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Resonance free regions in magnetic scattering by two solenoidal fields at large separation

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Abstract

We consider the problem of quantum resonances in magnetic scattering by two solenoidal fields at large separation in two dimensions. This system has trapped trajectories oscillating between two centers of the fields. We give a sharp lower bound on resonance widths when the distance between the two centers goes to infinity. The bound is described in terms of backward amplitudes calculated explicitly for scattering by each solenoidal field. The study is based on a new type of complex scaling method. As an application, we also discuss the relation to semiclassical resonances in scattering by two solenoidal fields.

Keywords: Resonances; Magnetic scattering; Solenoidal fields; Aharonov–Bohm effect

1. Introduction

In the present paper we study the problem of quantum resonances in magnetic scattering by two solenoidal fields at large separation. We work in two dimensions $\boldsymbol{R}^2$ throughout the entire discussion. We write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2} (-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j$$

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for the magnetic Schrödinger operator with potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ associated with the vector potential A is defined by

$$b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1$$

and the magnetic flux of b is defined by $\alpha = (2\pi)^{-1} \int b(x)\, dx$, where the integration with no domain attached is taken over the whole space.

Let $\Phi : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the potential defined by

$$\Phi(x) = \bigl(-x_2/|x|^2, x_1/|x|^2\bigr) = \bigl(-\partial_2 \log|x|, \partial_1 \log|x|\bigr), \tag{1.1}$$

which generates the point-like field (solenoidal field)

$$\nabla \times \Phi = \bigl(\partial_1\partial_1 \log|x| + \partial_2\partial_2 \log|x|\bigr) = \Delta \log|x| = 2\pi\delta(x)$$

with center at the origin. The quantum particle moving in the solenoidal field $2\pi\alpha\delta(x)$ with α as a magnetic flux is governed by the energy operator

$$P_\alpha = H(\alpha\Phi). \tag{1.2}$$

This is symmetric over $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$, but it is not necessarily essentially self-adjoint in the space $L^2 = L^2(\boldsymbol{R}^2)$ because of the strong singularity at the origin of Φ. We know [1,8] that it is a symmetric operator with type $(2, 2)$ of deficiency indices. The self-adjoint extension is realized by imposing a boundary condition at the origin. Its Friedrichs extension denoted by the same notation P_α has the domain

$$\mathcal{D}(P_\alpha) = \Bigl\{u \in L^2\colon\ (-i\nabla - \alpha\Phi)^2 u \in L^2,\ \lim_{|x|\to 0} \bigl|u(x)\bigr| < \infty\Bigr\}, \tag{1.3}$$

where $(-i\nabla - \alpha\Phi)^2 u$ is understood in $\mathcal{D}'(\boldsymbol{R}^2 \setminus \{0\})$ (in the sense of distribution).

The energy operator which governs quantum particles moving in a solenoidal field is often called the Aharonov–Bohm Hamiltonian in the physics literatures. This model was employed by Aharonov and Bohm [4] in 1959 in order to convince us theoretically that a magnetic potential itself has a direct significance in quantum mechanics. This phenomenon, unexplainable from a classical mechanical point of view, is now called the Aharonov–Bohm effect, which is known as one of the most remarkable quantum phenomena.

The scattering by one solenoidal field is also known as one of the exactly solvable quantum systems. We give a quick review of it in Section 2. In particular, the amplitude $f_\alpha(\theta \to \omega; E)$ for scattering from the initial direction $\omega \in S^1$ to the final direction θ at energy $E > 0$ is explicitly calculated as

$$f_\alpha(\omega \to \theta; E) = (2/\pi)^{1/2} e^{i\pi/4} E^{-1/4} \sin(\alpha\pi) e^{i[\alpha](\theta-\omega)} \frac{e^{i(\theta-\omega)}}{1 - e^{i(\theta-\omega)}}, \tag{1.4}$$

where the Gauss notation $[\alpha]$ denotes the greatest integer not exceeding α and the coordinates over the unit circle S^1 are identified with the azimuth angles from the positive x_1 axis. We also note that there are no resonances in the case of scattering by one solenoidal field, as seen in Section 2 below.

We formulate the problem which we want to discuss in this paper. We consider the energy operator

$$H_d = H(\Phi_d), \qquad \Phi_d(x) = \alpha_1 \Phi(x - d_1) + \alpha_2 \Phi(x - d_2), \tag{1.5}$$

which describes the quantum particle moving in the two solenoids $2\pi\alpha_1\delta(x - d_1)$ and $2\pi\alpha_2\delta(x - d_2)$. The operator H_d becomes self-adjoint under the boundary conditions $\lim_{|x-d_j|\to 0} |u(x)| < \infty$ for $j = 1, 2$, and the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2 \to L^2, \quad \zeta = E + i\eta,\ E > 0,\ \eta > 0,$$

is meromorphically continued over the lower half of the complex plane across the positive real axis where the spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\operatorname{Im}\zeta \leqslant 0$ is well defined as an operator from L^2_{comp} to L^2_{loc} in the sense that $\chi R(\zeta; H_d)\chi : L^2 \to L^2$ is bounded for every $\chi \in C_0^\infty(\boldsymbol{R}^2)$, where L^2_{comp} and L^2_{loc} denote the spaces of square integrable functions with compact support and of locally square integrable functions over $\boldsymbol{R}^2$, respectively. We refer to [14, Section 7] for the spectral properties of H_d: H_d has no bound states and the spectrum is absolutely continuous on $[0, \infty)$. The meromorphic continuation of $R(\zeta; H_d)$ over the unphysical sheet (the lower-half plane) follows as an application of the analytic perturbation theory of Fredholm for compact operators. For completeness, we shall show it in Appendix A.

The resonances of H_d are defined as the poles of the meromorphic function with values in operators from L^2_{comp} to L^2_{loc}. Our aim is to study to what extent $R(\zeta; H_d)$ can be analytically extended across the positive real axis as the distance $|d| = |d_2 - d_1|$ goes to infinity. We give a sharp lower bound on the resonance widths (imaginary parts of resonances) in terms of the backward amplitude $f_j(\omega \to -\omega; E)$ for scattering by each solenoidal field $2\pi\alpha_j\delta(x)$. As is seen from (1.4), the backward amplitude takes the form

$$f_j(\omega \to -\omega; E) = (2\pi)^{-1/2} e^{i\pi/4} E^{-1/4} (-1)^{[\alpha_j]+1} \sin(\alpha_j \pi),$$

which is independent of the direction ω. The main theorem is as follows.

Theorem 1.1. *Let the notation be as above and let $E > 0$. Assume that neither the flux α_1 nor α_2 is an integer. Set $\hat{d} = d/|d|$ for $d = d_2 - d_1$. Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ large enough such that $\zeta = E - i\eta$ with $0 < \eta < \eta_{\varepsilon d}(E)$ is not a resonance of H_d for $|d| > d_\varepsilon(E)$, where*

$$\eta_{\varepsilon d}(E) = \frac{E^{1/2}}{|d|} \bigl\{\log|d| - \log\bigl|f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E)\bigr| - \varepsilon\bigr\}.$$

Remark 1.1. If either of the two fluxes α_1 and α_2 is an integer, H_d is easily seen to be unitarily equivalent to the Hamiltonian with one solenoidal field, and hence H_d has no resonances. Since the scattering amplitude vanishes for an integer flux, Theorem 1.1 remains true in this special case also.

Remark 1.2. A slightly modified argument applies to magnetic Schrödinger operators with fields with compact supports at large separation. For example, such an argument applies to the operator

$$H_d = (-i\nabla - B_d)^2, \qquad B_d(x) = A_1(x - d_1) + A_2(x - d_2),$$

where $A_j \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ has the fields $b_j = \nabla \times A_j \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$. The result of Theorem 1.1 remains true with the backward amplitude for scattering by the fields b_j.

Corollary 1.1. *Assume that the same assumptions as in Theorem* 1.1 *are fulfilled. If* $\zeta_d(E) = E + i\,\mathrm{Im}\,\zeta_d(E)$ *is a resonance of* H_d*, then, for any* $\varepsilon > 0$ *small enough, there exists* $d_\varepsilon(E) \gg 1$ *such that the resonance width* $-\mathrm{Im}\,\zeta_d(E)$ *satisfies* $-\mathrm{Im}\,\zeta_d(E) > \eta_{\varepsilon d}(E)$ *for* $|d| > d_\varepsilon(E)$.

We make a comment on how to determine the constant $\eta_{\varepsilon d}(E)$ in the theorem. It is determined so that

$$\left| \frac{e^{2ik|d|}}{|d|} f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) \right| < 1 - \varepsilon/2, \quad k = \zeta^{1/2} \tag{1.6}$$

for $|d| \gg 1$, provided that $\zeta = E - i\eta$ satisfies $0 < \eta < \eta_{\varepsilon d}(E)$. We shall explain here from a heuristic point of view how sharp the bound in the theorem is and how reasonable

$$\rho_0 = \frac{e^{2ik|d|}}{|d|} f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) = 1 \tag{1.7}$$

is as an approximate relation to determine the location of the resonances near the real axis. We first consider the scattering by the solenoidal filed $2\pi\alpha\delta(x)$. As stated in Proposition 5.1 in Section 5, the Green function $R_\alpha(x, y; \zeta)$ of the resolvent $R(\zeta; P_\alpha) = (P_\alpha - \zeta)^{-1}$ with $\zeta = E - i\eta$ in the lower-half plane behaves like

$$R_\alpha(x, y; \zeta) \sim e^{ik|x-y|}|x-y|^{-1/2} + e^{ik(|y|+|x|)}\big(|y||x|\big)^{-1/2} f_\alpha(-\hat{y} \to \hat{x}; E) \tag{1.8}$$

with $\hat{y} = y/|y|$ and $\hat{x} = x/|x|$ when $|x|, |y| \gg 1$ and $|x - y| \gg 1$, where $k = \zeta^{1/2}$ and some numerical factors are ignored for brevity. The first term on the right side corresponds to the free trajectory which goes from y to x directly without being scattered at the origin, while the second term comes from the scattering trajectory which starts from y and arrives at x after being scattered by $2\pi\alpha\delta(x)$.

We now turn to scattering by the two solenoidal fields $2\pi\alpha_1\delta(x)$ and $2\pi\alpha_2\delta(x - d)$ with the origin and $d \in \boldsymbol{R}^2$ as centers. We denote by $f_j(\omega \to \theta)$ the amplitude for scattering from the direction ω to θ by $2\pi\alpha_j\delta(x)$, and in particular, we write simply f_1 and f_2 for the backward amplitudes $f_1(-\hat{d} \to \hat{d})$ and $f_2(\hat{d} \to -\hat{d})$, respectively. According to the asymptotic formula (1.8), the quantity associated with the trajectory starting from the origin and coming back to the origin after being scattered by $2\pi\alpha_2\delta(x - d)$ takes the form $(e^{2ik|d|}/|d|) f_2$, which is seen by setting $x = y = -d$ in the second term on the right side of (1.8). Let $\tau_0(x, y)$ be the trajectory which starts from y, hits the origin and arrives at x from the origin after oscillating between the origin and d several times. Then the contribution from $\tau_0(x, y)$ to the asymptotic form of the Green function is formally given by the series

$$e^{ik|x-y|}|x-y|^{-1/2} + e^{ik(|y|+|x|)}\big(|y||x|\big)^{-1/2} f_1(-\hat{y} \to \hat{x})$$
$$+ e^{ik|y|}|y|^{-1/2} f_1(-\hat{y} \to \hat{d}) \left(\sum_{n=0}^{\infty} \rho_0^n \right) \big\{ \big(e^{2ik|d|}/|d|\big) f_2 \big\} f_1(-\hat{d} \to \hat{x}) e^{ik|x|}|x|^{-1/2},$$

where ρ_0 is defined by (1.7). For example, the term with ρ_0^n describes the contribution from the trajectory oscillating $n+1$ times. Thus the location of the resonance is approximately determined by the relation $\rho_0 = 1$, and this intuitive idea clarifies the mechanism by which trapping trajectories generate the resonances near the real axis.

The rigorous proof of Theorem 1.1 is based on a new type of complex scaling method. The details are explained in Section 3 where we prove the theorem, accepting some lemmas as proved, and Sections 4, 5 and 6 are devoted to proving those lemmas. One of the difficulties in the resonance problem is that we have to control quantities growing exponentially at infinity. Such quantities cannot be controlled simply by integration by parts using oscillatory properties. We use a new method of complex scaling to avoid these difficulties.

We discuss the relation to the semiclassical theory for quantum resonances in scattering by two solenoidal fields. We now consider the self-adjoint operator

$$\tilde{H}_h = (-ih\nabla - \Psi)^2, \qquad \Psi(x) = \alpha_1 \Phi(x - p_1) + \alpha_2 \Phi(x - p_2), \quad 0 < h \ll 1,$$

under the boundary conditions $\lim_{|x-p_j|\to 0} |u(x)| < \infty$ at the two centers p_1 and p_2. We denote by $\gamma(x)$ the azimuth angle from the positive x_1 axis to $\hat{x} = x/|x|$ and define the two unitary operators

$$(U_1 f)(x) = h^{-1} f\big(h^{-1}x\big), \qquad (U_2 f)(x) = \exp\big(ig_h(x)\big) f(x)$$

acting on L^2, where $g_h = [\alpha_1/h]\gamma(x-d_1) + [\alpha_2/h]\gamma(x-d_2)$ with $d_j = p_j/h$. Since $\nabla\gamma(x) = \Phi(x)$, $g_h(x)$ satisfies

$$\nabla g_h = [\alpha_1/h]\Phi(x-d_1) + [\alpha_2/h]\Phi(x-d_2),$$

and $\exp(ig_h(x))$ is well defined as a single valued function. Then $\tilde{H}_h$ turns out to be unitarily equivalent to $H(\Psi_d) = (U_1U_2)^* \tilde{H}_h (U_1U_2)$, where

$$\Psi_d(x) = \beta_1 \Phi(x-d_1) + \beta_2 \Phi(x-d_2), \qquad \beta_j = \alpha_j/h - [\alpha_j/h], \quad d_j = p_j/h.$$

Thus the semiclassical resonance problem in scattering by two solenoidal fields is reduced to the resonance problem for magnetic Schrödinger operators with two solenoidal fields with centers at large separation

$$|d| = |d_2 - d_1| = |p_2 - p_1|/h = |p|/h \gg 1.$$

We denote by $\tilde{f}_j(\omega \to -\omega; E)$, $j = 1, 2$, the amplitude for the backward scattering by the field $2\pi\beta_j\delta(x)$ at energy $E > 0$ and by $\tilde{f}_{hj}(\omega \to -\omega; E)$ the semiclassical amplitude for the scattering by the field $2\pi\alpha_j\delta(x)$. The two amplitudes are related through $\tilde{f}_{hj}(\omega \to -\omega; E) =$

$h^{1/2}\tilde{f}_j(\omega \to -\omega; E)$ by (1.4) with E and α replaced by E/h^2 and α/h, respectively, and hence it follows that

$$\log\bigl|\tilde{f}_1(-\hat{p} \to \hat{p}; E)\tilde{f}_2(\hat{p} \to -\hat{p}; E)\bigr| = \log\bigl|\tilde{f}_{h1}(-\hat{p} \to \hat{p}; E)\tilde{f}_{h2}(\hat{p} \to -\hat{p}; E)\bigr| - \log h.$$

The fluxes β_1 and β_2 vary with h. If at least one of the two fluxes β_1 and β_2 is an integer, then

$$-\log\bigl|\tilde{f}_1(-\hat{p} \to \hat{p}; E)\tilde{f}_2(\hat{p} \to -\hat{p}; E)\bigr| = \infty, \quad \hat{p} = p/|p|,$$

because the scattering amplitude vanishes for an integer flux. The choice of $d_\varepsilon(E)$ in Theorem 1.1 depends on the fluxes α_1 and α_2 as well as on the energy $E > 0$. We require the additional assumption that β_1 and β_2 stay away from 0 and 1 uniformly in h; $c < \beta_1, \beta_2 < 1 - c$ for some $0 < c < 1/2$. Then we obtain the following result as an immediate consequence of Theorem 1.1.

Corollary 1.2. *Let the notation be as above. Assume that $\beta_j = \alpha_j/h - [\alpha_j/h]$, $j = 1, 2$, fulfills the flux condition above. Then, for any $\varepsilon > 0$ small enough, there exists $h_\varepsilon(E) > 0$ such that $\zeta = E - i\eta$ with*

$$0 < \eta < \frac{E^{1/2}h}{|p|}\bigl\{-\log\bigl|\tilde{f}_{h1}(-\hat{p} \to \hat{p}; E)\tilde{f}_{h2}(\hat{p} \to -\hat{p}; E)\bigr| + \log|p| - \varepsilon\bigr\}$$

is not a resonance of $\tilde{H}_h$ for $0 < h < h_\varepsilon(E) \ll 1$.

