d_-|)$, so that ∇w_d has support in

$$\Omega_d = \{ |d_-|/4 < |x_-| = |x - d_-| < |d_-|/2 \}.$$

We now consider the behavior of the kernel $G_{1d}(\xi, y; \zeta)$ of $G_{1d}(\zeta)$ for $y \in \Omega = \{|x_-| < 4\}$ and for $\xi \in \Omega_d$. The kernel $G_{1d}(\xi, y; \zeta)$ is written as $G_{1d}(\xi, y; \zeta) = G_1(\xi_-, y_-; \zeta)$ in terms of the kernel $G_1(x, y; \zeta)$ of $G_1(\zeta) = R(\zeta; K_1)$ with $K_1 = K_0 + V_1$. The operator $G_1(\zeta)$ is represented as

$$G_1(\zeta) = G_0(\zeta) - G_0(\zeta) V_1 \Big(G_1(E) + (G_1(\zeta) - G_1(E)) \Big)$$

by the resolvent identity, and the operator $G_0(\zeta) = R(\zeta; K_0)$ has the Hankel function $(i/4) H_0(k(|x - y|)$ as its integral kernel. Since

$$|\xi_- - y_-| = |\xi_-| - \hat{\xi}_- \cdot y_- + O(|d|^{-1}),$$

it follows from (2.12) that $G_0(\xi_-, y_-; \zeta)$ takes the asymptotic form

$$G_0(\xi_-, y_-; \zeta) = c_0(E) e^{ik|\xi_-|} |\xi_-|^{-1/2} \Big(\varphi_0(-y_-; \hat{\xi}_-, E) + O((\log |d|)/|d|) \Big),$$

where $c_0(E)$ is defined by (2.13) and

$$\varphi_0(-y_-; \hat{\xi}_-, E) = \exp\left(-iE^{1/2} y_- \cdot \hat{\xi}_- \right) = \overline{\varphi}_0(y_-; \hat{\xi}_-, E).$$

The incoming eigenfunction $\psi_{1-}(x; \omega, E)$ of K_1 has the representation

$$\psi_{1-}(x; \omega, E) = \varphi_0(x; \omega, E) - \left(G_1(E)^* V_1 \varphi_0 \right)(x; \omega, E).$$

Hence $G_{1d}(\xi, y; \zeta)$ takes the asymptotic form

$$G_{1d}(\xi, y; \zeta) \sim c_0(E) e^{ik|\xi_-|} |\xi_-|^{-1/2} \Big(\overline{\psi}_{1-}(y_-; \hat{\xi}_-, E) + \rho_{3N} \Big),$$

where $\rho_{3N} = \rho_{3N}(\xi, y; \zeta, |d_-|)$ obeys $\left|\partial_\xi^n \partial_y^m \rho_{3N}\right| = O\left((\log|d|)|d|^{-1-|n|}\right)$ and the error estimate $O(|d|^{-N})$ is negligible. On the other hand, it follows from Proposition 4.2 with $\lambda = |d_-|$ that the kernel $R_\alpha(x, \xi; \zeta)$ of $R_\alpha(\zeta)$ admits the decomposition

$$R_\alpha(x, \xi; \zeta) = Z_0(x, \xi; \zeta) + Z_1(x, \xi; \zeta) + O(|d|^{-N})$$

for $(x, \xi) \in \Omega \times \Omega_d$, where $Z_0 = (i/4)e^{i\alpha(\gamma(\hat{x}; -\hat{\xi}) - \pi)} H_0(k|x - \xi|)$ and

$$Z_1 = c_0(E) e^{ik(|x| + |\xi|)} (|x||\xi|)^{-1/2} \Big(g_\alpha(-\hat{\xi} \to \hat{x}; E) + \rho_{2N}\Big)$$

with $\rho_{2N} = \rho_{2N}(x, \xi; \zeta, |d_-|)$ obeying the bound (3.11) with $\lambda = |d_-|$. Thus the leading term of the asymptotic form of $T_d(x, y; \zeta)$ in question is determined by the sum of the two integrals

$$I_j(x, y) = \int [u_1, K_0] e^{-ig} Z_j(x, \xi; \zeta) e^{ig} [w_d, K_0] G_{1d}(\xi, y; \zeta)\, d\xi, \quad j = 0, 1,$$

where $(x, y) \in \Omega \times \Omega$.

(2) The integrands of the above integrals have support in Ω_d as a function of ξ. It is easy to show that $I_0(x, y) = O(|d|^{-N})$ is negligible. In fact, this is verified by repeated use of partial integration, since $\left|e^{ik|x-\xi|} e^{ik|\xi_-|}\right| = O\left(|d|^\nu\right)$ for some $\nu > 1$ and since

$$|\nabla_\xi \left(|x - \xi| + |\xi_-|\right)| \ge c > 0.$$

For the integral $I_1(x, y)$, we calculate only the leading term of the asymptotic form without referring to any contribution from remainder terms, which is seen to be at most of order $O\left((\log|d|)^2/|d|\right)$. We prove in the next two steps that $I_1(x, y)$ behaves like

$$\begin{aligned} I_1(x, y) \quad &\sim \quad c_0(E) \left(\frac{e^{2ik|d_-|}}{|d_-|}\right) g_\alpha(\hat{d} \to -\hat{d}; E) \times \\ &\times \quad \Big([u_1, K_0] \varphi_0(x_-; -\hat{d}, E)\Big) \times \overline{\psi}_{1-}(y_-; \hat{d}, E), \end{aligned} \tag{5.17}$$

which implies that $T_d(\zeta)$ is approximated by an integral operator of rank one. Since $u_1 V_{1d} = V_{1d}$, the L^2 scalar product

$$c_0(E) \Big([u_1, K_0] \varphi_0(\cdot - d_-; -\hat{d}, E), \psi_{1-}(\cdot - d_-; \hat{d}, E)\Big)$$

equals

$$c_0(E) \Big(\varphi_0(\cdot; -\hat{d}, E), V_1 \psi_{1-}(\cdot; \hat{d}, E)\Big) = -f_1(-\hat{d} \to \hat{d}; E).$$

Thus the integral operator of rank one has

$$-\left(e^{i2k|d_-|}/|d_-|\right) g_\alpha(\hat{d} \to -\hat{d}; E) f_1(-\hat{d} \to \hat{d}; E)$$

as a nontrivial eigenvalue. Thus we obtain from (5.16) that $Id + T_d(\zeta)$ has a bounded inverse, provided that (5.17) is established.

(3) We shall show (5.17). By (5.3), $g(x)$ behaves like $g(x) \sim \alpha\pi$ for $x \in \Omega$. If $\xi \in \Omega_d$, then $w_d(\xi) = \chi(4|\xi_-|/|d_-|)$ and $[w_d, K_0]$ takes the form

$$[w_d, K_0] = 2\nabla w_d \cdot \nabla + O(|d|^{-2}) \sim 8|d_-|^{-1}\chi'(4|\xi_-|/|d_-|)\left(\hat{\xi}_- \cdot \nabla\right).$$

Hence we have

$$[w_d, K_0]e^{ik|\xi_-|} \sim 8iE^{1/2}|d_-|^{-1}\chi'(4|\xi_-|/|d_-|)e^{ik|\xi_-|}.$$

Thus the leading term of $I_1(x,y)$ is determined as

$$I_1(x,y) \sim 8iE^{1/2}e^{-i\alpha\pi}c_0(E)^2|d_-|^{-1}\Big([u_1, K_0]J(x,y)\Big), \tag{5.18}$$

where $J(x,y)$ is defined by

$$J(x,y) = e^{ik|x|}|x|^{-1/2}\int e^{ik(|\xi|+|\xi_-|)}\left(|\xi||\xi_-|\right)^{-1/2} h(x,\xi,y)\, d\xi$$

with

$$h(x,\xi,y) = e^{ig(\xi)}\chi'\left(4|\xi_-|/|d_-|\right) g_\alpha(-\hat{\xi} \to \hat{x}; E)\overline{\psi}_{1-}(y_-; \hat{\xi}_-, E).$$