The resonance problem is one of the most active subjects in scattering theory at the present. There is a large number of works devoted to the semiclassical theory of resonances near the real axis generated by closed classical trajectories. An extensive list of references can be found in the book [12], and the paper [19] of Sjöstrand is an excellent exposition on this subject. In particular, the semiclassical problem of shape resonances has been studied in detail, and upper or lower bounds on the resonance width and its asymptotic expansion in h have been obtained by many authors [6,7,9–12,15] under various assumptions. Among these works is the one by Martinez [15] where he has established the following result in potential scattering: For any $M \gg 1$, there exists $h_M(E)$ such that $\zeta = E - i\eta$ with $\eta < -Mh\log h$ is not a resonance of $-h^2\Delta + V$ for $0 < h < h_M(E)$, if E is in the nontrapping energy range. As far as we know, there are no works dealing with the semiclassical bounds on resonance widths for scattering systems by solenoidal fields. Corollary 1.2 gives a new type of lower bound in which backward scattering amplitudes are involved, and it suggests the existence of resonances with the width of order $O(h|\log h|)$ in the trapping energy range.

We end this section by referring to the possibility of generalizing the results here to the case of scattering by several solenoidal fields. It seems to be possible to extend our ideas to such cases, although much more elaborate arguments are required. The results would depend heavily on the location of the centers of the fields, and the Aharonov–Bohm quantum effect is closely related to the bound on the resonance widths. If, for example, the three centers d_1, d_2 and d_3 are collinear with d_2 as the middle point, then the bound on the resonance width is determined by the longest trajectory oscillating between d_1 and d_3, but the potential $\alpha_2\Phi(x - d_2)$ generated by the field $2\pi\alpha_2\delta(x - d_2)$ with the middle point d_2 as a center has a direct significance on the trajectory oscillating between the two centers d_1 and d_3 by the Aharonov–Bohm effect. It seems to be an interesting problem to study how the Aharonov–Bohm effect is reflected in the location of the resonances in scattering by several solenoidal fields.

2. The scattering amplitude by one solenoidal field

Here we make a brief review of the scattering by one solenoidal field. As stated in the previous section, the scattering by such a field is known as one of the exactly solvable models in quantum mechanics. We refer to [1,2,4,8,17] for more detailed expositions.

Let P_α be the self-adjoint operator defined by (1.2) with domain (1.3). We calculate the generalized eigenfunction of the problem $P_\alpha \varphi = E\varphi$ with energy $E > 0$ as an eigenvalue. Since P_α is rotationally invariant, we work in the polar coordinate system (r, θ). Let U be the unitary mapping defined by

$$(Uu)(r,\theta) = r^{1/2} u(r\theta) : L^2 \to L^2\big((0,\infty); dr\big) \otimes L^2\big(S^1\big).$$

We write $\sum_l$ for the summation ranging over all integers l. Then U enables us to decompose P_α into the partial wave expansion

$$P_\alpha \simeq U P_\alpha U^* = \sum_l \oplus (P_{l\alpha} \otimes Id), \tag{2.1}$$

where Id is the identity operator and

$$P_{l\alpha} = -\partial_r^2 + \big(\nu^2 - 1/4\big) r^{-2}, \quad \nu = |l - \alpha|$$

is self-adjoint in $L^2((0,\infty); dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2}|u(r)| < \infty$ at $r = 0$. We denote by $\gamma(x;\omega)$ the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$ and use the notation $\cdot$ to denote the scalar product in $\boldsymbol{R}^2$. Then the outgoing eigenfunction $\varphi_+(x;\omega,E)$ with ω as an incident direction at energy $E > 0$ is calculated as

$$\varphi_+(x;\omega,E) = \sum_l \exp(-i\nu\pi/2)\exp\big(il\gamma(x;-\omega)\big) J_\nu\big(E^{1/2}|x|\big) \tag{2.2}$$

with $\nu = |l - \alpha|$, where $J_\mu(z)$ denotes the Bessel function of order μ. The eigenfunction φ_+ behaves like

$$\varphi_+(x;\omega,E) \sim \varphi_0(x;\omega,E) = \exp\big(iE^{1/2} x \cdot \omega\big)$$

as $|x| \to \infty$ in the direction $-\omega$ ($x = -|x|\omega$), and the difference $\varphi_+ - \varphi_0$ satisfies the outgoing radiation condition at infinity.

We decompose $\varphi_+(x;\omega,E)$ into the sum $\varphi_+ = \varphi_{\mathrm{in}} + \varphi_{\mathrm{sc}}$ of incident and scattering waves to calculate the scattering amplitude through the asymptotic behavior at infinity of the scattering wave $\varphi_{\mathrm{sc}}(x;\omega,E)$. The idea is due to Takabayashi [16]. If we set $\sigma = \sigma(x;\omega) = \gamma(x;\omega) - \pi$, then

$$\varphi_+ = \sum_l e^{-i\nu\pi/2} e^{il\sigma} J_\nu\big(E^{1/2}|x|\big), \quad \nu = |l - \alpha|.$$

If we further make use of the formula $e^{-i\mu\pi/2}J_\mu(iw) = I_\mu(w)$ for the Bessel function

$$I_\mu(w) = (1/\pi)\left(\int_0^\pi e^{w\cos\rho}\cos(\mu\rho)\,d\rho - \sin(\mu\pi)\int_0^\infty e^{-w\cosh p-\mu p}\,dp\right) \tag{2.3}$$

with $\operatorname{Re} w \geqslant 0$ [22, p. 181], then $\varphi_+(x;\omega,E)$ takes the form

$$\varphi_+ = (1/\pi)\sum_l e^{il\sigma}\int_0^\pi e^{-i\sqrt{E}|x|\cos\rho}\cos(\nu\rho)\,d\rho$$
$$- (1/\pi)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{i\sqrt{E}|x|\cosh p}e^{-\nu p}\,dp. \tag{2.4}$$

We take $\varphi_{\mathrm{in}}(x;\omega,E)$ as

$$\varphi_{\mathrm{in}} = e^{i\alpha\sigma}\varphi_0(x;\omega,E) = e^{i\alpha\sigma}e^{i\sqrt{E}|x|\cos\gamma(x;\omega)} = e^{i\alpha\sigma}e^{-i\sqrt{E}|x|\cos\sigma},$$

which is different from the usual plane wave $\varphi_0(x;\omega,E)$. The modified factor $e^{i\alpha\sigma}$ appears because of the long-range property of the potential $\Phi(x)$ defined by (1.1). The incident wave admits the Fourier expansion

$$\varphi_{\mathrm{in}}(x;\omega,E) = (1/\pi)\sum_l\left(\int_0^\pi e^{-i\sqrt{E}|x|\cos\rho}\cos(\nu\rho)\,d\rho\right)e^{il\sigma(x;\omega)}.$$

This, together with (2.4), yields

$$\varphi_{\mathrm{sc}}(x;\omega,E) = -(1/\pi)\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{i\sqrt{E}|x|\cosh p}e^{-\nu p}\,dp.$$

We compute the series

$$\sum_l e^{il\sigma}e^{-\nu p}\sin(\nu\pi) = \left\{\sum_{l\leqslant[\alpha]} + \sum_{l\geqslant[\alpha]+1}\right\}e^{il\sigma}e^{-\nu p}\sin(\nu\pi)$$
$$= \sin(\alpha\pi)(-1)^{[\alpha]}\left\{\frac{e^{-\alpha p}(e^{i\sigma}e^{p})^{[\alpha]}}{1+e^{-i\sigma}e^{-p}} + \frac{e^{\alpha p}(e^{i\sigma}e^{-p})^{[\alpha]}}{1+e^{-i\sigma}e^{p}}\right\}$$

for $|\sigma| < \pi$. Thus we have

$$\varphi_{\mathrm{sc}} = -\frac{\sin(\alpha\pi)}{\pi}(-1)^{[\alpha]}e^{i[\alpha]\sigma(x;\omega)}\int_{-\infty}^\infty e^{i\sqrt{E}|x|\cosh p}\frac{e^{-\beta p}}{1+e^{-i\sigma}e^{-p}}\,dp$$

with $\beta = \alpha - [\alpha]$. We apply the stationary phase method to the integral on the right side. Since $e^{i\sigma(x;\omega)} = e^{i(\gamma(x;\omega)-\pi)} = -e^{i(\theta-\omega)}$ by identifying $\theta = x/|x| = \hat{x} \in S^1$ with the azimuth angle θ, we see that $\varphi_{\mathrm{sc}}(x;\omega,E)$ obeys

$$\varphi_{\mathrm{sc}} = f_\alpha(\omega \to \hat{x}; E)e^{i\sqrt{E}|x|}|x|^{-1/2} + o\big(|x|^{-1/2}\big), \quad |x| \to \infty.$$

Here $f_\alpha(\omega \to \theta; E)$ defined by (1.4) for $\theta \neq \omega$ is called the amplitude for scattering from the initial direction $\omega \in S^1$ to the final one θ at energy $E > 0$. If, in particular, α is an integer, then $f_\alpha(\omega \to \theta; E)$ vanishes.

We calculate the Green function of the resolvent $R(\zeta; P_\alpha) = (P_\alpha - \zeta)^{-1}$ with $\operatorname{Im}\zeta > 0$. Let $k = \zeta^{1/2}$, $\operatorname{Im} k > 0$, and let $P_{l\alpha}$ be as in (2.1). Then the equation $(P_{l\alpha} - \zeta)u = 0$ has $\{r^{1/2}J_\nu(kr),\ r^{1/2}H_\nu(kr)\}$ with Wronskian $2i/\pi$ as a pair of linearly independent solutions, where $H_\mu(z) = H_\mu^{(1)}(z)$ denotes the Hankel function of the first kind. Thus $(P_{l\alpha} - \zeta)^{-1}$ has the integral kernel

$$R_{l\alpha}(r,\rho;\zeta) = (i\pi/2)r^{1/2}\rho^{1/2}J_\nu\big(k(r\wedge\rho)\big)H_\nu\big(k(r\vee\rho)\big), \quad \nu = |l-\alpha|,$$

where $r \wedge \rho = \min(r,\rho)$ and $r \vee \rho = \max(r,\rho)$. Hence the Green function $R_\alpha(x,y;\zeta)$ of $R(\zeta; P_\alpha)$ is given by

$$R_\alpha(x,y;\zeta) = (i/4)\sum_l e^{il(\theta-\omega)}J_\nu\big(k\big(|x|\wedge|y|\big)\big)H_\nu\big(k\big(|x|\vee|y|\big)\big), \tag{2.5}$$

where $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates. This makes sense even for ζ in the lower half of the complex plane by analytic continuation. Then $R(\zeta; P_\alpha)$ with $\operatorname{Im}\zeta \leqslant 0$ is well defined as an operator from L^2_{comp} to L^2_{loc}. Thus $R(\zeta; P_\alpha)$ does not have any poles as a function with values in operators from L^2_{comp} to L^2_{loc}. We can say that P_α with one solenoidal field $2\pi\alpha\delta(x)$ has no resonances. We do not discuss the possibility of resonances at zero energy.

3. Proof of Theorem 1.1 by the complex scaling method

The proof of Theorem 1.1 is based on the complex scaling method initiated by [3,5] and further developed by [18,20] (see [12] also). In this section we complete the proof of Theorem 1.1, accepting the five lemmas (Lemmas 3.1–3.5) formulated in the course of the proof as proved.

We first reformulate the problem to which the complex scaling method can be applied in a more convenient way and fix some basic notation used throughout the entire discussion in the sequel. We work in the coordinate system in which the two centers d_1 and d_2 are represented as

$$d_1 = d_- = (-d/2, 0), \qquad d_2 = d_+ = (d/2, 0), \quad d \gg 1,$$

and we set $\alpha_- = \alpha_1$ and $\alpha_+ = \alpha_2$ for two given fluxes α_1 and α_2. Then the operator $H_d = H(\Phi_d)$ under consideration is self-adjoint with domain

$$\mathcal{D} = \Big\{u \in L^2\colon\ (-i\nabla - \Phi_d)^2 u \in L^2,\ \lim_{|x-d_\pm|\to 0}\big|u(x)\big| < \infty \text{ at } d_- \text{ and } d_+\Big\} \tag{3.1}$$

and the potential $\Phi_d(x)$ takes the form

$$\Phi_d(x) = \Phi_{-d}(x) + \Phi_{+d}(x) = \alpha_-\Phi(x - d_-) + \alpha_+\Phi(x - d_+). \tag{3.2}$$

We denote by $H_0 = -\Delta$ the free Hamiltonian with domain $H^2(\boldsymbol{R}^2)$ (Sobolev space of order two) and define the auxiliary operators by

$$H_{\pm d} = H(\Phi_{\pm d}), \tag{3.3}$$

which are self-adjoint with domain

$$\mathcal{D}_\pm = \Big\{u \in L^2\colon\ (-i\nabla - \Phi_{\pm d})^2 u \in L^2,\ \lim_{|x-d_\pm|\to 0}\big|u(x)\big| < \infty\Big\}. \tag{3.4}$$

We fix $E_0 > 0$. We always assume that ζ is restricted to the complex neighborhood

$$D_d = \big\{\zeta = E + i\eta \in \boldsymbol{C}\colon\ |E - E_0| < \delta E_0,\ |\eta| < 2E_0^{1/2}(\log d)/d\big\} \tag{3.5}$$

with $0 < \delta \ll 1$ small enough, and we set

$$D_{\pm d} = D_d \cap \{\zeta \in \boldsymbol{C}\colon\ \pm \operatorname{Im}\zeta > 0\}.$$

We also introduce smooth cut-off functions χ_0, χ_∞ and $\chi_\pm$ over the real line $\boldsymbol{R} = (-\infty, \infty)$ with the following properties: $0 \leqslant \chi_0, \chi_\infty, \chi_\pm \leqslant 1$ and

$$\chi_0(t) = 1 \quad \text{for } |t| \leqslant 1, \qquad \chi_0(t) = 0 \quad \text{for } |t| \geqslant 2, \qquad \chi_\infty(t) = 1 - \chi_0(t),$$
$$\chi_+(t) = 1 \quad \text{for } t \geqslant 1, \qquad \chi_+(t) = 0 \quad \text{for } t \leqslant -1, \qquad \chi_-(t) = 1 - \chi_+(t).$$

We often use these functions without further references throughout the future discussion.

We define $j_d(x) : \boldsymbol{R}^2 \to \boldsymbol{C}^2$ by

$$j_d(x_1, x_2) = \big(x_1, x_2 + i\eta_d(x_2)x_2\big), \qquad \eta_d(t) = \eta_{0d}\chi_\infty(t/d), \tag{3.6}$$

with $\eta_{0d} = 5E_0^{-1/2}(\log d)/d$ and consider the complex scaling mapping

$$(J_d f)(x) = \big[\det(\partial j_d/\partial x)\big]^{1/2} f\big(j_d(x)\big)$$

associated with $j_d(x)$. The Jacobian $\det(\partial j_d/\partial x)$ of $j_d(x)$ does not vanish for $d \gg 1$, and therefore J_d is invertible. Since the coefficients of H_d are analytic in $\boldsymbol{R}^2 \setminus \{d_-, d_+\}$, we can define the operator

$$K_d = J_d H_d J_d^{-1}. \tag{3.7}$$

This becomes a closed operator under the same boundary condition as in (3.1), but it is not necessarily self-adjoint. The domain of K_d coincides with $\mathcal{D}$. We do not require the explicit form of K_d in the future discussion.

We define the complex scaled operator as above for the auxiliary operators $H_{\pm d}$ defined by (3.3). Recall that $\gamma(x;\omega)$ denotes the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. The potential $\Phi(x)$ defined by (1.1) satisfies the relation $\Phi(x) = \nabla\gamma(x;\omega)$. Hence it follows that

$$\Phi_{\pm d}(x) = \alpha_\pm \Phi(x - d_\pm) = \alpha_\pm \nabla\gamma(x - d_\pm;\omega_\pm), \quad \omega_\pm = (\pm 1, 0).$$

The angle function $\gamma(x;\omega_+)$ is represented as

$$\gamma(x;\omega_+) = -(i/2)\log\bigl((x_1 + ix_2)/(x_1 - ix_2)\bigr) + \pi,$$

so that it is well defined for complex variables also. We take $\arg z$, $0 \leqslant \arg z < 2\pi$, to be a single valued function over the complex plane slit along the direction ω_+ and define

$$\gamma\bigl(j_d(x);\omega_+\bigr) = \frac{1}{2}\bigl(\arg\bigl(b_{+d}(x)\bigr) - \arg\bigl(b_{-d}(x)\bigr)\bigr) + \pi - i\log\bigl|b_d(x)\bigr|^{1/2}, \tag{3.8}$$

where $b_d(x) = b_{+d}(x)/b_{-d}(x)$ and

$$b_{+d}(x) = x_1 - \eta_d(x_2)x_2 + ix_2, \qquad b_{-d}(x) = x_1 + \eta_d(x_2)x_2 - ix_2.$$

The function $\gamma(j_d(x);\omega_-)$ is similarly defined by taking $\arg z$ to be a single valued function over the complex plane slit along the direction ω_-.

We define $g_{\pm d}$ by

$$g_{\pm d}(x) = \alpha_\pm \chi_\mp\bigl(32(x_1 \mp 13d/32)/d\bigr)\gamma\bigl(j_d(x) - d_\pm;\omega_\pm\bigr) \tag{3.9}$$

and g_{0d} by

$$g_{0d}(x) = \chi_0(4x_1/d)\bigl(\alpha_-\gamma\bigl(j_d(x) - d_-;\omega_-\bigr) + \alpha_+\gamma\bigl(j_d(x) - d_+;\omega_+\bigr)\bigr). \tag{3.10}$$

By definition, $\operatorname{supp} g_{-d} \subset \{x\colon\ x_1 > -7d/16\}$ and $g_{-d} = \alpha_-\gamma(j_d(x) - d_-;\omega_-)$ on $\Pi_+ = \{x\colon\ x_1 > -3d/8\}$. Hence $\exp(ig_{-d})$ acts as

$$\exp(ig_{-d})f(x) = \bigl(J_d\exp\bigl(i\alpha_-\gamma(x - d_-;\omega_-)\bigr)J_d^{-1}f\bigr)(x)$$

on functions $f(x)$ with support in Π_+. On the other hand, $g_{+d}(x)$ has support in $\{x\colon\ x_1 < 7d/16\}$ and $g_{+d} = \alpha_+\gamma(j_d(x) - d_+;\omega_+)$ on $\Pi_- = \{x\colon\ x_1 < 3d/8\}$, so that $\exp(ig_{+d})$ acts as

$$\exp(ig_{+d})f(x) = \bigl(J_d\exp\bigl(i\alpha_+\gamma(x - d_+;\omega_+)\bigr)J_d^{-1}f\bigr)(x)$$

on functions $f(x)$ with support in Π_-. We take into account these relations to define the following closed operator

$$K_{\pm d} = \exp(ig_{\mp d})\bigl(J_d H_{\pm d} J_d^{-1}\bigr)\exp(-ig_{\mp d}) \tag{3.11}$$

with the same boundary condition as in (3.4). Since

$$K_{+d} = J_d H\bigl(\alpha_-\nabla\gamma(x - d_-;\omega_-) + \Phi_{+d}\bigr)J_d^{-1}$$

on Π_+, we have

$$K_{+d} = K_d \quad \text{on } \Pi_+ = \{x\colon x_1 > -3d/8\}. \tag{3.12}$$

Similarly we have

$$K_{-d} = K_d \quad \text{on } \Pi_- = \{x\colon x_1 < 3d/8\}. \tag{3.13}$$

The function $g_{0d}(x)$ defined by (3.10) has support in $\{x\colon |x_1| < d/2\}$ and satisfies

$$g_{0d} = \alpha_-\gamma\big(j_d(x) - d_-;\omega_-\big) + \alpha_+\gamma\big(j_d(x) - d_+;\omega_+\big)$$

on $\Pi_0 = \{x\colon |x_1| \leqslant d/4\}$. If we define the operator K_{0d} by

$$K_{0d} = \exp(ig_{0d})\big(J_d H_0 J_d^{-1}\big)\exp(-ig_{0d}), \tag{3.14}$$

then we obtain

$$K_{0d} = K_{\pm d} = K_d \quad \text{on } \Pi_0 = \big\{x\colon |x_1| \leqslant d/4\big\}. \tag{3.15}$$

We make some comments on the complex scaling mapping J_d defined above before going into the proof of the theorem. This mapping takes a form different from the standard mapping

$$(\tilde{J}_\theta f)(x) = \big[\det\big(1 + i\theta\, dF(x)\big)\big]^{1/2} f\big(x + i\theta F(x)\big), \quad \theta > 0,$$

used in the existing complex scaling method (for example see [12]), where $F : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a smooth vector field satisfying $F(x) = x$ for $|x| \gg 1$. If we define $\tilde{K}_{d\theta} = \tilde{J}_\theta H_d \tilde{J}_\theta^{-1}$, then it follows by the Weyl perturbation theorem that the essential spectrum of $\tilde{K}_{d\theta}$ is given by

$$\sigma_{\mathrm{ess}}(\tilde{K}_{d\theta}) = \big\{\zeta \in \boldsymbol{C}\colon \arg\zeta = -2\arg(1 + i\theta)\big\},$$

and the resonances of H_d in question are defined as eigenvalues near the positive real axis of the distorted operator $\tilde{K}_{d\theta}$. The spectrum $\sigma(\tilde{K}_{d\theta})$ is discrete in the sector

$$S_\theta = \big\{\zeta \in \boldsymbol{C}\colon \operatorname{Re}\zeta > 0,\ -2\arg(1 + i\theta) < \arg\zeta \leqslant 0\big\}$$

and it is known that $\sigma(\tilde{K}_{d\theta}) \cap S_\theta$ is independent of the vector field F and of θ. On the other hand, the distorted operator $K_d = J_d H_d J_d^{-1}$ defined by the mapping J_d has its essential spectrum in the region

$$\sigma_{\mathrm{ess}}(K_d) = \big\{\zeta \in \boldsymbol{C}\colon -2\arg(1 + i\eta_{0d}) \leqslant \arg\zeta \leqslant 0\big\}, \quad \eta_{0d} = 5E_0^{-1/2}(\log d)/d,$$

and has no discrete eigenvalues in this sector. This follows from the Weyl perturbation theorem, if we consider K_d as a perturbation of the operator $-\partial_1^2 - (1 + i\eta_{0d})^{-2}\partial_2^2$. Hence we have to define the resonances of H_d directly as the poles of the resolvent $R(\zeta; H_d)$ continued analytically over the unphysical sheet and not as the eigenvalues of K_d. It seems to be difficult to apply the standard complex scaling method to our resonance problem in scattering by two solenoidal fields with centers at large separation. In particular, it is difficult to separate the two centers from each other without introducing auxiliary operators such as $K_{\pm d}$ with one solenoidal field. For this reason, we develop the new type of complex scaling method which changes only the variable x_2 into the complex variable to separate the two centers from each other. We note that Wang [21] has already studied resonances in strong uniform magnetic fields in three dimensions by making use of a complex scaling method depending only on one variable (direction perpendicular to the magnetic field). However it seems that the motivation in the background is different from that in the present work. In particular, our complex scaled operator has a quite different structure in the essential spectrum. With the notation above, we are now in a position to prove the main theorem.

Proof of Theorem 1.1. The proof is divided into five steps. Throughout the proof, we use the notation $R(\zeta; K)$ to denote the resolvent $(K-\zeta)^{-1}$ of K, where K is not necessarily assumed to be self-adjoint. We also denote by the same notation $R(\zeta; K)$ the resolvent obtained by analytic continuation.

Step 1. At first we assume that $\zeta = E + i\eta \in D_{+d}$. Let $H_\pm = H(\alpha_\pm \Phi)$ be the self-adjoint operator with the boundary condition (1.3) at the origin and let $R_\pm(x, y; \zeta)$ be the kernel of the resolvent $R(\zeta; H_\pm)$. Then the kernel of the resolvent $R(\zeta; H_{\pm d})$ is given by $R_\pm(x - d_\pm, y - d_\pm; \zeta)$. We now consider the integral operator $\tilde{R}_{\pm d}(\zeta)$ with the kernel

$$\tilde{R}_{\pm d}(x, y; \zeta) = \tilde{j}_d(x, y) R_\pm\big(j_d(x) - d_\pm, j_d(y) - d_\pm; \zeta\big), \tag{3.16}$$

where

$$\tilde{j}_d(x, y) = \big[\det\big(\partial j_d(x)/\partial x\big)\big]^{1/2}\big[\det\big(\partial j_d(y)/\partial y\big)\big]^{1/2}.$$

If we set $\tilde{H}_{\pm d} = J_d H_{\pm d} J_d^{-1}$, then $\tilde{H}_{\pm d}$ becomes a closed operator with the boundary condition as in (3.4) and a formal argument using a change of variables shows that

$$\tilde{R}_{\pm d}(\zeta) = J_d R(\zeta; H_{\pm d}) J_d^{-1} = R(\zeta; \tilde{H}_{\pm d}).$$

The rigorous justification is based on the density of analytic vectors in L^2. The first step is to show the following lemma.

Lemma 3.1. *Assume that $\zeta \in D_{+d}$. Let $\tilde{H}_{\pm d}$ and $\tilde{R}_{\pm d}(\zeta)$ be as above. Then*

$$\tilde{R}_{\pm d}(\zeta) : L^2 \to L^2$$

is bounded, and ζ belongs to the resolvent set of $\tilde{H}_{\pm d}$ with $\tilde{R}_{\pm d}(\zeta)$ as a resolvent.

Remark 3.1. We can show that the adjoint operator $\tilde{R}_{\pm d}(\zeta)^* : L^2 \to L^2$ is similarly obtained from the resolvent $R(\bar{\zeta}; H_{\pm d}) : L^2 \to L^2$ with $\zeta \in D_{+d}$ and coincides with the resolvent $R(\bar{\zeta}; \tilde{H}^*_{\pm d})$.

Since $g_{\pm d}(x)$ defined by (3.9) is a bounded function, the lemma, together with (3.11), implies that $\zeta \in D_{+d}$ belongs to the resolvent set of $K_{\pm d}$ and the resolvent $R(\zeta; K_{\pm d})$ is given by

$$R(\zeta; K_{\pm d}) = \exp(ig_{\mp d})\tilde{R}_{\pm d}(\zeta)\exp(-ig_{\mp d}) : L^2 \to L^2$$

for $\zeta \in D_{+d}$.

Step 2. The second step is to show that $\zeta \in D_{+d}$ is also in the resolvent set of K_d and to derive the representation for the resolvent $R(\zeta; K_d)$ in terms of $R(\zeta; K_{\pm d})$. To see this, we define $\Lambda_d(\zeta)$ by

$$\Lambda_d(\zeta) = \chi_{-d} R(\zeta; K_{-d}) + \chi_{+d} R(\zeta; K_{+d}) : L^2 \to L^2,$$

where $\chi_{\pm d}(x) = \chi_\pm(16x_1/d)$. Since $K_d = K_{\pm d}$ on $\operatorname{supp}\chi_{\pm d}$ by (3.12) and (3.13), we compute

$$\begin{aligned}(K_d - \zeta)\Lambda_d(\zeta) &= (K_{-d} - \zeta)\chi_{-d} R(\zeta; K_{-d}) + (K_{+d} - \zeta)\chi_{+d} R(\zeta; K_{+d}) \\ &= Id + [K_{-d}, \chi_{-d}]R(\zeta; K_{-d}) + [K_{+d}, \chi_{+d}]R(\zeta; K_{+d}).\end{aligned}$$

The function $\chi_{\pm d}$ depends on x_1 only, and the derivative $\chi'_{\pm d}$ has support in

$$\Sigma_0 = \big\{x = (x_1, x_2):\ |x_1| < d/16\big\}. \tag{3.17}$$

By (3.15), $K_{\pm d} = K_{0d}$ on Π_0, so that the two commutators $[K_{-d}, \chi_{-d}]$ and $[K_{+d}, \chi_{+d}]$ on the right side equal $[K_{0d}, \chi_{-d}]$ and $-[K_{0d}, \chi_{-d}]$, respectively. Hence we have

$$(K_d - \zeta)\Lambda_d(\zeta) = Id + X\big(R(\zeta; K_{-d}) - R(\zeta; K_{+d})\big), \tag{3.18}$$

where

$$X = [K_{0d}, \chi_{-d}], \qquad \chi_{-d} = \chi_-(16x_1/d). \tag{3.19}$$

We further compute the operator on the right side of (3.18). If we set $\chi_{0d}(x) = \chi_0(8x_1/d)$, then $\chi_{0d} = 1$ on Σ_0 and $K_{\pm d} = K_{0d}$ on $\operatorname{supp}\chi_{0d}$ by (3.15). Hence it equals

$$T_d(\zeta) := X\big(R(\zeta; K_{-d}) - R(\zeta; K_{+d})\big) = XR(\zeta; K_{+d})YR(\zeta; K_{-d}) \tag{3.20}$$

as an operator acting on $L^2(\Sigma_0)$, where

$$Y = [K_{0d}, \chi_{0d}], \qquad \chi_{0d} = \chi_0(8x_1/d). \tag{3.21}$$

Then we can prove the following lemma.

Lemma 3.2. *Assume that $\zeta \in D_{+d}$. If $T_d(\zeta)$ is considered as an operator from $L^2(\Sigma_0)$ into itself, then*

$$Id + T_d(\zeta) : L^2(\Sigma_0) \to L^2(\Sigma_0)$$

has a bounded inverse.

We shall show that $\zeta \in D_{+d}$ belongs to the resolvent set of K_d. Let $L^2_{\mathrm{comp}}(\Sigma_0)$ denote the set of L^2 functions with support in Σ_0. We often identify $L^2_{\mathrm{comp}}(\Sigma_0)$ with $L^2(\Sigma_0)$, including its topology. It follows from (3.18) and (3.20) that

$$(K_d - \zeta)\Lambda_d(\zeta) = Id + T_d(\zeta)$$

on $L^2_{\mathrm{comp}}(\Sigma_0)$. Hence Lemma 3.2 implies that

$$(K_d - \zeta)\Lambda_d(\zeta)\big(Id + T_d(\zeta)\big)^{-1} f = f$$

for $f \in L^2_{\mathrm{comp}}(\Sigma_0)$, so that the operator $R(\zeta)$ defined by

$$R(\zeta) = \Lambda_d(\zeta) - \Lambda_d(\zeta)\big(Id + T_d(\zeta)\big)^{-1} X\big(R(\zeta; K_{-d}) - R(\zeta; K_{+d})\big) : L^2 \to L^2$$

satisfies $(K_d - \zeta)R(\zeta) f = f$ on L^2. Thus we have that the range $\mathrm{Ran}(K_d - \zeta)$ of $K_d - \zeta$ coincides with L^2. Similarly we can prove that $\mathrm{Ran}(K_d^* - \bar{\zeta}) = L^2$ (see Remark 3.1). This shows that $\zeta \in D_{+d}$ belongs to the resolvent set of K_d, and $R(\zeta; K_d)$ is represented as

$$R(\zeta; K_d) = \Lambda_d(\zeta) - \Lambda_d(\zeta)\big(Id + T_d(\zeta)\big)^{-1} X\big(R(\zeta; K_{-d}) - R(\zeta; K_{+d})\big). \tag{3.22}$$

Step 3. We still assume that $\zeta \in D_{+d}$. Let

$$\Omega_0 = \{x\colon\ |x_1| < d,\ |x_2| < r_0\} \tag{3.23}$$

for $r_0 \gg 1$ fixed large enough but independently of d. If $f \in L^2_{\mathrm{comp}}(\Omega_0)$ is an L^2 function with support in Ω_0, then $R(\zeta; H_d) f$ is analytic outside Ω_0, because the coefficients of H_d are analytic there. We can prove the following lemma.

Lemma 3.3. *Assume that $\zeta \in D_{+d}$. If $f \in L^2_{\mathrm{comp}}(\Omega_0)$, then $J_d R(\zeta; H_d) f \in L^2$.*

Since J_d acts as the identity operator on $L^2_{\mathrm{comp}}(\Omega_0)$, $J_d R(\zeta; H_d) f$ with $f \in L^2_{\mathrm{comp}}(\Omega_0)$ satisfies the boundary conditions in (3.4) and solves the equation

$$(K_d - \zeta) J_d R(\zeta; H_d) f = J_d (H_d - \zeta) R(\zeta; H_d) f = f$$

by (3.7). Since such a solution is unique in L^2, we have $J_d R(\zeta; H_d) = R(\zeta; K_d)$ on $L^2_{\mathrm{comp}}(\Omega_0)$ for $\zeta \in D_{+d}$. Thus we obtain

$$R(\zeta; H_d) = \Lambda_d(\zeta) - \Lambda_d(\zeta)\big(Id + T_d(\zeta)\big)^{-1} X\big(R(\zeta; K_{-d}) - R(\zeta; K_{+d})\big) \tag{3.24}$$

from (3.22), when considered as an operator from $L^2_{\mathrm{comp}}(\Omega_0)$ into itself.

Step 4. The relation (3.24) plays a basic role in studying the analytic continuation of $R(\zeta; H_d)$ as a function of ζ with values in operators from $L^2_{\mathrm{comp}}(\Omega_0)$ into itself over the lower-half plane. As stated above, $L^2_{\mathrm{comp}}(\Omega_0)$ is identified with $L^2(\Omega_0)$ together with its topology, and similarly for $L^2_{\mathrm{comp}}(\Sigma_0)$. We can prove the following two lemmas.