We analyze the behavior of $J(x,y)$. To do this, we work in the polar coordinates

$$\xi_- = \xi - d_- = (r\cos\theta, r\sin\theta), \quad r = |\xi_-|, \quad \theta = \gamma(\hat{\xi}_-; \hat{d}),$$

with $d_- = -\kappa d$ as the center, where $\gamma(\hat{\xi}_-; \hat{d})$ denotes the azimuth angle from $\hat{d}$ to $\hat{\xi}_-$. If we make the change of variable $r = |d_-|\rho$, then ρ ranges over the interval $1/4 < \rho < 1/2$, and

$$|\xi| = \left(|d_-|^2 + r^2 - 2|d_-|r\cos\theta\right)^{1/2} = |d_-|\left(1 + \rho^2 - 2\rho\cos\theta\right)^{1/2}.$$

The integral $J(x,y)$ takes the form

$$J(x,y) = e^{ik|x|}|x|^{-1/2}|d_-|\int\left\{\int e^{ik|d_-|\varphi(\rho,\theta)}\tilde{h}(x,\rho,\theta,y)\, d\theta\right\} d\rho,$$

where $\varphi(\rho,\theta) = (1+\rho^2-2\rho\cos\theta)^{1/2} + \rho$ and

$$\tilde{h} = e^{ig(\xi)}\chi'(4\rho)\left((1+\rho^2-2\rho\cos\theta)/\rho\right)^{-1/2} g_\alpha(-\hat{\xi} \to \hat{x}; E)\overline{\psi}_{1-}(y_-; \hat{\xi}_-, E)$$

with $\xi = \xi_- + d_- = |d_-|\rho\,\hat{\xi}_- + d_-$.

(4) The proof is completed in this step. We apply the stationary phase method ([6, Theorem 7.7.5]) to the integral with respect to θ in the brackets. The function

$$\theta \mapsto |d_-|\varphi(\rho,\theta) = |\xi| + |\xi_-|$$

attains its minimum at $\theta = 0$ for ρ fixed, when the segment joining d_- and the origin intersects the circle $|\xi_-| = |d_-|\rho$. The phase function $\varphi(\rho, \theta)$ satisfies

$$e^{ik|d_-|\varphi(\rho,0)} = e^{ik|d_-|}, \quad \varphi''(\rho, 0) = (\partial/\partial\theta)^2\varphi(\rho, 0) = \rho/(1-\rho)$$

at $\theta = 0$. The second relation, together with (2.13), yields

$$\begin{aligned}(k|d_-|\varphi''(\rho,0)/2\pi i)^{-1/2} &= (2\pi)^{1/2}\exp(i\pi/4)k^{-1/2}\left((1-\rho)/\rho\right)^{1/2}|d_-|^{-1/2} \\ &= (i/2)\left(E^{-1/2}/c_0(E)\right)\left((1-\rho)/\rho\right)^{1/2}|d_-|^{-1/2}\left(1 + O((\log|d|)/|d|)\right).\end{aligned}$$

We look at the value of $\tilde{h}(x, \rho, \theta, y)$ at $\theta = 0$. We have $\hat{\xi} = -\hat{d}$ and $\hat{\xi}_- = \hat{d}$ at $\theta = 0$, and hence $\exp(ig(\xi)) = \exp(i\alpha\pi)$ at $\xi = (1-\rho)d_-$ by (5.3). We also have that $g_\alpha(-\hat{\xi} \to \hat{x}; E)$ equals $g_\alpha(\hat{d} \to \hat{x}; E)$ at $\theta = 0$. We further note that

$$\int \chi'(4\rho)\, d\rho = \int_0^\infty \chi'(4\rho)\, d\rho = -1/4.$$

Thus the leading term of $J(x, y)$ takes the form

$$J \sim -\frac{i}{8}\left(\frac{E^{-1/2}}{c_0(E)}\right) e^{i\alpha\pi} e^{ik(|x|+|d_-|)} \left(|d_-|/|x|\right)^{1/2} g_\alpha(\hat{d} \to \hat{x}; E)\overline{\psi}_{1-}(y_-; \hat{d}, E).$$

If $x \in \Omega$, then $\hat{x} = -\hat{d} + O(|d|^{-1})$, $\;|d_-|/|x| = 1 + O(|d|^{-1})$ and

$$\begin{aligned}e^{ik|x|} &= e^{ik|d_-|}e^{-ikx_-\cdot\hat{d}}\left(1 + O(|d|^{-1})\right) \\ &= e^{ik|d_-|}\varphi_0(x_-; -\hat{d}, E)\left(1 + O((\log|d|)/|d|)\right).\end{aligned}$$

We insert these relations into (5.18) to obtain the desired leading term (5.17), and the proof of the lemma is now complete. □

5.4. Proof of Lemma 5.2 We move to the proof of Lemma 5.2. We use the notation u_j and v_j, $j = 0, 1, 2$, with the meaning ascribed in subsection 5.2. The proof uses the following lemma.

Lemma 5.4 *Let* $Y_{d0}(\zeta) : L^2(\Omega) \to L^2(\Omega)$ *be defined by* (5.9). *Then we have the relation*

$$(Id + Y_{d0}(E)^*)\, e^{ig}\psi_{1-}(x_-; \omega, E) = e^{ig}v_0\varphi_0(x_-; \omega; E)$$

over Ω, *where the incoming eigenfunction* ψ_{1-} *of* $K_1 = K_0 + V_1$ *is restricted to* Ω.

Proof. According to (2.18), (5.7) and (5.8), we have

$$\begin{aligned}Y_{d0}(E)^* &= u_2\tilde{G}_{1d}(E)^*[u_1, Q_{1d}] + v_0\tilde{G}_0(E)^*[v_1, Q_{1d}] \\ &= u_2e^{ig}G_{1d}(E)^*[u_1, K_0]e^{-ig} + v_0e^{ig}G_0(E)^*[v_1, K_0]e^{-ig}.\end{aligned}$$

We note that $\operatorname{supp} u_1 \subset \Omega$. This enables us to calculate

$$u_2 e^{ig} G_{1d}(E)^* [u_1, K_0] \psi_{1-} = u_2 e^{ig} G_{1d}(E)^* [u_1, K_{1d}] \psi_{1-} = -e^{ig} u_1 \psi_{1-}$$

over Ω, where $\psi_{1-} = \psi_{1-}(x_-; \omega, E)$. Similarly we have

$$\begin{aligned} v_0 e^{ig} G_0(E)^* [v_1, K_0] \psi_{1-} &= v_0 e^{ig} G_0(E)^* [K_0, u_1] \psi_{1-} \\ &= e^{ig} v_0 u_1 \psi_{1-} + e^{ig} v_0 G_0(E)^* V_{1d} \psi_{1-} \\ &= e^{ig} v_0 (1 - v_1) \psi_{1-} + e^{ig} v_0 G_0(E)^* V_{1d} \psi_{1-} \\ &= -e^{ig} v_1 \psi_{1-} + e^{ig} v_0 \left(\psi_{1-} + G_0(E)^* V_{1d} \psi_{1-} \right) \\ &= -e^{ig} v_1 \psi_{1-} + e^{ig} v_0 \varphi_0. \end{aligned}$$

This yields the desired relation. $\square$

Proof of Lemma 5.2. We have

$$\left| e^{i2k|d_-|}/|d| \right| + \left| e^{i2k|d_+|}/|d| \right| = O(|d|^{-c}) \tag{5.19}$$

for some $c > 0$, where $d_+ = (1-\kappa)d$ denotes the center of $\operatorname{supp} V_{2d}$. By assumption, $\zeta = E + i\eta \in D_d$ and α is not a half integer. Hence it follows from (5.1) that

$$|\eta| < \eta_d(E) < \min\left(\eta_{1d}(E), \eta_{2d}(E)\right)$$

for $|d| \gg 1$. We consider only the operator $\Pi_{1d}(\zeta) = V_{2d} R_{1d}(\zeta) \chi_{1d}$ in detail. We have already established the following relations :

$$R_{1d}(\zeta) = \Gamma_d(\zeta) \left(Id + Y_d(\zeta) \right)^{-1} : L^2(\Omega) \to L^2_{\mathrm{loc}}$$

$$\left(Id + Y_d(\zeta) \right)^{-1} = \left(Id + Y_{d0}(\zeta) \right)^{-1} \left(Id + e^{ig} T_d(\zeta) e^{-ig} \right)^{-1} : L^2(\Omega) \to L^2(\Omega)$$

where $\Gamma_d(\zeta)$ is defined by (5.5). The first relation follows from (5.6) and the second one from (5.12) and (5.14). Since $V_{2d} u_1 = 0$ and $V_{2d} v_1 = V_{2d}$, we have $V_{2d} \Gamma_d(\zeta) = V_{2d} R_\alpha(\zeta) v_0$, so that

$$\Pi_{1d}(\zeta) = V_{2d} R_\alpha(\zeta) v_0 \left(Id + Y_{d0}(\zeta) \right)^{-1} \left(Id + e^{ig} T_d(\zeta) e^{-ig} \right)^{-1} \chi_{1d}.$$

We have also shown

$$\| T_d(\zeta) \| = O\left(\left| e^{i2k|d_-|}/|d| \right| \right)$$

as a bounded operator acting on $L^2(\Omega)$ in the course of the proof of Lemma 5.3 (see (5.17)), and hence

$$\| T_d(\zeta) \| = O(|d|^{-c}) \tag{5.20}$$

for $c > 0$ as in (5.19).

We use Proposition 4.1 with $\mu = 0$ and $\lambda = |d|$ to analyze the behavior of the kernel $\tilde{\Pi}_{1d}(x, y; \zeta)$ of the operator

$$\tilde{\Pi}_{1d}(\zeta) = V_{2d} R_\alpha(\zeta) v_0 \left(Id + Y_{d0}(\zeta) \right)^{-1} : L^2(\Omega) \to L^2(\Sigma_{2d}),$$

where $\Sigma_{2d} = \{|x - d_+| < 1\} \supset \operatorname{supp} V_{2d}$. If $x \in \Sigma_{2d}$ and $y \in \Omega$, then $|\hat{x} \cdot \hat{y} + 1| < c\,|d|^{-2}$ for some $c > 0$, and

$$|x - y| = |d| + \hat{d} \cdot (x_+ - y_-) + O(|d|^{-1}),$$

where $x_+ = x - d_+$. We also have

$$|x| \sim |d|, \quad |y| \sim |d|, \quad |x| + |y| = |d| + O(1).$$

If $\zeta \in D_d$, then $\left| e^{ik|d|}/|d|^{1/2} \right| = O(1)$ is bounded uniformly in $|d| \gg 1$. Thus it follows from (2.10) and Proposition 4.1 that

$$V_{2d}(x) R_\alpha(x, y; \zeta) \sim c_0(E) \cos(\alpha\pi) \left(e^{ik|d|}/|d|^{1/2} \right) \varphi_0(x_+; \hat{d}, E) \overline{\varphi}_0(y_-; \hat{d}, E) \quad (5.21)$$

with the remainder term obeying the bound $O\left((\log|d|)/|d|\right)$. As already stated, $(Id + Y_{d0}(\zeta))^{-1} : L^2(\Omega) \to L^2(\Omega)$ is bounded uniformly in d. Since $|\zeta - E| = O((\log|d|)/|d|)$ for $\zeta = E + i\mathrm{Im}\,\zeta \in D_d$, we have

$$\|Y_{d0}(\zeta) - Y_{d0}(E)\| = O((\log|d|)/|d|)$$

as a bounded operator on $L^2(\Omega)$, and hence

$$\left\| (Id + Y_{d0}(\zeta))^{-1} - (Id + Y_{d0}(E))^{-1} \right\| = O((\log|d|)/|d|)$$

We are now in a position to apply Lemma 5.4 to (5.21). If $y \in \Omega$, then $e^{ig} \sim e^{i\alpha\pi}$ by (5.3), and we see that $\tilde{\Pi}_{1d}(x, y; \zeta)$ behaves like

$$\tilde{\Pi}_{1d}(x, y; \zeta) \sim c_0(E) \cos(\alpha\pi) \left(e^{ik|d|}/|d|^{1/2} \right) V_{2d}(x) \varphi_0(x_+; \hat{d}, E) \overline{\psi}_{1-}(y_-; \hat{d}, E).$$

This, together with (5.20), yields that the leading term of $\Pi_{1d}(x, y; \zeta)$ takes the form

$$\Pi_{1d}(x, y; \zeta) \sim c_0(E) \cos(\alpha\pi) \left(e^{ik|d|}/|d|^{1/2} \right) V_{2d}(x) \varphi_0(x_+; \hat{d}, E) \overline{\psi}_{1-}(y_-; \hat{d}, E)$$

Similarly the kernel $\Pi_{2d}(x, y; \zeta)$ of $\Pi_{2d}(\zeta) = V_{1d} R_{2d}(\zeta) \chi_{2d}$ behaves like

$$\Pi_{2d}(x, y; \zeta) \sim c_0(E) \cos(\alpha\pi) \left(e^{ik|d|}/|d|^{1/2} \right) V_{1d}(x) \varphi_0(x_-; -\hat{d}, E) \overline{\psi}_{2-}(y_+; -\hat{d}, E)$$

for $x \in \Sigma_{1d} \subset \Omega$ and for $y \in \{|y - d_+| < 4\}$. The L^2 scalar products equal

$$c_0(E) \left(V_1 \varphi_0(\cdot; -\hat{d}, E), \psi_{1-}(\cdot; \hat{d}, E) \right) = -f_1(-\hat{d} \to \hat{d}; E),$$
$$c_0(E) \left(V_2 \varphi_0(\cdot; \hat{d}, E), \psi_{2-}(\cdot; -\hat{d}, E) \right) = -f_2(\hat{d} \to -\hat{d}; E).$$

If we now observe that

$$\left| \left(e^{i2k|d|}/|d| \right) \cos^2(\alpha\pi) f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) \right| < 1 - \varepsilon/2$$

for ζ as in the lemma, then we see that the operator in the lemma is invertible, and the proof is complete. □

5.5. Proof of Theorem 1.3 We end the section by proving Theorem 1.3. The proof is done in almost the same way as that of Theorem 1.2.

Proof of Theorem 1.3. We first note that if $|d| \gg 1$, then $\eta_{2d}(E) < \eta_{1d}(E)$ or $\eta_{1d}(E) < \eta_{2d}(E)$ according as $0 < \kappa < 1/2$ or $1/2 < \kappa < 1$. We also note that

$$\left(E^{1/2}/|d|\right)\left((2-\varepsilon)\log|d|\right) < \min\left(\eta_{1d}(E), \eta_{2d}(E)\right), \quad \varepsilon > 0,$$

for $\kappa = 1/2$.

(1) Let $0 < \kappa < 1/2$. Assume that $\eta > -\eta_{2d}(E) + \varepsilon/|d|$ for $\zeta = E + i\eta$ with $|E - E_0| < \delta_0$. Then $\eta > -\eta_{1d}(E) + \varepsilon/|d|$, and

$$Id + V_{1d}R_\alpha(\zeta)\chi_{1d} : L^2(\Omega) \to L^2(\Omega)$$

is invertible and the inverse is bounded uniformly in $|d| \gg 1$, as is shown in the proof of Lemma 5.1. We can show a similar result for $Id + V_{2d}R_\alpha(\zeta)\chi_{2d}$. Thus it suffices to show that

$$\begin{pmatrix} Id & V_{1d}R_{2d}(\zeta)\chi_{2d} \\ V_{2d}R_{1d}(\zeta)\chi_{1d} & Id \end{pmatrix} : \mathcal{X} \to \mathcal{X}$$

is invertible for ζ as above. As is seen from the proof of Lemma 5.2, $\Pi_{1d}(\zeta) = V_{2d}R_{1d}(\zeta)\chi_{1d}$ is represented as

$$\Pi_{1d}(\zeta) = V_{2d}R_\alpha(\zeta)v_0\left(Id + Y_{d0}(\zeta)\right)^{-1}\left(Id + e^{ig}T_d(\zeta)e^{-ig}\right)^{-1}\chi_{1d}.$$

We again use Proposition 4.1 with $\mu = 0$ to see the behavior of the kernel $R_\alpha(x, y; \zeta)$ with the half integer flux α. If $x \in \operatorname{supp} V_{2d}$ and $y \in \Omega$, it behaves like

$$R_\alpha(x, y; \zeta) \sim e^{ik|d|}/|d|,$$

so that $\|\Pi_{1d}(\zeta)\| = O\left(\left|e^{ik|d|}/|d|\right|\right) = o(1)$ as $|d| \to \infty$, provided ζ satisfies the assumption above. Similarly we obtain $\|V_{1d}R_{2d}(\zeta)\chi_{2d}\| = o(1)$. Hence the statement (1) is verified.

(2) This is proved in exactly the same way as (1).