Lemma 3.4. *Let $\zeta \in D_d$. Then $R(\zeta; K_{\pm d})$ is bounded when it is considered as an operator from $L^2_{\mathrm{comp}}(\Omega_0)$ into $L^2(\Sigma_0)$ or from $L^2_{\mathrm{comp}}(\Sigma_0)$ into $L^2(\Omega_0)$, and it depends analytically on $\zeta \in D_d$.*

Lemma 3.5. *Let $T_d(\zeta)$ be defined by* (3.20). *Assume that $\zeta = E - i\eta$ fulfills the assumption $0 \leqslant \eta < \eta_{\varepsilon d}(E)$ in the theorem. Then*

$$Id + T_d(\zeta) : L^2(\Sigma_0) \to L^2(\Sigma_0)$$

has a bounded inverse.

The operator $R(\zeta; K_{\pm d})$ depends analytically on ζ when considered as an operator from $L^2_{\mathrm{comp}}(\Omega_0)$ into itself. The two lemmas above, together with (3.24), imply that $R(\zeta; H_d)$ is analytically continued as a function of ζ with values in operators from $L^2_{\mathrm{comp}}(\Omega_0)$ into itself over the region

$$D_{\varepsilon d} = \{\zeta = E - i\eta \in D_d \colon 0 \leqslant \eta < \eta_{\varepsilon d}(E)\}$$

in the lower half-plane.

Step 5. The proof is completed in this step. Once the analytic continuation of

$$R(\zeta; H_d) : L^2_{\mathrm{comp}}(\Omega_0) \to L^2_{\mathrm{comp}}(\Omega_0)$$

is established, we can show that $R(\zeta; H_d)$ is analytically continued as a function of ζ with values in operators from L^2_{comp} to L^2_{loc} over the above region $D_{\varepsilon d}$. To see this, we introduce the auxiliary operator $P_0 = H(\alpha_0 \Phi)$ with $\alpha_0 = \alpha_- + \alpha_+$, the self-adjoint extension (Friedrichs extension) of which is realized by imposing the boundary condition $\lim_{|x| \to 0} |u(x)| < \infty$ at the origin. We use the same notation P_0 to denote this self-adjoint realization. As is easily seen, the line integral

$$\int_C \big(\Phi_d(x) - \alpha_0 \Phi(x)\big) \cdot dx = 0$$

vanishes along any curve C outside Ω_0 by the Stokes formula. This makes it possible to construct a smooth real function $g(x)$ in such a way that

$$\Phi_d(x) = \alpha_0 \Phi(x) + \nabla g(x) \tag{3.25}$$

outside Ω_0. In fact, it is given by the line integral

$$g(x) = -\int_1^\infty \Big(\big(\Phi_d(tx) - \alpha_0 \Phi(tx)\big) \cdot \hat{x}\Big)\, dt, \quad \hat{x} = x/|x|,$$

for $|x| \gg 1$ and obeys $g(x) = O(|x|^{-1})$ as $|x| \to \infty$. This function $g(x)$ is also analytic outside Ω_0, because g solves

$$\Delta g = \nabla \cdot (\Phi_d - \alpha_0 \Phi)$$

and the function on the right side is analytic there. Let $\{\psi_0, \psi_1\}$ be a smooth partition of unity over $\boldsymbol{R}^2$ such that

$$\psi_0 + \psi_1 = 1, \qquad \operatorname{supp} \psi_0 \subset \Omega_0,$$

and let ψ_2 be a smooth function such that it has a slightly wider support than ψ_1 and satisfies $\psi_2\psi_1 = \psi_1$. We may assume that (3.25) remains true on $\operatorname{supp}\psi_2$ (and hence on $\operatorname{supp}\psi_1$ also). If we define $\hat{P}_0 = e^{ig} P_0 e^{-ig}$, then it follows that

$$H_d = \hat{P}_0 \quad \text{on } \operatorname{supp}\psi_2. \tag{3.26}$$

This relation enables us to decompose $R(\zeta; H_d) = R(\zeta; H_d)(\psi_0 + \psi_1)$ into the sum of three terms as follows:

$$R(\zeta; H_d) = R(\zeta; H_d)\psi_0 + \psi_2 R(\zeta; \hat{P}_0)\psi_1 - R(\zeta; H_d)[\hat{P}_0, \psi_2]R(\zeta; \hat{P}_0)\psi_1.$$

Since $R(\zeta; \hat{P}_0) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ depends analytically on ζ and since the commutator $[\hat{P}_0, \psi_2]$ vanishes outside Ω_0, we see that $R(\zeta; H_d) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}(\Omega_0)$ depends analytically on ζ. Similarly we obtain the relation

$$R(\zeta; H_d) = \psi_0 R(\zeta; H_d) + \psi_1 R(\zeta; \hat{P}_0)\psi_2 + \psi_1 R(\zeta; \hat{P}_0)[\hat{P}_0, \psi_2]R(\zeta; H_d)$$

on L^2_{comp}. This yields the analytic dependence on ζ of $R(\zeta; H_d) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ and the proof of the theorem is now complete. □

The proofs of the five lemmas which remain unproved are all based on the asymptotic analysis of the behavior at infinity of the Green function for the Schrödinger operator with one solenoidal field. In particular, the proof of Lemma 3.5, which has played an essential role in proving the theorem, occupies the main body of the paper.

4. Proof of Lemmas 3.1 and 3.3

The present section is devoted to proving Lemmas 3.1 and 3.3 among the five lemmas.

4.1. Preliminary proposition and lemmas

We begin by introducing the new notation

$$r_d(x, y)^2 = \bigl|j_d(x) - j_d(y)\bigr|^2, \qquad r_d(x)^2 = r_d(x, 0)^2 = \bigl|j_d(x)\bigr|^2, \tag{4.1}$$

$$\theta_d(x, y) = \gamma\bigl(j_d(x); \omega_+\bigr) - \gamma\bigl(j_d(y); \omega_+\bigr), \quad \omega_+ = (1, 0), \tag{4.2}$$

where $|z|^2 = x_1^2 + (x_2 + iy_2)^2$ for $z = (x_1, x_2 + iy_2) \in \boldsymbol{R} \times \boldsymbol{C}$. The branch $r_d(x, y)$ of $r_d(x, y)^2$ is taken in such a way that $\operatorname{Re} r_d(x, y) > 0$. We recall that the kernel $R_\alpha(x, y; \zeta)$ of the resolvent $R(\zeta; P_\alpha)$ with $\operatorname{Im}\zeta > 0$ is given by (2.5) for the self-adjoint operator $P_\alpha = H(\alpha\Phi)$ defined by (1.2) with domain (1.3). The argument here is based on the following proposition.

Proposition 4.1. *Assume that $\zeta \in D_{+d}$. Let $R_\alpha(x, y; \zeta)$ be the kernel of the resolvent $R(\zeta; P_\alpha)$. Set $k = \zeta^{1/2}$ with $\operatorname{Im} k > 0$. If $x_2 > c$ and $y_2 > c$ for some $c > 1$, then*

$$R_\alpha\big(j_d(x), j_d(y); \zeta\big) = (i/4)e^{i\alpha\theta_d(x,y)} H_0\big(kr_d(x, y)\big) + O\big(\big(|x| + |y|\big)^{-L}\big)$$

as $|x| + |y| \to \infty$ for any $L \gg 1$, where $H_0(z) = H_0^{(1)}(z)$ denotes the Hankel function of the first kind, and the order estimate depends on ζ. A similar relation holds true in the case where $x_2 < -c$ and $y_2 < -c$.

We prove the proposition at the end of this section. We complete the proof of the two lemmas in question after showing two preliminary lemmas. We define $\tilde{R}_{\alpha d}(\zeta) = J_d R(\zeta; P_\alpha) J_d^{-1}$ as the integral operator with kernel

$$\tilde{R}_{\alpha d}(x, y; \zeta) = \tilde{j}_d(x, y) R_\alpha\big(j_d(x), j_d(y); \zeta\big),$$

where $\tilde{j}_d(x, y)$ is defined in (3.16).

Lemma 4.1. *If $q_\pm \in C^\infty(\boldsymbol{R}^2)$ is a bounded function with support in $\{x\colon \pm x_2 > c\}$ for some $c > 1$, then*

$$q_+ \tilde{R}_{\alpha d}(\zeta) q_+, \qquad q_- \tilde{R}_{\alpha d}(\zeta) q_- : L^2 \to L^2$$

is bounded.

Proof. Let $\eta_d(t)$ be defined in (3.6) and set $\tilde{\eta}_d(t) = \eta_d(t)t$. We may assume that $\tilde{\eta}_d'(t) \geqslant 0$. According to (4.1), we calculate

$$r_d(x, y)^2 = (x_1 - y_1)^2 + \big(1 + i\eta_d(x_2, y_2)\big)^2 (x_2 - y_2)^2,$$

where

$$\eta_d(x_2, y_2) = \int_0^1 \tilde{\eta}_d'\big(y_2 + s(x_2 - y_2)\big)\, ds \geqslant 0$$

and $\eta_d(x_2, y_2) = O((\log d)/d)$. Hence we have

$$\operatorname{Im}\big(kr_d(x, y)\big) \geqslant c\eta^{1/2}|x - y|, \quad |x - y| \gg 1,$$

for some $c > 0$, so that the Hankel function $H_0(kr_d(x, y))$ falls off exponentially as $|x - y| \to \infty$. This, together with Proposition 4.1, proves the lemma. □

We denote by $\tilde{P}_{\alpha d} = J_d P_\alpha J_d^{-1}$ the complex scaled operator obtained from P_α. The coefficients of P_α are analytic in $\boldsymbol{R}^2 \setminus \{0\}$. Hence $\tilde{P}_{d\alpha}$ has coefficients smooth in $\boldsymbol{R}^2 \setminus \{0\}$ and becomes a closed operator under the same boundary condition as in (1.3). Let $\mathcal{A}$ be the dense space in L^2 spanned by all functions of the form

$$f(x_1, x_2) = h(x_1)p(x_2)\exp(-cx_2^2), \quad c > 0,$$

where $h \in C_0^\infty(\boldsymbol{R})$ and $p(x_2)$ is a polynomial. According to [12, Proposition 17.10], we know that $J_d\mathcal{A}$ is also dense in L^2. If $f \in J_d\mathcal{A}$, then $\tilde{R}_{\alpha d}(\zeta)f$ satisfies the boundary condition in (1.3) and the relation

$$(\tilde{P}_{\alpha d} - \zeta)\tilde{R}_{\alpha d}(\zeta) = Id \tag{4.3}$$

holds on the dense set $J_d\mathcal{A}$. This is shown by making a change of variables and by deforming the contour by analyticity.

Lemma 4.2. *Assume that $\zeta \in D_{+d}$. Let $\tilde{P}_{\alpha d}$ and $\tilde{R}_{\alpha d}(\zeta)$ be as above. Then $\tilde{R}_{\alpha d}(\zeta)$ is bounded on L^2, and ζ belongs to the resolvent set of $\tilde{P}_{\alpha d}$ with $\tilde{R}_{\alpha d}(\zeta)$ as a resolvent.*

Proof. Let $\{u_-, u_0, u_+\}$ be a nonnegative smooth partition of unity such that $u_-(x_2) + u_0(x_2) + u_+(x_2) = 1$ and

$$\operatorname{supp} u_0 \subset (-2c, 2c), \qquad \operatorname{supp} u_+ \subset (3c/2, \infty), \qquad \operatorname{supp} u_- \subset (-\infty, -3c/2)$$

for $c > 1$ fixed. We shall show that $\tilde{R}_{\alpha d}(\zeta)u_0$ and $\tilde{R}_{\alpha d}(\zeta)u_\pm$ are bounded on L^2. We first consider $\tilde{R}_{\alpha d}(\zeta)u_0$. If $v_0 \in C_0^\infty(\boldsymbol{R})$ has support in $\{|x_2| < 4c\}$, then we have $v_0\tilde{R}_{\alpha d}(\zeta)u_0 = v_0R(\zeta; P_\alpha)u_0$, and $v_0\tilde{R}_{\alpha d}(\zeta)u_0$ is bounded. Let v_+ and $\tilde{v}_+$ be smooth functions of x_2 such that they have support in $(3c, \infty)$, and $v_+ = 1$ on $[4c, \infty)$, $\tilde{v}_+ = 1$ on $\operatorname{supp} v_+$. Then u_0 vanishes on $\operatorname{supp} v_+$ and $J_d^{-1}v_+J_d = v_+$ for $d \gg 1$. Thus we can calculate

$$v_+\tilde{R}_{\alpha d}(\zeta)u_0 = \tilde{v}_+\tilde{R}_{\alpha d}(\zeta)(\tilde{P}_{\alpha d} - \zeta)v_+\tilde{R}_{\alpha d}(\zeta)u_0 = \tilde{v}_+\tilde{R}_{\alpha d}(\zeta)[\tilde{P}_{\alpha d}, v_+]\tilde{R}_{\alpha d}(\zeta)u_0$$

on the dense set $J_d\mathcal{A}$, and it follows from Lemma 4.1 that $v_+\tilde{R}_{\alpha d}(\zeta)u_0$ is bounded on L^2. A similar argument applies to $v_-\tilde{R}_{\alpha d}(\zeta)u_0$, where v_- is supported in $(-\infty, 3c)$ and has properties similar to v_+. Hence we obtain that $\tilde{R}_{\alpha d}(\zeta)u_0$ is bounded. Next we show that $\tilde{R}_{\alpha d}(\zeta)u_+$ is bounded. The boundedness of $\tilde{R}_{\alpha d}(\zeta)u_-$ is shown in a similar way. Let $\{w_-, w_0, w_+\}$ be a nonnegative smooth partition of unity such that $w_-(x_2) + w_0(x_2) + w_+(x_2) = 1$ and

$$\operatorname{supp} w_0 \subset (-c/2, c/2), \qquad \operatorname{supp} w_+ \subset (c/3, \infty), \qquad \operatorname{supp} w_- \subset (-\infty, -c/3).$$

By Lemma 4.1, $w_+\tilde{R}_{\alpha d}(\zeta)u_+$ is bounded. Let $\tilde{u}_+ \in C^\infty(\boldsymbol{R})$ be a function such that $\operatorname{supp} \tilde{u}_+ \subset (3c/4, \infty)$ and it satisfies $\tilde{u}_+u_+ = u_+$. Then we have the relation

$$w_0\tilde{R}_{\alpha d}(\zeta)u_+ = w_0\tilde{R}_{\alpha d}(\zeta)u_+(\tilde{P}_{\alpha d} - \zeta)\tilde{R}_{\alpha d}(\zeta)\tilde{u}_+ = w_0\tilde{R}_{\alpha d}(\zeta)[u_+, \tilde{P}_{\alpha d}]\tilde{R}_{\alpha d}(\zeta)\tilde{u}_+$$

on $J_d\mathcal{A}$. This, together with Lemma 4.1, implies that $w_0\tilde{R}_{\alpha d}(\zeta)u_+$ is bounded. We repeat the commutator calculus on $J_d\mathcal{A}$ to obtain

$$w_-\tilde{R}_{\alpha d}(\zeta)u_+ = \tilde{w}_-\tilde{R}_{\alpha d}(\zeta)[\tilde{P}_{\alpha d}, w_-]\tilde{R}_{\alpha d}(\zeta)[u_+, \tilde{P}_{\alpha d}]\tilde{R}_{\alpha d}(\zeta)\tilde{u}_+,$$

where $\tilde{w}_- \in C^\infty(\boldsymbol{R})$ has support in $(-\infty, -c/4)$ and satisfies $\tilde{w}_-w_- = w_-$. Hence Lemma 4.1 again shows that $w_-\tilde{R}_{\alpha d}(\zeta)u_+$ is bounded. Thus we have shown that $\tilde{R}_{\alpha d}(\zeta)$ is bounded on L^2.