(3) Let $\kappa = 1/2$. Assume that $\eta > -\left(E^{1/2}/|d|\right)\left((2-\varepsilon)\log|d|\right)$ for $\zeta = E + i\eta$ with $|E - E_0| < \delta_0$. Then $\eta > -\min\left(\eta_{1d}(E), \eta_{2d}(E)\right) + \varepsilon/|d|$, and hence

$$Id + V_{1d}R_\alpha(\zeta)\chi_{1d}, \quad Id + V_{2d}R_\alpha(\zeta)\chi_{2d} : L^2 \to L^2$$

are invertible for ζ as above. We can show in the same way as above that

$$\|V_{2d}R_{1d}(\zeta)\chi_{1d}\| + \|V_{1d}R_{2d}(\zeta)\chi_{2d}\| = o(1)$$

as $|d| \to \infty$. This proves the statement (3), and the proof is complete. $\square$

6. Asymptotic properties of the Green function

This section is devoted to proving Propositions 3.1, 3.2 and 3.3. We prove these propositions only for ζ with $\operatorname{Im}\zeta \le 0$ and write $\zeta = E - i\eta$ with $0 \le \eta \le c_1(\log\lambda)/\lambda$. Before going into the proof of the propositions, we first establish the basic integral representation for the kernel $R_0(x, y; \zeta)$. The derivation is based on the following formula

$$H_\nu(Z)J_\nu(z) = \frac{1}{i\pi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{Z^2+z^2}{2t}\right) I_\nu\left(\frac{Zz}{t}\right)\frac{dt}{t}, \quad |z| \le |Z|,$$

for the product of Bessel functions ([15, p.439]), where $I_\nu(w)$ is defined by (3.6) and the contour is taken to be rectilinear with corner at $\kappa + i0$, $\kappa > 0$ being fixed arbitrarily. We use the notation κ with the meaning ascribed above throughout the section. We apply to (3.9) this formula with $Z = k(|x| \vee |y|)$ and $z = k(|x| \wedge |y|)$, where $k = \zeta^{1/2}$ with $\operatorname{Im} k \le 0$. If we write $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates, then $R_0(x, y; \zeta)$ is represented as

$$R_0(x, y; \zeta) = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta\left(|x|^2+|y|^2\right)}{2t}\right) I_\nu\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t} \qquad (6.1)$$

with $\nu = |l - \alpha|$, where $\psi = \theta - \omega$. If, in particular, $\alpha = 0$, then the resolvent $R(\zeta; K_0)$ of the free Hamiltonian K_0 has the kernel $(i/4)H_0(k|x-y|)$ represented as

$$\frac{i}{4}H_0(k|x-y|) = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta\left(|x|^2+|y|^2\right)}{2t}\right) I_l\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t},$$

where $I_{|l|}(w) = I_l(w) = (1/\pi)\int_0^\pi e^{w\cos\rho}\cos(l\rho)\,d\rho$ (see (3.6)). By the Fourier expansion, the series $\sum_l e^{il\psi}I_l(w)$ converges to $e^{w\cos\psi}$. Since

$$|x-y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\psi, \qquad (6.2)$$

the kernel $(i/4)H_0(k|x-y|)$ has the representation

$$\frac{i}{4}H_0(k|x-y|) = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right)\frac{dt}{t}. \qquad (6.3)$$

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Throughout the proof, we assume that α is not an integer, so that $\beta = \alpha - [\alpha]$ always satisfies $0 < \beta < 1$. If α is an integer, $R_0(x, y; \zeta)$ is easily shown to coincide with

$$R_0(x, y; \zeta) = (i/4)\cos(\alpha\pi)e^{i\alpha(\gamma(\hat{x};\hat{y})-\pi)}H_0(k|x-y|)$$

and hence the proposition holds true. We fix $M \gg 1$ large enough and take

$$\kappa = M^2 \log \lambda$$

in the contour of the integral (6.1). We divide (6.1) into the sum of integrals over the following four intervals by a smooth partition of unity :

$$(0)\ 0 < t < \kappa,\ \ (i)\ 0 < s < 2\lambda/M,\ \ (ii)\ \lambda/M < s < 2M\lambda,\ \ (iii)\ s > M\lambda \quad (6.4)$$

for $t = \kappa + is$. We evaluate the integral over each interval. The main contribution comes from the integral over interval (ii). If $t = \kappa + is$ satisfies (i) or (ii), then

$$\mathrm{Re}(\zeta/t) = \left(\kappa^2 + s^2\right)^{-1} (E\kappa - \eta s) > 0 \quad (6.5)$$

for κ as above. The proof is long and is divided into six steps.

(1) We recall that $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ satisfy

$$\lambda/c \le |x|,\ |y|,\ |x-y| \le c\lambda, \quad |\hat{x}\cdot\hat{y} + 1| < c\lambda^{2(\mu-1)}$$

for some $c > 1$ and for some $0 \le \mu < 1/2$. We begin by evaluating the integral over the interval (iii) in (6.4) and show that it obeys the bound $O(\lambda^{-N})$ for any $N \gg 1$. To see this, we employ the formula

$$I_\nu(w) = \frac{e^{-i\nu\pi/2}}{\pi}\left\{\int_0^\pi \cos\left(\nu\rho - iw\sin\rho\right)\, d\rho - \sin(\nu\pi)\int_0^\infty e^{-iw\sinh p - \nu p}\, dp\right\} \quad (6.6)$$

for $\mathrm{Im}\, w \le 0$, which is obtained as an immediate consequence of the formula $I_\nu(w) = e^{-i\nu\pi/2} J_\nu(iw)$ ([15, p.176]). We note that

$$\mathrm{Im}(\zeta/t) = -\left(\kappa^2 + s^2\right)^{-1}(Es + \eta\kappa) < 0$$

for $t = \kappa + is$. We insert $I_\nu(\zeta|x||y|/t)$ into (6.1) and evaluate the resulting integral by partial integration for each l with $|l| < \lambda$. If $M \gg 1$, then

$$\left|\partial_t\left(t - \zeta(|x|^2 + |y|^2)/t \pm (\zeta|x||y|/t)\sin\rho\right)\right| > c > 0$$
$$\left|\partial_t\left(t - \zeta(|x|^2 + |y|^2)/t - (2i\zeta|x||y|/t)\sinh p\right)\right| > c > 0$$

for $t = \kappa + is$ with $s > M\lambda$ uniformly in ρ, $0 < \rho < \pi$, and in p, $0 < p < 1$. If $p > 1$, then we use $\left|\partial_t(t - \zeta(|x|^2 + |y|^2)/t)\right| > c > 0$ and

$$\partial_t e^{-i(\zeta|x||y|/t)\sinh p} = -t^{-1}\left(\sinh p/\cosh p\right)\partial_p e^{-i(\zeta|x||y|/t)\sinh p}. \quad (6.7)$$

We take these relations into account to repeat the integration by parts. Then the sum of the integrals with $|l| < \lambda$ obeys $O(\lambda^{-N})$. To see that the sum over l with $|l| > \lambda$ is also negligible, we make use of the other representation formula

$$I_\nu(w) = \frac{(w/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)}\int_{-1}^1 e^{-w\rho}\left(1 - \rho^2\right)^{\nu - 1/2}\, d\rho \quad (6.8)$$

for $I_\nu(w)$ with $\nu \geq 0$ ([15, p.172]). Since $|x| + |y| = O(\lambda)$, we have $|w| = \left|\zeta|x||y|/t\right| = O(\lambda)/M$ for $s = \operatorname{Im} t > M\lambda$ and

$$\left|e^{-w\rho}\right| = O\left(e^{|\operatorname{Re}(\zeta|x||y|/t)|}\right) = O\left(e^{\lambda}\right), \quad |\rho| < 1.$$

By the Stirling formula, $\Gamma(\nu) \sim (2\pi)^{1/2} e^{-\nu} \nu^{\nu-(1/2)}$ for $\nu \gg 1$. Thus we can take $M \gg 1$ so large that $|w^\nu/\Gamma(\nu)| \leq (1/2)^{|l|}$, $\nu = |l - \alpha|$, for $|l| > \lambda$. Hence the sum of integrals with $|l| > \lambda$ also obeys $O(\lambda^{-N})$, and it follows that the integral (6.1) over interval (iii) is negligible.

(2) Let $\chi_M(t)$ be defined by $\chi_M(t) = \chi(\operatorname{Im} t/(M\lambda))$ over the contour of the integral (6.1), so that $\chi_M(t) = 1$ for $0 \leq t \leq \kappa$ and for $t = \kappa + is$ with $0 \leq s \leq M\lambda$. Then we consider the integral

$$R(x, y; \zeta) = \frac{1}{4\pi} \sum_l e^{il\psi} \int_0^{\kappa+i\infty} \chi_M(t) \exp\left(\frac{t}{2} - \frac{\zeta\left(|x|^2 + |y|^2\right)}{2t}\right) I_\nu\left(\frac{\zeta|x||y|}{t}\right) \frac{dt}{t}.$$

The representation (3.6) for $I_\nu(w)$ is used to calculate the series

$$L(w, \psi) = \sum_l e^{il\psi} I_\nu(w), \quad \nu = |l - \alpha|,$$

in the integrand above, where $\psi = \theta - \omega$ and $w = \zeta|x||y|/t$. It is decomposed into $L(w, \psi) = L_{\mathrm{fr}}(w, \psi) + L_{\mathrm{sc}}(\omega, \psi)$, where

$$L_{\mathrm{fr}}(w, \psi) = (1/\pi) \sum_l e^{il\psi} \int_0^\pi e^{w\cos\rho} \cos(\nu\rho)\, d\rho,$$
$$L_{\mathrm{sc}}(w, \psi) = -\,(1/\pi) \sum_l e^{il\psi} \sin(\nu\pi) \int_0^\infty e^{-w\cosh p - \nu p}\, dp.$$

We have $L_{\mathrm{fr}}(w) = e^{i\alpha\psi} e^{w\cos\psi}$ by the Fourier expansion and

$$L_{\mathrm{sc}}(w, \psi) = -\frac{\sin(\alpha\pi)}{\pi} (-1)^{[\alpha]} e^{i[\alpha]\psi} \int e^{-w\cosh p} \frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\, dp \tag{6.9}$$

with $0 < \beta = \alpha - [\alpha] < 1$ by the same argument as used to calculate the eigenfunction φ_+ in section 3. It should be noted that the two relations hold true only for $|\psi| < \pi$. If $\psi = \pm\pi$, the denominator $e^p + e^{-i\psi}$ vanishes at $p = 0$. By (6.2), $R(x, y; \zeta)$ is written as the sum of the two integrals

$$R(x, y; \zeta) = R_{\mathrm{fr}}(x, y; \zeta) + R_{\mathrm{sc}}(x, y; \zeta),$$

where

$$R_{\mathrm{fr}} = \frac{1}{4\pi} e^{i\alpha\psi} \int_0^{\kappa+i\infty} \chi_M(t) \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right) \frac{dt}{t},$$
$$R_{\mathrm{sc}} = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \chi_M(t) \exp\left(\frac{t}{2} - \frac{\zeta\left(|x|^2 + |y|^2\right)}{2t}\right) L_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t}.$$

We now consider $R_{\mathrm{fr}}(x,y;\zeta)$. We may assume that $\psi \sim \pi$ $(\psi < \pi)$ or $\psi \sim -\pi$ $(\psi > -\pi)$ because $|\hat{x}\cdot\hat{y}+1| = O\left(\lambda^{2(\mu-1)}\right)$ by assumption. We set

$$\sigma = \sigma(x,y) = \gamma(\hat{x};\hat{y}) - \pi.$$

Then it follows that σ obeys $|\sigma| = O(\lambda^{\mu-1})$ for $0 \le \mu < 1/2$. If $\psi < \pi$, then $\sigma = \psi - \pi < 0$ and $e^{i\alpha\psi} = (\cos(\alpha\pi) + i\sin(\alpha\pi))\, e^{i\alpha\sigma}$. If $\psi > -\pi$, then $\sigma = \psi + \pi > 0$ and $e^{i\alpha\psi} = (\cos(\alpha\pi) - i\sin(\alpha\pi))\, e^{i\alpha\sigma}$. Since $|x-y| < c\lambda$ by assumption, we have, by partial integration, that

$$\int_0^{\kappa+i\infty} (1-\chi_M(t)) \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right) \frac{dt}{t} = O(\lambda^{-N}) \qquad (6.10)$$

for any $N \gg 1$. This, together with (6.3), yields

$$R_{\mathrm{fr}}(x,y;\zeta) = (i/4)\,(\cos(\alpha\pi) \mp i\sin(\alpha\pi))\, e^{i\alpha\sigma} H_0(k|x-y|) + O(\lambda^{-N}) \qquad (6.11)$$

for $\pm\sigma > 0$.

(3) Let σ be as above and let $L_{\mathrm{sc}}(w,\psi)$ be defined by (6.9) with $w = \zeta|x||y|/t$. Then we have

$$L_{\mathrm{sc}}(w,\psi) = -\frac{\sin(\alpha\pi)}{\pi} e^{i[\alpha]\sigma} \int_{-\infty}^{\infty} e^{-w\cosh p} \frac{e^{(1-\beta)p}}{e^p - e^{-i\sigma}}\, dp$$

with $\beta = \alpha - [\alpha]$. By analyticity, we represent the above integral as integrals over contours in the complex plane. Let

$$l_\pm = \{p < -\lambda^{-1/2}\} \cup \pi_\pm \cup \{p > \lambda^{-1/2}\}$$

and let $c_\pm$ be the closed curves formed by the two parts $c_\pm = \pi_\pm \cup \{|p| < \lambda^{-1/2}\}$, where

$$\pi_+ = \left\{p = \lambda^{-1/2}e^{i\theta} : \pi > \theta > 0\right\}, \quad \pi_- = \left\{p = \lambda^{-1/2}e^{i\theta} : -\pi < \theta < 0\right\}.$$

The semicircles π_+ in l_+ and π_- in l_- are directed negatively and positively respectively, but both the closed curves c_+ and c_- are positively directed. We now assume that $\sigma < 0$. Then it follows by Cauchy's integral formula that

$$\int_{c_+} e^{-w\cosh p} \frac{e^{(1-\beta)p}}{e^p - e^{-i\sigma}}\, dp = 2\pi i\, e^{i\beta\sigma} e^{-w\cos\sigma} = 2\pi i\, e^{i(\alpha-[\alpha])\sigma} e^{-w\cos\sigma}$$

and $\int_{c_-} e^{-w\cosh p} \frac{e^{(1-\alpha)p}}{e^p - e^{-i\sigma}}\, dp = 0$. Hence $L_{\mathrm{sc}}(w,\psi)$ equals

$$L_{\mathrm{sc}} = -\sin(\alpha\pi)\left(ie^{i\alpha\sigma}e^{-w\cos\sigma} + \frac{e^{i[\alpha]\sigma}}{2\pi}\left\{\int_{l_+} + \int_{l_-}\right\} e^{-w\cosh p} \frac{e^{(1-\beta)p}}{e^p - e^{-i\sigma}}\, dp\right).$$

Since $-\cos\sigma = \cos\psi$, we again use (6.2) and (6.3) together with (6.10) to see that $R_{\mathrm{sc}}(x,y;\zeta)$ takes the form

$$R_{\mathrm{sc}} = \frac{1}{4}\sin(\alpha\pi)e^{i\alpha\sigma}H_0(k|x-y|) - \frac{\sin(\alpha\pi)}{8\pi^2}e^{i[\alpha]\sigma}\tilde{R}_{\mathrm{sc}}(x,y;\zeta) + O(\lambda^{-N})$$

for $\sigma < 0$, where $\tilde{R}_{\mathrm{sc}}(x,y;\zeta)$ is defined by the integral

$$\int_0^{\kappa+i\infty}\chi_M(t)\exp\left(\frac{t}{2}-\frac{\zeta}{2t}\left(|x|^2+|y|^2\right)\right)\left(\left\{\int_{l_+}+\int_{l_-}\right\}e^{-w\cosh p}\frac{e^{(1-\beta)p}}{e^p-e^{-i\sigma}}\,dp\right)\frac{dt}{t}$$

with $w = \zeta|x||y|/t$. Similarly we have

$$R_{\mathrm{sc}} = -\frac{1}{4}\sin(\alpha\pi)e^{i\alpha\sigma}H_0(k|x-y|) - \frac{\sin(\alpha\pi)}{8\pi^2}e^{i[\alpha]\sigma}\tilde{R}_{\mathrm{sc}}(x,y;\zeta) + O(\lambda^{-N})$$

for $\sigma > 0$. This, together with (6.11), implies that the kernel $R_0(x,y;\zeta)$ in question behaves like

$$R_0 = \frac{i}{4}\cos(\alpha\pi)e^{i\alpha\sigma}H_0(k|x-y|) - \frac{\sin(\alpha\pi)}{8\pi^2}e^{i[\alpha]\sigma}\tilde{R}_{\mathrm{sc}}(x,y;\zeta) + O(\lambda^{-N})$$

for $\sigma \neq 0$. As is easily seen, this relation remains true even for $\sigma = 0$. Thus the first term on the right side of the relation in the proposition is obtained.