Since $\tilde{P}_{\alpha d}$ is a closed operator, it follows from (4.3) that the range $\mathrm{Ran}(\tilde{P}_{\alpha d} - \zeta)$ coincides with L^2. We can also obtain $\mathrm{Ran}(\tilde{P}^*_{\alpha d} - \bar{\zeta}) = L^2$ for the adjoint operator $\tilde{P}^*_{\alpha d}$ (see Remark 3.1). This shows that ζ is in the resolvent set of $\tilde{P}_{\alpha d}$ and that the resolvent $R(\zeta; \tilde{P}_{\alpha d})$ equals $\tilde{R}_{\alpha d}(\zeta)$, and the proof is complete. □

4.2. *Proof of Lemmas 3.1 and 3.3*

We prove Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. If we apply Lemma 4.2 to $H_{\pm d} = H(\Phi_{\pm d})$ with $\Phi_\pm(x) = \alpha_\pm \Phi(x - d_\pm)$, then $\tilde{R}_{\pm d}(\zeta) = J_d R(\zeta; H_{\pm d}) J_d^{-1}$ with $\zeta \in D_{+d}$ is bounded on L^2. Since $g_{\mp d}(x)$ defined by (3.9) is bounded, it follows from (3.11) that

$$R(\zeta; K_{\pm d}) = \exp(i g_{\mp d}) \tilde{R}_{\pm d}(\zeta) \exp(-i g_{\mp d})$$

turns out to be the resolvent of $K_{\pm d}$ for $\zeta \in D_{+d}$. This proves the lemma. □

Proof of Lemma 3.3. We use the notation with the same meaning as ascribed in Step 5 of the proof of Theorem 1.1. In particular, g satisfies (3.25). In addition, we introduce a smooth function $\psi_3 \in C^\infty(\boldsymbol{R}^2)$ such that $\psi_3 \psi_2 = \psi_2$. We may assume that (3.25) remains true on $\operatorname{supp} \psi_3$ also. We decompose $R(\zeta; H_d) f = (\psi_0 + \psi_1) R(\zeta; H_d) f$ with $f \in L^2_{\mathrm{comp}}(\Omega_0)$ into the sum of three terms in the following way:

$$R(\zeta; H_d) f = \psi_0 R(\zeta; H_d) f + \psi_1 R(\zeta; H_d) \psi_2 f + \psi_1 R(\zeta; H_d)[H_d, \psi_2] R(\zeta; H_d) f.$$

The first term on the right side fulfills $J_d \psi_0 R(\zeta; H_d) f = \psi_0 R(\zeta; H_d) f \in L^2$. If we take the relation (3.26) into account, then the second term on the right side is further calculated as

$$\psi_1 R(\zeta; H_d) \psi_2 f = \psi_1 R(\zeta; \hat{P}_0) \psi_2 f + \psi_1 R(\zeta; \hat{P}_0)[\hat{P}_0, \psi_3] R(\zeta; H_d) \psi_2 f.$$

We note that

$$J_d \exp\big(\pm i g(x)\big) J_d^{-1} = \exp\big(\pm i g\big(j_d(x)\big)\big) : L^2 \to L^2$$

is bounded and $[\hat{P}_0, \psi_3] R(\zeta; H_d) \psi_2 f \in L^2_{\mathrm{comp}}(\Omega_0)$. Since $J_d \psi_1 = \psi_1 J_d$ and $J_d \psi_2 f = \psi_2 f$ for $f \in L^2_{\mathrm{comp}}(\Omega_0)$, Lemma 4.2 with $P_\alpha = P_0$ yields that $J_d \psi_1 R(\zeta; H_d) \psi_2 f$ is in L^2. Since $\psi_3 = 1$ both on $\operatorname{supp} \psi_1$ and on $\operatorname{supp} \nabla \psi_2$, a similar argument applies to the third term, and we obtain

$$J_d \psi_1 R(\zeta; H_d)[\psi_2, H_d] R(\zeta; H_d) f \in L^2.$$

Thus the proof is complete. □

4.3. Proof of Proposition 4.1

Before going into the proof, we derive the integral representation for the kernel $R_\alpha(x, y; \zeta)$. The derivation is based on the following formula

$$H_\mu(Z)J_\mu(z) = \frac{1}{i\pi}\int_0^{\kappa+i\infty} \exp\left\{\frac{t}{2} - \frac{Z^2+z^2}{2t}\right\} I_\mu\left(\frac{Zz}{t}\right)\frac{dt}{t}, \quad |z| \leqslant |Z|,$$

for the product of Bessel functions [22, p. 439], where the contour is taken to be rectilinear with corner at $\kappa + i0$, $\kappa > 0$ being fixed arbitrarily. We apply to (2.5) this formula with $Z = k(|x| \vee |y|)$ and $z = k(|x| \wedge |y|)$, where $\operatorname{Im} k = \operatorname{Im} \zeta^{1/2} > 0$. If we write $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in polar coordinates, then $R_\alpha(x, y; \zeta)$ is represented as

$$R_\alpha = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right) I_\nu\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t} \tag{4.4}$$

with $\nu = |l - \alpha|$, where $\psi = \theta - \omega$. If, in particular, $\alpha = 0$, then the resolvent $(H_0 - \zeta)^{-1}$ of the free Hamiltonian H_0 has the kernel $(i/4)H_0(k|x - y|)$ represented as the integral

$$\frac{i}{4}H_0\big(k|x-y|\big) = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right) I_l\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t},$$

where $I_l(w) = I_{|l|}(w)$ is defined by $I_l(\omega) = (1/\pi)\int_0^\pi e^{w\cos\rho}\cos(l\rho)\,d\rho$ (see (2.3)). Since the series $\sum_l e^{il\psi} I_l(w)$ converges to $e^{w\cos\psi}$ by the Fourier expansion and since

$$|x-y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\psi,$$

the kernel $(i/4)H_0(k|x - y|)$ has the integral representation

$$\frac{i}{4}H_0\big(k|x-y|\big) = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right)\frac{dt}{t}. \tag{4.5}$$

We are now in a position to prove the proposition.

Proof of Proposition 4.1. We consider only the case when $x_2 > c$ and $y_2 > c$ and assume throughout the proof that $\zeta \in D_{+d}$. The proof is divided into three steps.

(i) Let $w = Zz/t = \zeta|x||y|/t$ with $Z = k(|x| \vee |y|)$ and $z = k(|x| \wedge |y|)$. Then $\operatorname{Re} w \geqslant 0$ for t on the contour in the integral (4.4), and the integral representation (2.3) for $I_\nu(w)$ is well defined. We make use of this representation to calculate the series $\sum_l e^{il\psi} I_\nu(w)$ in the integral. Then it admits the decomposition

$$\sum_l e^{il\psi} I_\nu(w) = \sum_l e^{il\psi} I_{\mathrm{fr},\nu}(w) + \sum_l e^{il\psi} I_{\mathrm{sc},\nu}(w), \tag{4.6}$$

where $I_{\mathrm{fr},\nu}(w)$ and $I_{\mathrm{sc},\nu}(w)$ are defined by

$$I_{\mathrm{fr},\nu}(w) = \frac{1}{\pi}\int_0^\pi e^{w\cos\xi}\cos(\nu\xi)\,d\xi, \qquad I_{\mathrm{sc},\nu}(w) = -\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-w\cosh p-\nu p}\,dp$$

with $\nu = |l-\alpha|$. A simple calculation yields

$$I_{\mathrm{fr},\nu}(w) = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{w\cos\xi}e^{i\alpha\xi}e^{-il\xi}\,d\xi$$

and hence we have

$$I_{\mathrm{fr}}(w,\psi) = \sum_l e^{il\psi}I_{\mathrm{fr},\nu}(w) = e^{w\cos\psi}e^{i\alpha\psi}, \quad |\psi|<\pi, \tag{4.7}$$

by the Fourier expansion. On the other hand, the second series on the right side of (4.6) is computed in the same way as in Section 2, and we see that it converges to

$$I_{\mathrm{sc}}(w,\psi) = -\frac{\sin(\alpha\pi)}{\pi}e^{i[\alpha](\psi+\pi)}\int_{-\infty}^{\infty} e^{-w\cosh p}\frac{e^{(1-\beta)p}}{e^p+e^{-i\psi}}\,dp \tag{4.8}$$

with $\beta = \alpha-[\alpha]$, $0<\beta<1$. By assumption, $x_2>c$ and $y_2>c$, so that $0<\theta,\omega<\pi$. This implies that $-\pi<\psi=\theta-\omega<\pi$, and hence the denominator $e^p+e^{-i\psi}$ in (4.8) never vanishes even for $p=0$. Thus $R_\alpha(x,y;\zeta)$ admits the decomposition

$$R_\alpha(x,y;\zeta) = R_{\mathrm{fr},\alpha}(x,y;\zeta)+R_{\mathrm{sc},\alpha}(x,y;\zeta),$$

where $R_{\mathrm{fr},\alpha}$ and $R_{\mathrm{sc},\alpha}$ are defined by

$$R_{\mathrm{fr},\alpha}(x,y;\zeta) = \frac{e^{i\alpha\psi}}{4\pi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{\zeta|x-y|^2}{2t}\right)\frac{dt}{t} = \frac{ie^{i\alpha\psi}}{4}H_0\bigl(k|x-y|\bigr),$$

$$R_{\mathrm{sc},\alpha}(x,y;\zeta) = \frac{1}{4\pi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{\zeta(|x|^2+|y|^2)}{2t}\right)I_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t},\psi\right)\frac{dt}{t}.$$

The function $R_\alpha(j_d(x),j_d(y);\zeta)$ in question also admits the corresponding decomposition

$$R_\alpha\bigl(j_d(x),j_d(y);\zeta\bigr) = R_{\mathrm{fr},\alpha}\bigl(j_d(x),j_d(y);\zeta\bigr)+R_{\mathrm{sc},\alpha}\bigl(j_d(x),j_d(y);\zeta\bigr). \tag{4.9}$$

If we recall the notation in (4.1) and (4.2), the functions on the right side are defined with $|x|$, $|y|$ and ψ replaced by $r_d(x)$, $r_d(y)$ and $\theta_d(x,y)$, respectively. In fact, if $x_2>c>0$ and $y_2>c>0$, then ψ equals

$$\psi = \gamma(x; -\hat{y}) - \pi = \gamma(x; \omega_+) - \gamma(y; \omega_+)$$

and it is changed into $\theta_d(x, y)$. In particular, we have

$$R_{\mathrm{fr},\alpha}\big(j_d(x), j_d(y); \zeta\big) = (i/4)e^{i\alpha\theta_d(x,y)} H_0\big(kr_d(x, y)\big). \tag{4.10}$$

(ii) We prove that $R_{\mathrm{sc},\alpha}(j_d(x), j_d(y); \zeta)$ obeys

$$\big|R_{\mathrm{sc},\alpha}\big(j_d(x), j_d(y); \zeta\big)\big| = O\big(\big(|x| + |y|\big)^{-L}\big), \quad |x| + |y| \to \infty. \tag{4.11}$$

By definition, $R_{\mathrm{sc},\alpha}(j_d(x), j_d(y); \zeta)$ is written as

$$\frac{1}{4\pi} \int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta(r_d(x)^2 + r_d(y)^2)}{2t}\right) I_{\mathrm{sc}}(t, x, y; \zeta)\frac{dt}{t}, \tag{4.12}$$

and $I_{\mathrm{sc}}(t, x, y; \zeta) = I_{\mathrm{sc}}(\zeta r_d(x) r_d(y)/t, \theta_d(x, y))$ takes the form

$$I_{\mathrm{sc}}(t, x, y; \zeta) = -\frac{\sin(\alpha\pi)}{\pi} e^{i[\alpha](\theta_d(x,y)+\pi)} L_{\mathrm{sc}}(t, x, y; \zeta)$$

by (4.8), where $L_{\mathrm{sc}}(t, x, y; \zeta)$ is defined by

$$L_{\mathrm{sc}}(t, x, y; \zeta) = \int_{-\infty}^{\infty} e^{-(\zeta r_d(x) r_d(y)/t)\cosh p} \frac{e^{(1-\beta)p}}{e^p + e^{-i\theta_d(x,y)}}\, dp. \tag{4.13}$$

We prove the two lemmas below after completing the proof of the proposition.

Lemma 4.3. *Assume that $x_2 > c$ and $y_2 > c$ for some $c > 0$. If $x_1 \gg 1$ and $y_1 \ll -1$ or if $x_1 \ll -1$ and $y_1 \gg 1$, then there exists $c_1 > 0$ such that*

$$\big|\mathrm{Im}\, e^{-i\theta_d(x,y)}\big| \geqslant c_1\big(|x_1| + |y_1|\big)^{-1}, \quad |x_1| + |y_1| \gg 1.$$

Lemma 4.4. *If $0 < t < \kappa$, then*

$$\big|\exp\big(-\zeta\big(r_d(x)^2 + r_d(y)^2\big)/2t\big)\big| \leqslant \exp\big(-c_2\big(|x|^2 + |y|^2\big)/t\big), \quad |x| + |y| \gg 1,$$

for some $c_2 > 0$, and if $0 < s < M(|x| + |y|)$ for $t = \kappa + is$, $M \gg 1$ being fixed, then

$$\big|\exp\big(-\zeta\big(r_d(x)^2 + r_d(y)^2\big)/2t\big)\big| \leqslant \exp\big(-c_3\big(|x| + |y|\big)\big), \quad |x| + |y| \gg 1,$$

for some $c_3 > 0$, where c_3 may depend on η.

The denominator $e^p + e^{-i\theta_d(x,y)}$ in the integral (4.13) does not vanish, but it can take values close to zero around $p = 0$, provided that $\theta_d(x, y) \sim \pm\pi$. This is the case where $x_1 \gg 1$ and $y_1 \ll -1$ or where $x_1 \ll -1$ and $y_1 \gg 1$. However, Lemma 4.3 implies that $|L_{\mathrm{sc}}(t, x, y; \zeta)| = O(|x| + |y|)$, and hence it follows from Lemma 4.4 that

$$\left| \int_0^{\kappa+iM} \exp\left(\frac{t}{2} - \frac{\zeta(r_d(x)^2 + r_d(y)^2)}{2t}\right) I_{\mathrm{sc}}(t, x, y; \zeta)\frac{dt}{t} \right| = O\bigl(\bigl(|x| + |y|\bigr)^{-L}\bigr).$$

(iii) The proof is completed in this step by showing that the integral

$$\int_{\kappa+i0}^{\kappa+i\infty} \chi_M(t, x, y) \exp\left(\frac{t}{2} - \frac{\zeta(r_d(x)^2 + r_d(y)^2)}{2t}\right) I_{\mathrm{sc}}(t, x, y; \zeta)\frac{dt}{t}$$

obeys $O((|x| + |y|)^{-L})$, where

$$\chi_M(t, x, y) = \chi_\infty\bigl(s/\bigl(M\bigl(|x| + |y|\bigr)\bigr)\bigr), \quad |x| + |y| \gg 1,$$

for $s = \mathrm{Im}\, t$. To see this, we decompose $L_{\mathrm{sc}}(t, x, y; \zeta)$ defined by (4.13) into the sum

$$L_{\mathrm{sc}}(t, x, y, ; \zeta) = \int \bigl(\chi_0(p) + \chi_\infty(p)\bigr)e^{-(\zeta r_d(x) r_d(y)/t)\cosh p} \frac{e^{(1-\beta)p}}{e^p + e^{-i\theta_d(x,y)}}\, dp.$$

If we set $a_0(t, x, y) = t/2 - \zeta(r_d(x)^2 + r_d(y)^2)/2t$ and

$$a_1(t, x, y, p) = a_0(t, x, y) - \bigl(\zeta r_d(x) r_d(y)/t\bigr)\cosh p, \quad |p| < 2,$$

then we can take $M \gg 1$ so large that $|\partial_t a_0| \geqslant c$ and $|\partial_t a_1| \geqslant c$ for some $c > 0$. The desired bound is obtained by partial integration. We use $|\partial_t a_1| \geqslant c$ for the integral with $\chi_0(p)$. On the other hand, we make use of $|\partial_t a_0| \geqslant c$ and of the relation

$$\partial_t e^{-(\zeta r_d(x) r_d(y)/t)\cosh p} = -t^{-1}(\cosh p/\sinh p)\partial_p e^{-(\zeta r_d(x) r_d(y)/t)\cosh p}, \quad |p| > 2,$$

to evaluate the integral with $\chi_\infty(p)$. Thus (4.11) is obtained, and the proposition follows from (4.9), (4.10) and (4.11). □

We end the section by proving Lemmas 4.3 and 4.4.

Proof of Lemma 4.3. We consider only the case when $x_1 \gg 1$ and $y_1 \ll -1$, so that $\theta_d(x, y)$ behaves like $\theta_d \sim -\pi$. We write

$$\bigl|\mathrm{Im}\, e^{-i\theta_d(x,y)}\bigr| = e^{\mathrm{Im}\,\theta_d(x,y)}\bigl|\sin\bigl(\mathrm{Re}\,\theta_d(x, y)\bigr)\bigr|.$$

We recall the representation (3.8) for $\gamma(j_d(x); \omega_+)$. We note that $\eta_d(t)$ defined in (3.6) satisfies $\eta_d(t) \geqslant 0$ and $\eta_d(t) = O((\log d)/d)$ uniformly in t. If $x_2 > c$ and $y_2 > c$ and if $x_1 \gg 1$ and

$y_1 \ll -1$, then it follows that $\operatorname{Re}\gamma(j_d(x);\omega_+) \geqslant c_1/x_1$ and $\operatorname{Re}\gamma(j_d(y);\omega_+) \leqslant \pi + c_1/y_1$ for some $c_1 > 0$. Hence we have

$$\operatorname{Re}\theta_d(x,y) = \operatorname{Re}\big(\gamma\big(j_d(x);\omega_+\big) - \gamma\big(j_d(y);\omega_+\big)\big) \geqslant -\pi + c_1\big(|x_1| + |y_1|\big)^{-1}.$$

This shows that $|\sin(\operatorname{Re}\theta_d(x,y))| \geqslant c_1(|x_1| + |y_1|)^{-1}$. As is easily seen from (3.8),

$$\operatorname{Im}\gamma\big(j_d(x);\omega_+\big) = O\big((\log d)/d\big) \tag{4.14}$$

uniformly in x with $|x| > c_2 > 0$, and hence we have $e^{\operatorname{Im}\theta_d(x,y)} \geqslant c_3$ for some $c_3 > 0$. Thus the lemma is verified. □

Proof of Lemma 4.4. By (4.1), we have

$$r_d(x)^2 = x_1^2 + \big(1 + 2i\eta_d(x_2) - \eta_d(x_2)^2\big)x_2^2.$$

Since $\eta_d(t) = O((\log d)/d)$, we can easily see that

$$\operatorname{Re}\big(r_d(x)^2 + r_d(y)^2\big)/t \geqslant c\big(|x|^2 + |y|^2\big)/t, \quad c > 0,$$

for $0 < t < \kappa$. Thus the first statement is obtained. If we compute

$$\zeta/t = \big(\kappa^2 + s^2\big)^{-1}\big((E\kappa + \eta s) + i(\eta\kappa - Es)\big)$$

by setting $t = \kappa + is$ and $\zeta = E + i\eta$, then we have

$$\operatorname{Re}\big((\zeta/t)\big(r_d(x)^2 + r_d(y)^2\big)\big) \geqslant c\big((1+\eta s)/\big(1+s^2\big)\big)\big(|x|^2 + |y|^2\big)$$

for some $c > 0$. This proves the second statement for $0 < s < M(|x| + |y|)$, and the proof is complete. □

5. Proof of Lemmas 3.2 and 3.5

In this section we prove Lemmas 3.2 and 3.5. The proof of both the lemmas is based on the same idea, but Lemma 3.5 is much more difficult to prove than Lemma 3.2. We give a detailed proof for Lemma 3.5 and only a sketch for Lemma 3.2.

5.1. Preliminary proposition and lemmas

We begin by formulating the proposition which plays an important role in proving Lemma 3.5.

Proposition 5.1. *Let $R_\alpha(x,y;\zeta)$ be the kernel of the resolvent $R(\zeta;P_\alpha)$ with $\zeta \in \overline{D}_{-d}$, $\overline{D}_{-d}$ being the closure of D_{-d}, and let $N \gg 1$ be fixed arbitrarily but large enough. Set $k = \zeta^{1/2}$ with* $\operatorname{Im} k \leqslant 0$. *Assume that*

$$-3d/4 < x_1, y_1 < -d/4, \qquad |x_1 - y_1| > cd$$

for some $c > 0$. Then we have the following statements:

(1) *If* $|x_2| + |y_2| \geqslant Nd$, *then* $R_\alpha(j_d(x), j_d(y); \zeta)$ *behaves like*

$$R_\alpha\big(j_d(x), j_d(y); \zeta\big) = (i/4)e^{i\alpha\theta_d(x,y)} H_0\big(kr_d(x,y)\big) + O\big(\big(|x| + |y|\big)^{-\sigma N}\big)$$

for some $\sigma > 0$ *independent of* N.

(2) *Let* $c(E)$ *be the constant defined by*

$$c(E) = (8\pi)^{-1/2} e^{i\pi/4} E^{-1/4}. \tag{5.1}$$

If $|x_2| + |y_2| \leqslant Nd$, *then* $R_\alpha(j_d(x), j_d(y); \zeta)$ *admits the decomposition*

$$R_\alpha\big(j_d(x), j_d(y); \zeta\big) = (i/4)e^{i\alpha\theta_d(x,y)} H_0\big(kr_d(x,y)\big) + G_\alpha(x, y; \zeta) + O\big(d^{-N}\big)$$

and $G_\alpha(x, y; \zeta)$ *takes the asymptotic form*

$$G_\alpha = c(E)e^{ik(r_d(x)+r_d(y))}\big(r_d(x)r_d(y)\big)^{-1/2}\big(f_\alpha(-\omega \to \theta; E) + e_N(x, y; \zeta)\big),$$

where $f_\alpha(-\omega \to \theta; E)$ *is the amplitude defined by* (1.4) *for scattering from* $-\omega = -y/|y|$ *to* $\theta = x/|x|$ *at energy* E *by the field* $2\pi\alpha\delta(x)$, *and* $e_N(x, y; \zeta)$ *obeys*

$$\partial_x^n \partial_y^m e_N = O\big((\log d)^2 d^{-1-|n|-|m|}\big)$$

uniformly in x, y *and* ζ.

(3) *Similar asymptotic formulas remain true for the derivatives*

$$\partial R_\alpha\big(j_d(x), j_d(y); \zeta\big)/\partial x_j, \qquad \partial R_\alpha\big(j_d(x), j_d(y); \zeta\big)/\partial y_j, \quad j = 1, 2,$$

with natural modification in both the cases (1) *and* (2) *above.*

Remark 5.1. If x_1 and y_1 satisfy $d/4 < x_1, y_1 < 3d/4$, then the same results remain true with $\theta_d(x, y)$ replaced by $\tilde{\theta}_d(x, y) = \gamma(j_d(x); \omega_-) - \gamma(j_d(y); \omega_-)$, where $\omega_- = (-1, 0)$.

We prove the proposition at the end of the section. We proceed with the argument, accepting the proposition as proved. We apply this proposition to the kernel $F_\pm(x, y; \zeta)$ of the resolvent $R(\zeta; K_{\pm d})$ with $\zeta \in \overline{D}_{-d}$ for the operator $K_{\pm d}$ defined by (3.11). Let $H_\pm = H(\alpha_\pm \Phi)$ be the self-adjoint operator with the boundary condition (1.3) at the origin and let $R_\pm(x, y; \zeta)$ be the kernel of the resolvent $R(\zeta; H_\pm)$ analytically continued over $\overline{D}_{-d}$. Then the kernel of $R(\zeta; H_{\pm d})$ is given by $R_\pm(x - d_\pm, y - d_\pm; \zeta)$ with $d_\pm = (\pm d/2, 0)$ for the auxiliary operator $H_{\pm d} = H(\Phi_{\pm d})$, and it follows from (3.11) that

$$F_\pm(x, y; \zeta) = \tilde{j}_d(x, y)e^{i(g_{\mp d}(x) - g_{\mp d}(y))} R_{\pm d}(x, y; \zeta),$$

where

$$R_{\pm d}(x, y; \zeta) = R_\pm\big(j_d(x) - d_\pm, j_d(y) - d_\pm; \zeta\big)$$

and $g_{\pm d}(x)$ is defined by (3.9). According to (4.1), it is obvious that $r_d(x - d_\pm, y - d_\pm) = r_d(x, y)$.

Let X and Y be the commutators defined by (3.19) and (3.21), respectively. The coefficients of X have support in Σ_0 (see (3.17)). On the other hand, the support of the coefficients of Y is divided into the two regions

$$\Sigma_- = \{x\colon -d/4 < x_1 < -d/8\}, \qquad \Sigma_+ = \{x\colon d/8 < x_1 < d/4\}. \tag{5.2}$$

We assume that $x \in \Sigma_0$ and $y \in \Sigma = \Sigma_- \cup \Sigma_+$. Then

$$-9d/16 < x_1 - d/2 < -7d/16, \qquad -3d/4 < y_1 - d/2 < -d/4$$

and $d/16 < |x_1 - y_1| < 5d/16$. If $|x_2| + |y_2| \geqslant Nd$ for $N \gg 1$ fixed, then we have

$$R_{+d}(x, y; \zeta) = (i/4)e^{i\alpha_+\theta_{+d}(x,y)} H_0\big(kr_d(x, y)\big) + O\big(\big(|x| + |y|\big)^{-\sigma N}\big)$$

by Proposition 5.1(1), where

$$\theta_{+d}(x, y) = \gamma\big(j_d(x) - d_+; \omega_+\big) - \gamma\big(j_d(y) - d_+; \omega_+\big).$$

We write $r_{\pm d}(x)$ for $r_d(x, d_\pm)$ and $\hat{x}_{\pm d}$ for $(x - d_\pm)/|x - d_\pm|$. Let $f_\pm(\omega \to \theta; E)$ denote the amplitude for scattering from ω to θ by the field $2\pi\alpha_\pm\delta(x)$. If $|x_2| + |y_2| \leqslant Nd$, then it follows from Proposition 5.1(2) that $R_{+d}(x, y; \zeta)$ behaves like

$$R_{+d}(x, y; \zeta) = (i/4)e^{i\alpha_+\theta_{+d}(x,y)} H_0\big(kr_d(x, y)\big) + G_{+d}(x, y; \zeta) + O\big(d^{-N}\big)$$

and $G_{+d}(x, y; \zeta)$ takes the form

$$G_{+d} = c(E)e^{ik(r_{+d}(x)+r_{+d}(y))}\big(r_{+d}(x)r_{+d}(y)\big)^{-1/2}\big(f_+(-\hat{y}_{+d} \to \hat{x}_{+d}; E) + e_{+N}\big),$$

where $e_{+N} = e_{+N}(x, y; \zeta)$ obeys the same bound as e_N in Proposition 5.1. We can derive a similar asymptotic form for $R_{-d}(x, y; \zeta)$. Assume that $x \in \Sigma = \Sigma_+ \cup \Sigma_-$ and $y \in \Sigma_0$. If $|x_2| + |y_2| \geqslant Nd$, then

$$R_{-d}(x, y; \zeta) = (i/4)e^{i\alpha_-\theta_{-d}(x,y)} H_0\big(kr_d(x, y)\big) + O\big(\big(|x| + |y|\big)^{-\sigma N}\big),$$

where

$$\theta_{-d}(x, y) = \gamma\big(j_d(x) - d_-; \omega_-\big) - \gamma\big(j_d(y) - d_-; \omega_-\big).$$

If $|x_2| + |y_2| \leqslant Nd$, then

$$R_{-d}(x, y; \zeta) = (i/4)e^{i\alpha_-\theta_{-d}(x,y)} H_0\big(kr_d(x, y)\big) + G_{-d}(x, y; \zeta) + O\big(d^{-N}\big)$$

and $G_{-d}(x, y; \zeta)$ takes the form

$$G_{-d} = c(E)e^{ik(r_{-d}(x)+r_{-d}(y))}\big(r_{-d}(x)r_{-d}(y)\big)^{-1/2}\big(f_-(-\hat{y}_{-d} \to \hat{x}_{-d}; E) + e_{-N}\big).$$

We summarize the asymptotic properties of the kernel $F_\pm(x, y; \zeta)$ of $R(\zeta; K_{\pm d})$ with $\zeta \in \overline{D}_{-d}$ in the lemma below.

Lemma 5.1. *Define*

$$F_{\pm 0}(x, y; \zeta) = (i/4)\tilde{j}_d(x, y)e^{i(g_{\mp d}(x)-g_{\mp d}(y))}e^{i\alpha_\pm \theta_{\pm d}(x,y)}H_0\bigl(kr_d(x, y)\bigr)$$

and set

$$F_{\pm 1}(x, y; \zeta) = \tilde{j}_d(x, y)e^{i(g_{\mp d}(x)-g_{\mp d}(y))}G_{\pm d}(x, y; \zeta)$$

for $G_{\pm d}(x, y; \zeta)$ as above.

(1) *Assume that $x \in \Sigma_0$ and $y \in \Sigma = \Sigma_- \cup \Sigma_+$. If $|x_2| + |y_2| \geqslant Nd$ for $N \gg 1$, then*

$$F_+(x, y; \zeta) = F_{+0}(x, y; \zeta) + O\bigl(\bigl(|x| + |y|\bigr)^{-\sigma N}\bigr),$$

and if $|x_2| + |y_2| \leqslant Nd$, then

$$F_+(x, y; \zeta) = F_{+0}(x, y; \zeta) + F_{+1}(x, y; \zeta) + O\bigl(d^{-N}\bigr).$$

These relations hold true in the C^1 topology.

(2) *Assume that $x \in \Sigma$ and $y \in \Sigma_0$. If $|x_2| + |y_2| \geqslant Nd$ for $N \gg 1$, then*

$$F_-(x, y; \zeta) = F_{-0}(x, y; \zeta) + O\bigl(\bigl(|x| + |y|\bigr)^{-\sigma N}\bigr),$$

and if $|x_2| + |y_2| \leqslant Nd$, then

$$F_-(x, y; \zeta) = F_{-0}(x, y; \zeta) + F_{-1}(x, y; \zeta) + O\bigl(d^{-N}\bigr).$$

We prove two preliminary lemmas.

Lemma 5.2. *Assume that $|x_1| \leqslant d/2$ and $|y_1| \leqslant d/2$. If $\zeta \in \overline{D}_{-d}$, then there exist $\mu > 0$ and $c > 0$ such that*

$$\bigl|e^{ikr_d(x,y)}\bigr| = O\bigl(d^\mu\bigr)\exp\bigl(-c\bigl((\log d)/d\bigr)|x_2 - y_2|\bigr)$$

for $k = \zeta^{1/2}$, and in particular, one has $|e^{ikr_{-d}(x)}| + |e^{ikr_{+d}(x)}| = O(d^\mu)$.

Proof. Let $\eta_d(t)$ be defined in (3.6). We set $\tilde{\eta}_d(t) = \eta_d(t)t$. For brevity, we assume that $y_2 \leqslant x_2$. Then we compute

$$r_d(x, y)^2 = |x - y|^2 + \bigl(2i\tilde{\eta}_d'(z_2) + O\bigl((\log d)^2/d^2\bigr)\bigr)(x_2 - y_2)^2$$

for some z_2 with $y_2 \leqslant z_2 \leqslant x_2$, where $\tilde{\eta}'_d(z_2) = O((\log d)/d)$. If $\zeta = E - i\eta \in \overline{D}_{-d}$, then $0 \leqslant \eta \leqslant 2E_0^{1/2}(\log d)/d$ and

$$k = \zeta^{1/2} = E^{1/2} - iE^{-1/2}\eta/2 + O\big((\log d)^2/d^2\big), \quad d \gg 1. \tag{5.3}$$

Hence we have

$$\mathrm{Im}\big(kr_d(x,y)\big) \sim \big(E^{1/2}\tilde{\eta}'_d(z_2)\big((x_2 - y_2)/|x-y|\big)^2 - E^{-1/2}\eta/2\big)|x-y|$$

for $d \gg 1$. We can take $c_1 > 0$ in such a way that

$$\tilde{\eta}'_d(z_2) = \big(\tilde{\eta}_d(x_2) - \tilde{\eta}_d(y_2)\big)/(x_2 - y_2) \geqslant 4E_0^{-1/2}(\log d)/d$$

when $|x_2| + |y_2| > c_1 d$. If $|x_2 - y_2| > d$ and $|x_2| + |y_2| > c_1 d$, then

$$\big(E^{1/2}\tilde{\eta}'_d(z_2) - E^{-1/2}\eta\big)/2 \geqslant \big(2(E/E_0)^{1/2} - (E_0/E)^{1/2}\big)(\log d)/d \geqslant c(\log d)/d$$

for some $c > 0$. This implies that

$$\big|e^{ikr_d(x,y)}\big| = e^{-\mathrm{Im}(kr_d(x,y))} \leqslant e^{-c((\log d)/d)|x_2 - y_2|}$$

for x and y as above. On the other hand, if $|x_2 - y_2| < d$ or if $|x_2| + |y_2| < c_1 d$, then we have $|\mathrm{Im}(kr_d(x,y))| = O(\log d)$, and hence the estimate in the lemma is obtained. Thus the proof is complete. □

Lemma 5.3. *Assume that $\zeta \in \overline{D}_{-d}$. Let $u(x)$ be a smooth function such that u has support in $\{d/8 < |x_1| < d/4\}$ and satisfies $\partial_x^n u = O(d^{-|n|})$. Define $U(x,y)$ by*

$$U(x,y) = \int e^{ikr_d(x,\xi)} u(\xi) e^{ikr_d(\xi,y)}\, d\xi$$

for $k = \zeta^{1/2}$. If x and y are in Σ_0, then there exists $c > 0$ such that

$$\big|U(x,y)\big| = O\big(d^{-L}\big)\exp\big(-c\big((\log d)/d\big)|x_2 - y_2|\big)$$

for any $L \gg 1$.