(4) Next we evaluate the second term. We assert that

$$\tilde{R}_{\mathrm{sc}}(x,y;\zeta) = e^{ik(|x|+|y|)}\left(|x|+|y|\right)^{-1/2}e_{\pm N}(x,y;\zeta) + O(\lambda^{-N}), \tag{6.12}$$

where $e_{\pm N}$ satisfies (3.10). This yields the desired asymptotic form for $R_0(x,y;\zeta)$. To see this, we define $R_\pm(x,y;\zeta)$ by

$$R_\pm = \int_0^{\kappa+i\infty}\chi_M(t)\exp\left(\frac{t}{2}-\frac{\zeta}{2t}(|x|+|y|)^2\right)L_\pm\left(\frac{\zeta|x||y|}{t},\sigma\right)\frac{dt}{t},$$

where

$$L_\pm(w,\sigma) = \int_{l_\pm}e^{-w(\cosh p-1)}\frac{e^{(1-\beta)p}}{e^p-e^{-i\sigma}}\,dp.$$

Then

$$\tilde{R}_{\mathrm{sc}}(x,y;\zeta) = R_+(x,y;\zeta) + R_-(x,y;\zeta).$$

We divide R_+ and R_- into the sum of the integrals over intervals (0), (i) and (ii) in (6.4) and prove that the integrals over (0) and (i) are negligible. We consider $R_+(x,y;\zeta)$ only.

If $p \in l_+$, then $\left|e^p - e^{-i\sigma}\right| > c\lambda^{-1/2}$ for some $c > 0$, and if, moreover, $p \in \pi_+$, then $\cosh p - 1 = O(|p|^2) = O(\lambda^{-1})$. Assume that $0 < t < \kappa$. Then $\mathrm{Re}(\zeta/t) = E/t > 0$ and it follows that

$$\left|\exp(-\zeta(|x|+|y|)^2/2t)\exp\left(-(\zeta/t)(\cosh p-1)\right)\right| = O\left(e^{-c\lambda^2/t}\right)$$

uniformly in $p \in l_+$. This yields that the integral over the interval $(0, \kappa)$ obeys the bound $O(\lambda^{-N})$. We evaluate the integral over interval (i). By (6.5), $\mathrm{Re}\,(\zeta/t) > 0$ and $e^t = O\left(\lambda^{M^2}\right)$ is of polynomial growth in λ for $t = \kappa + is$ with $0 < s < 2\lambda/M$. Assume that $t = \kappa + is$ satisfies $0 < s < 2\lambda^{1-\delta}$ for $0 < \delta \ll 1$. Then

$$\left|\exp(-\zeta(|x| + |y|)^2/2t) \exp\left(-(\zeta/t)(\cosh p - 1)\right)\right| = O\left(e^{-c\lambda^{2\delta}/t}\right)$$

uniformly in $p \in l_+$. This shows that the integral over $\left(\kappa + i0, \kappa + i2\lambda^{1-\delta}\right)$ is negligible. The integral over $(\kappa + i\lambda^{1-\delta}, \kappa + i2\lambda/M)$ is evaluated by making use of partial integration. We can take $M \gg 1$ so large that

$$\left|\partial_t \left(t - \zeta(|x| + |y|)^2/t - (\zeta|x||y|/t)(\cosh p - 1)\right)\right| \geq c > 0$$

for t as above, provided that $p \in \pi_+$ or $\lambda^{-1/2} < |p| < 1$. If $|p| > 1$, we use the relations $\left|\partial_t \left(t - \zeta(|x| + |y|)^2/t\right)\right| \geq c > 0$ and

$$\partial_t e^{-(\zeta|x||y|/t)(\cosh p - 1)} = -t^{-1}\left((\cosh p - 1)/\sinh p\right) \partial_p e^{-(\zeta|x||y|/t)(\cosh p - 1)}.$$

Then we obtain that the integral over interval (i) is also negligible.

(5) The behavior of $\tilde{R}_{\mathrm{sc}}(x, y; \zeta)$ is determined by the integral over interval (ii) in (6.4). If $t = \kappa + is$ satisfies $\lambda/M < s < M\lambda$, then it follows from (6.5) that $\mathrm{Re}\,(\zeta|x||y|/t) > c\lambda$ with some $c > 0$, and it is easy to see that

$$\int e^{-(\zeta|x||y|/t)(\cosh p - 1)} \chi_\infty\left(\frac{p}{\lambda^{1/2-\delta}}\right) \frac{e^{(1-\beta)p}}{e^p - e^{-i\sigma}}\, dp = O(\lambda^{-N})$$

for $\delta > 0$, where $\chi_\infty(p) = 1 - \chi(|p|)$. In fact, the stationary point $p = 0$ is away from the support of the integrand. We define $q_M \in C_0^\infty[0, \infty)$ by

$$q_M(s) = \chi(2s/M)\left(1 - \chi(Ms)\right),$$

so that q_M has support in $(1/M, M)$ and $q_M = 1$ on $[2/M, M/2]$. We further define $\tilde{R}_\pm(x, y; \zeta)$ by

$$\tilde{R}_\pm = \int_0^{\kappa + i\infty} q_M\left(\frac{\mathrm{Im}\, t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta}{2t}(|x| + |y|)^2\right) \tilde{L}_\pm\left(\frac{\zeta|x||y|}{t}, \sigma\right) \frac{dt}{t},$$

where

$$\tilde{L}_\pm(w, \sigma) = \int_{l_\pm} e^{-w(\cosh p - 1)} \left(1 - \chi_\infty\left(\frac{p}{\lambda^{1/2-\delta}}\right)\right) \frac{e^{(1-\beta)p}}{e^p - e^{-i\sigma}}\, dp.$$

Then we have

$$\tilde{R}_{\mathrm{sc}}(x, y; \zeta) = \tilde{R}_+(x, y; \zeta) + \tilde{R}_-(x, y; \zeta) + O(\lambda^{-N}).$$

We deform the contour into the imaginary axis by analyticity to study the behavior of the integrals $\tilde{R}_+$ and $\tilde{R}_-$. Assume that $t \in \boldsymbol{C}$ satisfies $0 \le \operatorname{Re} t \le \kappa$ and either $\lambda/(cM) \le \operatorname{Im} t \le c\lambda/M$ or $M\lambda/c \le \operatorname{Im} t \le cM\lambda$ for some $c > 1$. Then we can take $M \gg 1$ so large that $\left|\partial_t \left(t - \zeta\left(|x| + |y|\right)^2/t\right)\right| > c_0 > 0$. We also have

$$\operatorname{Re}\left(\zeta/t\right) = (E \operatorname{Re} t - \eta \operatorname{Im} t)/|t|^2 = O((\log \lambda)/\lambda^2)$$

for t as above. This implies that $\exp(-\zeta(|x| + |y|)^2/t)$ is at most of polynomial growth in λ. If $p \in \pi_\pm$, then

$$|(\zeta|x||y|/t)(\cosh p - 1)| = O(\lambda)O(|p|^2) = O(1)$$

is bounded uniformly in λ, and if p satisfies $\lambda^{-1/2} < |p| < \lambda^{-1/2+\delta}$ for $0 < \delta \ll 1$, then

$$\Big(\operatorname{Re}(\zeta|x||y|/t)\Big)(\cosh p - 1) = O(\log \lambda)O(|p|^2) = O(1)$$

is also bounded uniformly in λ. Hence we can easily see that

$$\partial_t^l \tilde{L}_\pm(\zeta|x||y|/t, \sigma) = O\left(\lambda^{\delta-(1-2\delta)l}\right) \tag{6.13}$$

uniformly in t as above, because $|(\zeta|x||y|/t^2)(\cosh p - 1)| = O(\lambda^{-1+2\delta})$. Thus we can deform the contour into the imaginary axis to obtain

$$\tilde{R}_{\mathrm{sc}}(x, y; \zeta) = G_{\mathrm{sc}}(x, y; \zeta) + O(\lambda^{-N}),$$

where

$$G_{\mathrm{sc}}(x, y; \zeta) = \int_0^\infty q_M(t/\lambda) \exp\left(i\left(\frac{t}{2} + \frac{\zeta}{2t}(|x| + |y|)^2\right)\right) F\left(\frac{\zeta|x||y|}{t}, \sigma\right) \frac{dt}{t}$$

with $F(w, \sigma) = \tilde{L}_+(-iw, \sigma) + \tilde{L}_-(-iw, \sigma)$. The integral interval may be slightly shrunken or expanded, if necessary.