Proof. The proof is based on the property that $\exp(ikr_d(x,y))$ oscillates highly in the x_1 variable and falls off exponentially in the x_2 variable. By Lemma 5.2, we have

$$\big|e^{ikr_d(x,\xi)}e^{ikr_d(\xi,y)}\big| = O\big(d^{2\mu}\big)\exp\big(-c\big((\log d)/d\big)\big(|x_2 - \xi_2| + |\xi_2 - y_2|\big)\big)$$

for some $c > 0$. In particular, if $|x_2 - \xi_2| > Ld$ for $L \gg 1$ fixed arbitrarily, then it follows from Lemma 5.2 that

$$\big|e^{ikr_d(x,\xi)}\big| = O\big(d^{-\sigma L}\big)\exp\big(-c\big((\log d)/d\big)|x_2 - \xi_2|\big)$$

with some $\sigma > 0$ independent of L. Since $|x_2 - \xi_2| + |\xi_2 - y_2| > |x_2 - y_2|$ and since

$$\int \exp\bigl(-c\bigl((\log d)/d\bigr)|\xi_2 - y_2|\bigr)\, d\xi_2 = O(d/\log d),$$

the desired bound is obtained for the integral over the interval $|\xi_2 - x_2| > Ld$. A similar argument applies to the integral over the interval $|\xi_2 - y_2| > Ld$. We assume that $|\xi_2 - x_2| < Ld$ and $|\xi_2 - y_2| < Ld$. If x and y are in Σ_0, then $|x_1| < d/16$ and $|y_1| < d/16$, and hence it follows that $|x_1 - \xi_1| > d/16$ and $|y_1 - \xi_1| > d/16$ for $\xi \in \operatorname{supp} u$. We consider the function

$$\xi_1 \mapsto r_d(x, \xi) + r_d(\xi, y) = |x - \xi| + |\xi - y| + O(\log d).$$

Since $|x_2 - y_2| < 2Ld$ and since

$$\bigl|(\partial/\partial\xi_1)\bigl(|x - \xi| + |\xi - y|\bigr)\bigr| > c > 0$$

for $\xi \in \operatorname{supp} u$, we make repeated use of partial integration to obtain the desired bound for the integral over the interval where $|\xi_2 - x_2| < Ld$ and $|\xi_2 - y_2| < Ld$. This completes the proof. □

5.2. Proof of Lemmas 3.2 and 3.5

The proof of Lemma 3.5 is done through a series of lemmas. We begin by recalling that $T_d(\zeta)$ is defined by

$$T_d(\zeta) = XR(\zeta; K_{+d})YR(\zeta; K_{-d}) : L^2(\Sigma_0) \to L^2(\Sigma_0)$$

for $\zeta = E - i\eta \in \overline{D}_{-d}$ (see (3.20)). The commutators X and Y are defined by $X = [K_{0d}, \chi_{-d}]$ with $\chi_{-d} = \chi_-(16x_1/d)$ and by $Y = [K_{0d}, \chi_{0d}]$ with $\chi_{0d} = \chi_0(8x_1/d)$, where $K_{0d} = e^{ig_{0d}}(J_d H_0 J_d^{-1})e^{-ig_{0d}}$ (see (3.14), (3.19) and (3.21)). By definition, the map J_d commutes with operators depending only on the x_1 variable. Hence X is calculated as

$$X = e^{ig_{0d}} J_d\bigl[-\partial_1^2, \chi_{-d}\bigr]J_d^{-1} e^{-ig_{0d}} = e^{ig_{0d}}\bigl[-\partial_1^2, \chi_{-d}\bigr]e^{-ig_{0d}} \tag{5.4}$$

and similarly we have $Y = e^{ig_{0d}}[-\partial_1^2, \chi_{0d}]e^{-ig_{0d}}$. We may write χ_{0d} as the product

$$\chi_{0d}(x_1) = \chi_+\bigl((16x_1 + 3d)/d\bigr)\chi_-\bigl((16x_1 - 3d)/d\bigr) = \tilde{\chi}_{+d}(x_1)\tilde{\chi}_{-d}(x_1),$$

so that Y takes the form

$$Y = e^{ig_{0d}}\bigl\{\bigl[-\partial_1^2, \tilde{\chi}_{+d}\bigr] + \bigl[-\partial_1^2, \tilde{\chi}_{-d}\bigr]\bigr\}e^{-ig_{0d}} = Y^- + Y^+, \tag{5.5}$$

where the coefficients of $Y^\pm$ have support in $\Sigma_\pm$ defined by (5.2). We note that the function $g_{0d}(x)$ defined by (3.10) satisfies $\partial_x^n g_{0d} = O(d^{-|n|})$ and similarly for $g_{\pm d}(x)$ and $\det(\partial j_d(x)/\partial x)$.

We now consider the equation

$$\varphi + T_d(\zeta)\varphi = h, \tag{5.6}$$

for a given $h \in L^2(\Sigma_0)$. We show that this equation is solvable in $L^2(\Sigma_0)$, provided that η satisfies the assumption $0 \leqslant \eta < \eta_{\varepsilon d}(E)$ in Theorem 1.1. We fix $N \gg 1$ large enough and take $\rho > 1/2$ close enough to 1/2. Let $\{u_1, u_2, u_3\}$ be the partition of unity defined by

$$u_1 = \chi_0\big(x_2/d^\rho\big), \qquad u_2 = \chi_\infty\big(x_2/d^\rho\big)\chi_0(x_2/Nd), \qquad u_3 = \chi_\infty(x_2/Nd). \tag{5.7}$$

We further introduce smooth functions $\tilde{u}_j$ such that $\tilde{u}_j$ has a slightly larger support than u_j and satisfies the relation $\tilde{u}_j u_j = u_j$ for $j = 1, 2, 3$ and that all their derivatives obey the same bounds as those of u_j for $d \gg 1$. We decompose φ into

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 = (u_1 + u_2 + u_3)\varphi$$

and similarly for h. Then (5.6) is written in the matrix form

$$\begin{pmatrix} Id + S_{11} & S_{12} & S_{13} \\ S_{21} & Id + S_{22} & S_{23} \\ S_{31} & S_{32} & Id + S_{33} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},$$

where $S_{jk} = S_{jk}(\zeta) = u_j T_d(\zeta)\tilde{u}_k$, $1 \leqslant j, k \leqslant 3$. We use the notation $\|\cdot\|$ to denote the norm of a bounded operator acting on $L^2(\Sigma_0)$.

Lemma 5.4. *We have $\|S_{33}(\zeta)\| = O(d^{-\sigma N})$ for some $\sigma > 0$ independent of N.*

Proof. We show that the kernel $S_{33}(x, y; \zeta)$ of $S_{33}(\zeta)$ satisfies

$$\begin{aligned} \big|S_{33}(x, y; \zeta)\big| &= O\big(d^{-N}\big)\exp\big(-c\big((\log d)/d\big)|x_2 - y_2|\big) \\ &\quad + O\big(\big(|x_2| + d\big)^{-cN}\big)\exp\big(-c\big((\log d)/d\big)|y_2|\big) \\ &\quad + O\big(\big(|y_2| + d\big)^{-cN}\big)\exp\big(-c\big((\log d)/d\big)|x_2|\big) \\ &\quad + O\big(\big(|x_2| + |y_2| + d\big)^{-cN}\big) \end{aligned}$$

for some $c > 0$. If $x \in \Sigma_0$ and $\xi \in \Sigma = \Sigma_+ \cup \Sigma_-$, then $|x - \xi| > d/16$. Hence the Hankel function $H_0(kr_d(x, \xi))$ takes the asymptotic form

$$H_0\big(kr_d(x, \xi)\big) = \left(\frac{2}{\pi}\right)^{1/2} \frac{e^{-i\pi/4}e^{ikr_d(x,\xi)}}{(kr_d(x, \xi))^{1/2}}\big(1 + O\big(|r_d(x, \xi)|^{-1}\big)\big) \tag{5.8}$$

for $|r_d(x, \xi)| \gg 1$ by formula, and similarly for $H_0(kr_d(\xi, y))$ with $\xi \in \Sigma$ and $y \in \Sigma_0$. Thus the first bound on the right side is obtained by applying Lemma 5.3 to the integral

$$u_3(x_2)\left\{\int XF_{+0}(x, \xi)YF_{-0}(\xi, y)\, d\xi\right\}\tilde{u}_3(y_2).$$

The other bounds are obtained by evaluating integrals such as

$$\int \big(|x| + |\xi| + d\big)^{-cN} YF_{-0}(\xi, y)\tilde{u}_3(y_2)\, d\xi, \qquad \int \big(|x| + |\xi| + d\big)^{-cN}\big(|\xi| + |y| + d\big)^{-cN}\, d\xi.$$

If $|y_2 - \xi_2| < |y_2|/2$, then $|x_2| + |\xi_2| \sim |x_2| + |y_2|$, and if $|y_2 - \xi_2| > |y_2|/2$, then

$$\left|e^{ikr_d(\xi,y)}\right| = O\big(d^{\mu}\big)\exp\big(-c\big((\log d)/d\big)|y_2|\big)$$

by Lemma 5.2. If we take these facts into account, then we can establish the above bound on $S_{33}(x, y; \zeta)$, and hence the lemma is proved. □

Lemma 5.5. *The operators $S_{32}(\zeta)$, $S_{23}(\zeta)$, $S_{31}(\zeta)$ and $S_{13}(\zeta)$ obey*

$$\|S_{32}\| + \|S_{23}\| + \|S_{31}\| + \|S_{13}\| = O\big(d^{-\sigma N}\big)$$

for some $\sigma > 0$ independent of N.

Proof. The lemma is verified in almost the same way as Lemma 5.4. For example, we consider the kernel $S_{32}(x, y; \zeta)$ of $S_{32}(\zeta)$. Let $\{v_{N0}, v_{N\infty}\}$ be the partition of unity defined by $v_{N0} = \chi_0(4x_2/Nd)$ and $v_{N\infty} = \chi_\infty(4x_2/Nd)$. Then the integral

$$u_3(x_2)\left\{\int XF_+(x,\xi;\zeta)v_{N\infty}YF_-(\xi,y;\zeta)\,d\xi\right\}\tilde{u}_2(y_2)$$

is shown to obey the same bound as $S_{33}(x, y; \zeta)$ in the proof of Lemma 5.4. Let $F_{\pm 0}(x, y; \zeta)$ and $F_{\pm 1}(x, y; \zeta)$ be as in Lemma 5.1. We apply Lemma 5.3 to the integral

$$V_0(x,y;\zeta) = u_3(x_2)\left\{\int XF_{+0}(x,\xi;\zeta)v_{N0}YF_{-0}(\xi,y;\zeta)\,d\xi\right\}\tilde{u}_2(y_2)$$

and Lemma 5.2 to the integral

$$V_1(x,y;\zeta) = u_3(x_2)\left\{\int XF_{+0}(x,\xi;\zeta)v_{N0}YF_{-1}(\xi,y;\zeta)\,d\xi\right\}\tilde{u}_2(y_2).$$

Since $|x_2 - \xi_2| > Nd/2$ for $\xi_2 \in \operatorname{supp} v_{N0}$, it follows from Lemma 5.2 that

$$\left|u_3(x_2)XF_{+0}(x,\xi;\zeta)\right| \leqslant \big(|x_2| + d\big)^{-\sigma N}, \qquad \xi_2 \in \operatorname{supp} v_{N0},$$

for some $\sigma > 0$ independent of N, and we also have

$$\left|v_{N0}(\xi_2)YF_{-1}(\xi,y;\zeta)\tilde{u}_2(y_2)\right| = O\big(d^{\mu}\big)$$

for some $\mu > 0$ independent of N. Thus we make use of these lemmas to obtain

$$V_0(x,y;\zeta) = O\big(d^{-\sigma N}\big)\exp\big(-c\big((\log d)/d\big)|x_2 - y_2|\big)$$

and $V_1(x, y; \zeta) = O((|x_2| + d)^{-\sigma N})\tilde{u}_2(y_2)$. This yields $\|S_{32}\| = O(d^{-\sigma N})$. The other operators are also dealt with in a similar way. We skip the details. □

By Lemmas 5.4 and 5.5, the problem is now reduced to the solvability of equation

$$\begin{pmatrix} Id + S_{11} & S_{12} \\ S_{21} & Id + S_{22} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{5.9}$$

with another h_1 and h_2 in $L^2(\Sigma_0)$.

Lemma 5.6. *We have* $\|S_{22}(\zeta)^2\| = O(d^{-L})$ *for any* $L \gg 1$.

Proof. We present only an outline of the proof. A similar but more refined argument is used for proving Lemma 5.8 below. We evaluate the kernel of the operator

$$S_{22}(\zeta)^2 = u_2 X R(\zeta; K_{+d}) Y R(\zeta; K_{-d}) u_2 X R(\zeta; K_{+d}) Y R(\zeta; K_{-d}) \tilde{u}_2.$$

The idea is based on the fact that a particle which starts from $\operatorname{supp} \tilde{u}_2$ and passes over $\operatorname{supp} u_2$ again after being scattered by the fields $2\pi\alpha_\pm \delta(x - d_\pm)$ never returns to $\operatorname{supp} u_2$. Let $Y^\pm$ be as in (5.5) and let $F_{\pm 0}$ and $F_{\pm 1}$ be as in Lemma 5.1. Then S_{22} admits the decomposition

$$S_{22} = S_{22}^+ + S_{22}^-, \qquad S_{22}^\pm = u_2 X R(\zeta; K_{+d}) Y^\pm R(\zeta; K_{-d}) \tilde{u}_2$$

and it is shown in almost the same way as in the proof of Lemma 5.4 that the asymptotic form of the kernel $S_{22}^\pm(x, y; \zeta)$ is determined by the sum of the integrals

$$U_{jk}^\pm(x, y; \zeta) = u_2(x_2) \left\{ \int X F_{+j}(x, \xi; \zeta) v_{L0} Y^\pm F_{-k}(\xi, y; \zeta)\, d\xi \right\} \tilde{u}_2(y_2)$$

with $0 \leqslant j, k \leqslant 1$, where $v_{L0}(x_2)$ is defined by $v_{L0} = \chi_0(x_2/Ld)$ for $L \gg 1$ fixed arbitrarily. We make use of partial integration in the ξ_1 variable to evaluate the integrals U_{10}^-, U_{01}^+ and $U_{00}^\pm$. If $\xi = (\xi_1, \xi_2) \in \Sigma_-$ with $\xi_2 \in \operatorname{supp} v_{L0}$ and $y = (y_1, y_2) \in \Sigma_0$ with $y_2 \in \operatorname{supp} \tilde{u}_2$, then $y_1 > \xi_1$ and

$$\left| (\partial/\partial\xi_1)\big(|d_+ - \xi| + |\xi - y|\big) \right| > c > 0$$

and hence it follows that $U_{10}^-(x, y; \zeta) = O(d^{-L})$. A similar argument applies to U_{01}^+ and $U_{00}^\pm$. On the other hand, we make use of the stationary phase method in the ξ_2 variable to evaluate the other integrals. We consider $U_{01}^-(x, y; \zeta)$ for $x = (x_1, x_2) \in \Sigma_0$ with $x_2 \in \operatorname{supp} u_2$. We recall the behavior of $F_{+0}(x, \xi; \zeta)$ and $F_{-1}(\xi, y; \zeta)$ from Lemma 5.1. The phase function takes the form

$$\xi_2 \mapsto r_{-d}(\xi) + r_d(\xi, x) = |\xi - d_-| + |x - \xi| + O(\log d)$$

for ξ and x as above. For each ξ_1 fixed, the stationary point is attained at $\xi = (\xi_1, \xi_2)$ on the segment joining x and d_-. We see that the stationary point ξ_2 is non-degenerate and $|x - \xi| + |\xi - d_-| = |x - d_-|$ at the point $\xi = (\xi_1, \xi_2)$. Thus $U_{01}^-(x, y; \zeta)$ takes the asymptotic form

$$U_{01}^-(x, y; \zeta) \sim e^{ik(|x - d_-| + |y - d_-|)} u_{01}^-(x, y; \zeta)$$

and $u_{01}^-(x, y; \zeta)$ obeys $\partial_x^n \partial_y^m u_{01}^- = O(d^{\mu - \rho(|n| + |m|)})$ for some $\mu > 0$, where $\rho > 1/2$ is as in (5.7). The explicit representation for the leading term of u_{01}^- does not matter in the proof of the lemma. A similar argument applies to U_{10}^{+} and $U_{11}^{\pm}$. For the integral $U_{10}^{+}(x, y; \zeta)$, the stationary point is attained at the point $\xi = (\xi_1, \xi_2) \in \Sigma_+$ on the segment joining the two points y and d_+ for each ξ_1 fixed, and the integral takes the asymptotic form

$$U_{10}^{+}(x, y; \zeta) \sim e^{ik(|x-d_+|+|y-d_+|)} u_{10}^{+}(x, y; \zeta),$$

where $u_{10}^{+}(x, y; \zeta)$ satisfies the same type of estimates as $u_{01}^{-}(x, y; \zeta)$. For the integral $U_{11}^{\pm}(x, y; \zeta)$, the stationary point is attained at the point $\xi = (\xi_1, 0) \in \Sigma_{\pm}$, and we have

$$U_{11}^{\pm}(x, y; \zeta) \sim e^{ik(|x-d_+|+|y-d_-|)} u_{11}^{\pm}(x, y; \zeta).$$

We evaluate the kernel of the iterated operator $S_{22}(\zeta)^2$. For example, we consider the integral $\int U_{01}^{-}(x, \xi; \zeta) U_{10}^{+}(\xi, y; \zeta)\, d\xi$. If $\xi_2 \in \operatorname{supp} u_2$, then $|\xi_2| > d^{\rho}$, and we have

$$\left|(\partial/\partial\xi_2)\big(|\xi - d_-| + |\xi - d_+|\big)\right| > c d^{-1+\rho}$$

for some $c > 0$. Since $\rho > 1 - \rho$, we see by partial integration that the integral obeys the bound $O(d^{-L})$. A similar argument applies to other terms, and the proof is complete. □

It follows from Lemma 5.6 that $Id + S_{22}$ is invertible on $L^2(\Sigma_0)$, and we have

$$(Id + S_{22})^{-1} = \big(Id - S_{22}^2\big)^{-1}(Id - S_{22}).$$

Hence the first component φ_1 of Eq. (5.9) solves

$$\big(Id + S_{11} - S_{12}(Id + S_{22})^{-1} S_{21}\big)\varphi_1 = \tilde{h}_1,$$

where $\tilde{h}_1 = h_1 - S_{12}(Id + S_{22})^{-1} h_2$.