(6) In the last step, we analyze the behavior of $G_{\mathrm{sc}}(x, y; \zeta)$ by use of the stationary phase method and prove assertion (6.12). We make the change of variable $t = \rho\tau$ with $\rho = E^{1/2}(|x| + |y|) \sim \lambda$ to obtain that

$$G_{\mathrm{sc}}(x, y; \zeta) = \int_0^\infty \exp\left(i\rho f(\tau)\right) \exp(a(\tau)) g(\tau) \frac{d\tau}{\tau},$$

where $f(\tau) = \left(\tau + 1/\tau\right)/2$ and

$$a(\tau) = \frac{i}{2\rho\tau}(\zeta - E)(|x| + |y|)^2, \quad g(\tau) = q_M(\rho\tau/\lambda) F\left(\frac{\zeta|x||y|}{\rho\tau}, \sigma\right).$$

By (6.13), $g(\tau)$ satisfies $\partial_\tau^l g = O\left(\lambda^{\delta+2\delta l}\right)$. We apply the stationary phase method to the integral above. The phase function $f(\tau)$ has the unique stationary point $\tau = 1$. Then we have $e^{i\rho f(1)} = \exp(iE^{1/2}(|x| + |y|))$ and

$$\left(\rho f''(1)/2\pi i\right)^{-1/2} = (2\pi)^{1/2} e^{i\pi/4} E^{-1/4}(|x| + |y|)^{-1/2}.$$

We further have

$$\begin{aligned} i\rho f(1)+a(1) &= (i/2)\left(\zeta/E^{1/2}+E^{1/2}\right)(|x|+|y|) \\ &= i\left(E^{1/2}-i\eta/(2E^{1/2})\right)(|x|+|y|)=ik(|x|+|y|)+O((\log\lambda)^2/\lambda). \end{aligned}$$

and

$$\begin{aligned} g(1) &= F(\zeta|x||y|/\rho,\sigma)=\tilde{L}_+(-i\zeta|x||y|/\rho,\sigma)+\tilde{L}_-(-i\zeta|x||y|/\rho,\sigma) \\ &= \left\{\int_{l_+}+\int_{l_-}\right\} e^{i(\zeta|x||y|/\rho)(\cosh p-1)}\left(1-\chi_\infty\left(\frac{p}{\lambda^{1/2-\delta}}\right)\right)\frac{e^{(1-\beta)p}}{e^p-e^{-i\sigma}}\,dp. \end{aligned}$$

We take a look at only the leading term for the behavior as $\lambda\to\infty$ of $g(1)$. It behaves like

$$\begin{aligned} g(1) \sim & \left\{\int_{\pi_+}+\int_{\pi_-}\right\}\frac{1}{p+i\sigma}\,dp+\frac{i\zeta|x||y|}{\rho}\left\{\int_{\pi_+}+\int_{\pi_-}\right\}\frac{p^2}{p+i\sigma}\,dp \\ &+\int_{|p|>\lambda^{-1/2}} e^{i(\zeta|x||y|/\rho)(\cosh p-1)}\left(1-\chi_\infty\left(\frac{p}{\lambda^{1/2-\delta}}\right)\right)\frac{1}{p+i\sigma}\,dp. \quad (6.14) \end{aligned}$$

Since $\left\{\int_{\pi_+}+\int_{\pi_-}\right\}p^{-1}\,dp=0$, the first term on the right side of (6.14) obeys

$$-\left\{\int_{\pi_+}+\int_{\pi_-}\right\}\frac{i\sigma}{p(p+i\sigma)}\,dp=O(\lambda^{\mu-1})O(\lambda^{1/2})=O(\lambda^{\mu-1/2}).$$

Since $\int_{\pi_\pm} p\,dp=0$, the second term obeys

$$\begin{aligned} O(\lambda)\left\{\int_{\pi_+}+\int_{\pi_-}\right\}\frac{p^2}{p+i\sigma}\,dp &= O(\lambda)\left\{\int_{\pi_+}+\int_{\pi_-}\right\}\left(p-i\sigma-\frac{\sigma^2}{p+i\sigma}\right)dp \\ &= O(\lambda)\left(O\left(\lambda^{\mu-3/2}\right)+O\left(\lambda^{2\mu-2}\right)\right)=O\left(\lambda^{\mu-1/2}\right). \end{aligned}$$

The third term equals

$$\int_{\lambda^{-1/2}}^{\infty} e^{i(\zeta|x||y|/\rho)(\cosh p-1)}\left(1-\chi_\infty\left(\frac{p}{\lambda^{1/2-\delta}}\right)\right)\left(\frac{1}{p+i\sigma}-\frac{1}{p-i\sigma}\right)dp$$

and obeys $O\left(\lambda^{\mu-1}\right)\int_{\lambda^{-1/2}}^{\infty}\frac{1}{p^2+\sigma^2}\,dp=O\left(\lambda^{\mu-1/2}\right)$. Thus we have established

$$G_{\mathrm{sc}}(x,y;\zeta)=e^{ik(|x|+|y|)}(|x|+|y|)^{-1/2}O\left(\lambda^{\mu-1/2}\right),$$

which implies (3.10) with $|n|=|m|=0$. If we take account of the estimate

$$\partial_x^n\partial_y^m\left(e^p-e^{-i\sigma}\right)^{-1}=\left|\left(e^p-e^{-i\sigma}\right)^{-1}\right|O\left(\lambda^{-|n|/2-|m|/2}\right),$$

the other cases with $|n| + |m| \neq 0$ are dealt with in a similar way. We skip the details. The proof of the proposition is now complete. $\square$

The remaining two propositions are much easier to prove than Proposition 3.1. We give only a sketch of the proof.

Proof of Proposition 3.2. We repeat the same argument as in the proof of Proposition 3.1 to obtain the decomposition

$$R_0(x, y; \zeta) = R_{\mathrm{fr}}(x, y; \zeta) + R_{\mathrm{sc}}(x, y; \zeta) + O(\lambda^{-N}),$$

where

$$R_{\mathrm{fr}} = \frac{1}{4\pi}\, e^{i\alpha\psi} \int_0^{\kappa+i\infty} \chi_M(t) \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right) \frac{dt}{t},$$

$$R_{\mathrm{sc}} = \frac{1}{4\pi} \int_0^{\kappa+i\infty} q_M\left(\frac{\mathrm{Im}\, t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta}{2t}(|x|+|y|)^2\right) L_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t}$$

and

$$L_{\mathrm{sc}}(w, \psi) = -\frac{\sin(\alpha\pi)}{\pi}\, (-1)^{[\alpha]}\, e^{i[\alpha]\psi} \int e^{-w(\cosh p - 1)} \frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\, dp$$

with $0 < \beta = \alpha - [\alpha] < 1$. By assumption, ψ stays away from $\pm\pi$, so that the denominator $e^p + e^{-i\psi}$ does not vanish at $p = 0$. This makes it possible to prove the proposition more easily than Proposition 3.1. We may write

$$\psi = \theta - \omega = \gamma(\hat{x}; -\hat{y}) - \pi,$$

and hence it follows from (6.3) and (6.10) that $R_{\mathrm{fr}}(x, y; \zeta)$ behaves like

$$R_{\mathrm{fr}} = (i/4) e^{i\alpha(\gamma(\hat{x}; -\hat{y}) - \pi)} H_0(k|x-y|) + O(\lambda^{-N}).$$

Thus the first term of the asymptotic form is obtained. We further write

$$L_{\mathrm{sc}}(\zeta|x||y|/t, \psi) = -\frac{\sin(\alpha\pi)}{\pi} e^{i[\alpha]\gamma(\hat{x}; -\hat{y})} \int e^{i(i\zeta|x||y|/t)(\cosh p - 1)} \frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\, dp.$$

If $t = \kappa + is$ satisfies $\lambda/M < s < M\lambda$, then $\left|\zeta|x||y|/t\right| \sim \lambda$ and $\mathrm{Re}(\zeta/t) > 0$ by (6.5). We recall that the amplitude $g_0(\omega \to \theta; E)$ is defined by (3.8) and note that $\zeta^{-1/2}$ behaves like

$$\zeta^{-1/2} = (E - i\eta)^{-1/2} = E^{-1/2} + O\left((\log \lambda)/\lambda\right).$$

Hence the stationary phase method ([6, Theorem 7.7.5]) applied to the integral $L_{\mathrm{sc}}(\zeta|x||y|/t, \psi)$ yields

$$L_{\mathrm{sc}}(\zeta|x||y|/t, \psi) \sim E^{-1/4} e^{-i\pi/4} g_0(-\hat{y} \to \hat{x}; E)(|x||y|)^{-1/2} t^{1/2}.$$

We deform the contour to the imaginary axis to obtain that

$$R_{\rm sc}(x,y;\zeta) \sim (4\pi)^{-1}E^{-1/4}e^{-i\pi/4}g_0(-\hat{y}\to\hat{x};E)(|x||y|)^{-1/2}\times$$
$$\times\int_0^\infty q_M(t/\lambda)\exp\left(i\left(\frac{t}{2}+\frac{\zeta}{2t}(|x|+|y|)^2\right)\right)(it)^{1/2}\,\frac{dt}{t}.$$

We make the change of variable $t=\rho\tau$ with $\rho=E^{1/2}(|x|+|y|)\sim\lambda$ and use again the stationary phase method. Then $R_{\rm sc}(x,y;\zeta)$ behaves like

$$R_{\rm sc}\sim c_0(E)e^{ik(|x|+|y|)}(|x||y|)^{-1/2}g_0(-\hat{y}\to\hat{x};E).$$

Thus the leading term of the asymptotic form of $R_{\rm sc}(x,y;\zeta)$ is obtained. $\square$

The proof of Proposition 3.3 requires the simple lemma below.

Lemma 6.1 *Let $u\geq 0$. Then $\left(\sum_l |I_\nu(u)|\right)=O\left(e^{cu}\right)$ as $u\to\infty$ for some $c>0$, where $\nu=|l-\alpha|$.*

Proof. By (6.8), we have $|I_\nu(u)|=\dfrac{e^u}{2^\nu\Gamma(\nu+1/2)}\,O(u^\nu)$ for $\nu\gg 1$. It is easy to see that $e^{-cu}u^\nu\leq e^{-\nu}\left(\nu/c\right)^\nu$ for $c>1$. Thus the lemma follows from the Stirling formula. $\square$

Proof of Proposition 3.3. We give only a sketch for the proof of statement (1). We take $Z=k|x|$ and $z=k|y|$ in (6.1), so that

$$R_0=\frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{\zeta|x|^2}{2t}\right)\exp\left(-\frac{\zeta|y|^2}{2t}\right)I_\nu\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t}\qquad(6.15)$$

with $\kappa=M^2\log\lambda$. We again divide this integral into the sum of integrals over the four intervals (0) $\sim$ (*iii*) in (6.4). The main contribution comes from the integral over interval (*ii*).

By assumption, $\lambda/c<|x|<c\lambda$ and $1/c<|y|<c$. If $t\in(0,\kappa)$, then it follows from Lemma 6.1 that

$$\left|\exp(-\zeta|x|^2/4t)\right|\left(\sum_l |I_\nu(\zeta|x||y|/t)|\right)=O(1)\qquad(6.16)$$

is bounded uniformly in t. Hence the integral over the interval $(0,\kappa)$ obeys the bound $O(\lambda^{-N})$. The estimate (6.16) remains true for $t=\kappa+is$ with $0<s<2\lambda^{1-\delta}$, $0<\delta\ll 1$, and a similar argument applies to the integral over $(\kappa+i0,\kappa+i2\lambda^{1-\delta})$. If $\lambda^{1-\delta}<s<2\lambda/M$, we use the representation (6.6) for $I_\nu(w)$ with $w=Zz/t$. We insert $I_\nu(Zz/t)$ into (6.15). We evaluate the resulting integral for each l by repeated use of partial integration. If $M \gg 1$ and $|y| < c$, then

$$\left|\partial_t \left(t - \zeta|x|^2/t \pm (\zeta|x||y|/t)\sin\rho\right)\right| > c > 0$$
$$\left|\partial_t \left(t - \zeta|x|^2/t - (2i\zeta|x||y|/t)\sinh p\right)\right| > c > 0$$

for $t = \kappa + is$ with $\lambda^{1-\delta} < s < 2\lambda/M$ uniformly in ρ, $0 < \rho < \pi$, and in p, $0 < p < 1$. If $p > 1$, then we use $\left|\partial_t(t - \zeta|x|^2/t)\right| > c > 0$ and (6.7). Then the integral can be shown to obey $O(\lambda^{-N})$ uniformly in l with $|l| < \lambda$. The sum over l with $|l| > \lambda$ is controlled by the Stirling formula combined with (6.8). Thus it follows that the integral over interval (i) is negligible. Similarly it is shown that the integral over interval (iii) is also negligible. Hence $R_0(x, y; \zeta)$ behaves like

$$R_0(x, y; \zeta) = G_-(x, y; \zeta) + O(\lambda^{-N}), \qquad \lambda \to \infty,$$

where

$$G_- = \frac{1}{4\pi}\int_0^{\kappa+i\infty} q_M\left(\frac{\operatorname{Im} t}{\lambda}\right)\exp\left(\frac{t}{2} - \frac{\zeta|x|^2}{2t}\right)\exp\left(-\frac{\zeta|y|^2}{2t}\right) I\left(\frac{\zeta|x||y|}{t}, \psi\right)\frac{dt}{t}\,dt$$

and $I(w, \psi)$ is defined by $I(w, \psi) = \sum_l e^{il\psi} I_\nu(w)$. By (6.8), the series $I(w, \psi)$ converges absolutely and is a analytic function in w, provided that $1/c < |w| < c$ for some $c > 1$. If we deform the contour to the imaginary axis and make the change of variable $t = \rho\tau$ with $\rho = E^{1/2}|x| \sim \lambda$, then the behavior of $G_-(x, y; \zeta)$ is determined by the integral

$$\tilde{G}_-(x, y; \zeta) = \frac{1}{4\pi}\int_0^\infty \exp\left(i\rho f(\tau)\right)\exp(a(\tau))g(\tau)\frac{d\tau}{\tau},$$

where $f(\tau) = \left(\tau + 1/\tau\right)/2$ and

$$a(\tau) = \frac{i}{2\rho\tau}(\zeta - E)|x|^2, \quad g(\tau) = \exp\left(\frac{i\zeta|y|^2}{2\rho\tau}\right) q_M\left(\frac{\rho\tau}{\lambda}\right) I\left(\frac{\zeta}{i\rho\tau}|x||y|, \psi\right)$$

We apply the stationary phase method to the integral with $\tau = 1$ as a stationary point. Then we have

$$e^{i\rho f(1)} = \exp(iE^{1/2}|x|), \quad (\rho f''(1)/2\pi i)^{-1/2} = (2\pi)^{1/2} e^{i\pi/4} E^{-1/4}|x|^{-1/2}$$

and

$$i\rho f(1) + a(1) = (i/2)\left(\zeta/E^{1/2} + E^{1/2}\right)|x| = ik|x| + O((\log\lambda)^2/\lambda).$$

We also have

$$I\left(\zeta|x||y|/i\rho, \psi\right) = I(z/i, \psi) = \sum_l e^{il\psi} I_\nu(z/i)$$

at $\tau = 1$, where

$$z = \left(\zeta/E^{1/2}\right)|y| = E^{1/2}|y| + O((\log\lambda)/\lambda).$$

Since $I_\nu(z/i) = e^{-i\nu\pi/2}J_\nu(z)$ and since

$$e^{il\psi} = e^{il(\theta-\omega)} = e^{il\gamma(\hat{x};\hat{y})} = e^{-il\gamma(\hat{y};\hat{x})},$$

we have by (3.5) that

$$\sum_l e^{il\psi}I_\nu(z/i) = \sum_l e^{-il\gamma(\hat{y};\hat{x})}e^{-i\nu\pi/2}J_\nu(z) = \overline{\varphi}_-(y;\hat{x},E) + O((\log\lambda)/\lambda)$$

for z as above. Hence $g(1) = \overline{\varphi}_-(y;\hat{x},E) + O((\log\lambda)/\lambda)$ uniformly in y, $1/c < |y| < c$. This yields the desired asymptotic form for $R_0(x,y;\zeta)$, when x and y satisfy the assumption of statement (1). □

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