Lemma 5.7.

$$\big\| S_{12}(Id + S_{22})^{-1} S_{21} \big\| = O\big(d^{-L}\big)$$

for any $L \gg 1$.

Proof. We write

$$(Id + S_{22})^{-1} = Id - S_{22} + S_{22}(Id + S_{22})^{-1} S_{22}.$$

Then we have

$$S_{12}(Id + S_{22})^{-1} S_{21} = S_{12}S_{21} - S_{12}S_{22}S_{21} + S_{12}S_{22}(Id + S_{22})^{-1} S_{22}S_{21}.$$

For the same reason as in the proof of Lemma 5.6, we can show that

$$\|S_{12}S_{21}\| + \|S_{12}S_{22}\| = O\big(d^{-L}\big).$$

This proves the lemma. □

By Lemma 5.7, the solvability of (5.9) is obtained from the lemma below.

Lemma 5.8. *Let $\eta_{\varepsilon d}(E)$ be as in Theorem* 1.1. *If $\zeta = E - i\eta \in \overline{D}_{-d}$ satisfies $0 \leqslant \eta < \eta_{\varepsilon d}(E)$, then*

$$Id + S_{11}(\zeta) : L^2(\Sigma_0) \to L^2(\Sigma_0)$$

has a bounded inverse for $d \gg 1$.

Proof. As in the proof of Lemma 5.6, we decompose S_{11} into the sum

$$S_{11} = S_{11}^+ + S_{11}^-, \qquad S_{11}^\pm = u_1 X R(\zeta; K_{+d}) Y^\pm R(\zeta; K_{-d}) \tilde{u}_1.$$

Then the asymptotic form of the kernel $S_{11}^\pm(x, y; \zeta)$ is determined by the sum of the integrals

$$Q_{jk}^\pm(x, y; \zeta) = u_1(x_2) \left\{ \int X F_{+j}(x, \xi; \zeta) v_{L0} Y^\pm F_{-k}(\xi, y; \zeta)\, d\xi \right\} \tilde{u}_1(y_2)$$

with $0 \leqslant j, k \leqslant 1$, where v_{L0} is again defined by $v_{L0}(x_2) = \chi_0(x_2/Ld)$. Among these kernels, Q_{10}^-, Q_{01}^+ and $Q_{00}^\pm$ obey

$$\left|Q_{10}^-(x, y; \zeta)\right| + \left|Q_{01}^+(x, y; \zeta)\right| + \left|Q_{00}^+(x, y; \zeta)\right| + \left|Q_{00}^-(x, y; \zeta)\right| = O\bigl(d^{-L}\bigr).$$

The asymptotic behaviors as $d \to \infty$ of the other kernels are analyzed by use of the stationary phase method in the ξ_2 variable.

We analyze the behavior of $Q_{01}^-(x, y; \zeta)$ in some detail. Assume that $x = (x_1, x_2) \in \Sigma_0$ with $x_2 \in \operatorname{supp} u_1$ and $y = (y_1, y_2) \in \Sigma_0$ with $y_2 \in \operatorname{supp} \tilde{u}_1$. Let $\xi = (\xi_1, \xi_2) \in \Sigma_-$ with $\xi_2 \in \operatorname{supp} v_{L0}$. For each ξ_1 fixed, the stationary point of the phase function

$$\xi_2 \mapsto |\xi - d_-| + |x - \xi|$$

is attained at $\xi = (\xi_1, \xi_2)$ on the segment joining x and d_-. We note that

$$|\xi_1 + d/2| / |\xi - d_-| = |x_1 - \xi_1| / |x - \xi| = |x_1 + d/2| / |x - d_-|$$

and $|\xi - d_-| + |x - \xi| = |x - d_-|$ at $\xi = (\xi_1, \xi_2)$ with the stationary point ξ_2. Thus $Q_{01}^-(x, y; \zeta)$ takes the asymptotic form

$$Q_{01}^-(x, y; \zeta) \sim e^{ik(|x - d_-| + |y - d_-|)} q_{01}^-(x, y; \zeta).$$

We analyze the behavior as $d \to \infty$ of $q_{01}^-(x, y; \zeta)$. The Hessian is calculated as

$$\frac{(\xi_1 + d/2)^2}{|\xi - d_-|^3} + \frac{(x_1 - \xi_1)^2}{|x - \xi|^3} = \left(\frac{x_1 + d/2}{|x - d_-|}\right)^2 \frac{|x - d_-|}{|\xi - d_-||x - \xi|},$$

so that the contribution from the Hessian turns out to be

$$(2\pi)^{1/2}e^{i\pi/4}k^{-1/2}\big(|x-d_-|/(x_1+d/2)\big)\left(\frac{|\xi-d_-||x-\xi|}{|x-d_-|}\right)^{1/2}$$

according to the stationary phase method [13, Theorem 7.7.5]. Since $k=\zeta^{1/2}=E^{1/2}+O((\log d)/d)$ and since

$$|x-d_-|/(x_1+d/2)=1+O\big(d^{-2+2\rho}\big),$$

the above quantity behaves like

$$(2\pi)^{1/2}e^{i\pi/4}E^{-1/4}\big(|\xi-d_-||x-\xi|\big)^{1/2}|x-d_-|^{-1/2}\big(1+O\big(d^{-1+\rho}\big)\big). \tag{5.10}$$

We recall the behaviors of $F_{+0}(x,\xi;\zeta)$ and of $F_{-1}(\xi,y;\zeta)$ from Lemma 5.1 to calculate $XF_{+0}(x,\xi;\zeta)$ and $Y^-F_{-1}(\xi,y;\zeta)$ when ξ is on the segment joining d_- and x. We have $\tilde{j}_d(x,\xi)=1$ and

$$e^{i\alpha_+\theta_{+d}(x,\xi)}=1+O\big(d^{-1+\rho}\big),\qquad e^{i(g_{-d}(x)-g_{-d}(\xi))}=1+O\big(d^{-1+\rho}\big).$$

We further have

$$F_{+0}(x,\xi;\zeta)=c(E)e^{ik|x-\xi|}|x-\xi|^{-1/2}\big(1+O\big(d^{-1+\rho}\big)\big)$$

by (5.1) and (5.8). It follows from (5.4) and (5.5) that X and $Y^\pm$ take the forms $X\sim-2\chi'_{-d}(x_1)\partial_1$ and $Y^\pm\sim-2\tilde{\chi}'_{\mp d}(x_1)\partial_1$. Since $x_1>\xi_1$ for $x\in\Sigma_0$ and $\xi\in\Sigma_-$, we have $\partial_1|x-\xi|=1+O(d^{-1+\rho})$. Thus $XF_{+0}(x,\xi;\zeta)$ behaves like

$$XF_{+0}=-2iE^{1/2}c(E)e^{ik|x-\xi|}|x-\xi|^{-1/2}\big(\chi'_{-d}(x_1)+O\big(d^{-2+\rho}\big)\big). \tag{5.11}$$

We consider $Y^-F_{-1}(\xi,y;\zeta)$. Let $\xi\in\Sigma_-$ be as above. Assume that $y\in\Sigma_0$ with $y_2\in\operatorname{supp}\tilde{u}_1$. Then the amplitude $f_-(-\hat{y}_{-d}\to\hat{\xi}_{-d};E)$ for the scattering by the field $2\pi\alpha_-\delta(x)$ satisfies the relation

$$f_-(-\hat{y}_{-d}\to\hat{\xi}_{-d};E)=f_-(\omega_-\to\omega_+;E)+O\big(d^{-1+\rho}\big),\quad \omega_\pm=(\pm1,0).$$

Since $\partial_1|\xi-d_-|=1+O(d^{-1+\rho})$, we repeat a similar computation to obtain that $Y^-F_{-1}(\xi,y;\zeta)$ behaves like

$$-2iE^{1/2}c(E)e^{ik(|\xi-d_-|+|y-d_-|)}\big(|\xi-d_-||y-d_-|\big)^{-1/2}\big(\tilde{\chi}'_{+d}(\xi_1)f_-+O\big(d^{-2+\rho}\big)\big)$$

with $f_-=f_-(\omega_-\to\omega_+;E)$. We now note that

$$-2iE^{1/2}c(E)(2\pi)^{1/2}e^{i\pi/4}E^{-1/4}=1$$

by the definition (5.1) of $c(E)$ and that $\int\tilde{\chi}'_{+d}(\xi_1)\,d\xi_1=1$. Then we combine the above behavior of Y^-F_{-1} with (5.10) and (5.11) to see that $Q^-_{01}(x,y;\zeta)$ takes the form

$$Q_{01}^{-} = e^{ik(|x-d_-|+|y-d_-|)} q_{01}^{-}(x, y; \zeta),$$

where $q_{01}^{-}(x, y; \zeta)$ behaves like

$$q_{01}^{-} = -2iE^{1/2}c(E)u_1(x_2)\big(|x-d_-||y-d_-|\big)^{-1/2}\tilde{u}_1(y_2)\big(\chi'_{-d}(x_1)f_- + O\big(d^{-2+\rho}\big)\big).$$

The other integrals $Q_{10}^{+}(x, y; \zeta)$ and $Q_{11}^{\pm}(x, y; \zeta)$ are dealt with in a similar way. Since $\int \tilde{\chi}'_{-d}(\xi_1)\,d\xi_1 = -1$ and $\partial_1|x-d_+| = -1 + O(d^{-1+\rho})$, $Q_{10}^{+}(x, y; \zeta)$ takes the form

$$Q_{10}^{+} = e^{ik(|x-d_+|+|y-d_+|)} q_{10}^{+}(x, y; \zeta),$$

where $q_{10}^{+}(x, y; \zeta)$ behaves like

$$q_{10}^{+} = -2iE^{1/2}c(E)u_1(x_2)\big(|x-d_+||y-d_+|\big)^{-1/2}\tilde{u}_1(y_2)\big(\chi'_{-d}(x_1)f_+ + O\big(d^{-2+\rho}\big)\big)$$

with $f_+ = f_+(\omega_+ \to \omega_-; E)$. On the other hand, if we take into account the relations $\int \tilde{\chi}'_{\pm d}(\xi_1)\,d\xi_1 = \pm 1$ and $\partial_1|x-d_+| = -1 + O(d^{-1+\rho})$, we can show that $Q_{11}^{\pm}(x, y; \zeta)$ takes the form

$$Q_{11}^{\pm} = e^{ik(|x-d_+|+|y-d_-|)} q_{11}^{\pm}(x, y; \zeta),$$

where $q_{11}^{\pm}(x, y; \zeta)$ behaves like

$$\mp 2iE^{1/2}c(E)e^{ikd}d^{-1/2}u_1\big(|x-d_+||y-d_-|\big)^{-1/2}\tilde{u}_1\big(\chi'_{-d}(x_1)f_-f_+ + O\big(d^{-2+\rho}\big)\big).$$

Hence it follows that the sum $Q_{11}^{-}(x, y; \zeta) + Q_{11}^{+}(x, y; \zeta)$ behaves like

$$e^{ik(|x-d_+|+|y-d_-|)}u_1(x_2)\chi'_{-d}(x_1)\big(|x-d_+||y-d_-|\big)^{-1/2}\tilde{u}_1(y_2)O\big(d^{-2+\rho}\big),$$

because $|e^{ikd}d^{-1/2}| = O(1)$ is bounded uniformly in $d \gg 1$ when $\zeta = E - i\eta$ satisfies $0 \leqslant \eta < \eta_{\varepsilon d}(E)$ (see (1.6)).

We evaluate the norm of the integral operator with the remainder term

$$r(x, y; \zeta) = e^{ik(|x-d_-|+|y-d_-|)}u_1(x_2)\big(|x-d_-||y-d_-|\big)^{-1/2}\tilde{u}_1(y_2)O\big(d^{-2+\rho}\big)$$

of $Q_{01}^{-}(x, y; \zeta)$ as a kernel. Since $7d/16 < |x-d_-| < 9d/16$ and $|e^{2ikd}/d| = O(1)$, it follows that $|e^{2ik|x-d_-|}| = O(d^{1-\mu})$ for some $\mu > 0$. If we note that $|x_2| = O(d^{\rho})$ on the support of u_1, then we have

$$\int_{\Sigma_0}\int_{\Sigma_0} \big|r(x, y; \zeta)\big|^2\,dx\,dy = O\big(d^{-2-2\mu+4\rho}\big).$$

We can take $\rho > 1/2$ so close to $1/2$ that the norm of the integral operator under consideration obeys the bound $o(1)$ as $d \to \infty$. A similar argument applies to the remainder term of $Q_{10}^{+}(x, y; \zeta)$, and also the norm of the integral operator with the kernel $Q_{11}^{-}(x, y; \zeta) + Q_{11}^{+}(x, y; \zeta)$ obeys the bound $o(1)$ as $d \to \infty$.

We now combine all the results obtained above to see that the kernel of the operator S_{11} behaves like

$$S_{11}(x, y; \zeta) \sim -2iE^{1/2}c(E)\big(f_- s_-(x) \times \tilde{s}_-(y) + f_+ s_+(x) \times \tilde{s}_+(y)\big) + R(x, y; \zeta)$$

with $f_- = f_-(\omega_- \to \omega_+; E)$ and $f_+ = f_+(\omega_+ \to \omega_-; E)$, where

$$s_\pm = \chi'_{-d}(x_1) e^{ik|x-d_\pm|}|x - d_\pm|^{-1/2} u_1(x_2), \qquad \tilde{s}_\pm = e^{ik|x-d_\pm|}|x - d_\pm|^{-1/2} \tilde{u}_1(x_2)$$

and the error term $O((|x| + |y|)^{-N})$ is negligible. The remainder term $R(x, y; \zeta)$ on the right side takes the form

$$\begin{aligned} R(x, y; \zeta) = u_1(x_2)\big\{ & e^{ik(|x-d_-|+|y-d_-|)}\big(|x - d_-||y - d_-|\big)^{-1/2} r_-(x, y; \zeta) \\ & + e^{ik(|x-d_+|+|y-d_+|)}\big(|x - d_+||y - d_+|\big)^{-1/2} r_+(x, y; \zeta) \\ & + e^{ik(|x-d_+|+|y-d_-|)}\big(|x - d_+||y - d_-|\big)^{-1/2} r_0(x, y; \zeta)\big\} \tilde{u}_1(y_2), \end{aligned}$$

where $r_0(x, y; \zeta)$ satisfies $\partial_x^m \partial_y^n r_0 = O(d^{-2+\rho-\rho(|m|+|n|)})$ and similarly for $r_\pm$. We denote by R the integral operator with the kernel $R(x, y; \zeta)$ and consider the operator $S_0 : L^2(\Sigma_0) \to L^2(\Sigma_0)$ with the kernel

$$S_0(x, y; \zeta) = -2iE^{1/2}c(E)\big(f_- \hat{s}_-(x) \times \tilde{s}_-(y) + f_+ \hat{s}_+(x) \times \tilde{s}_+(y)\big),$$

where

$$\hat{s}_\pm = (Id + R)^{-1} s_\pm = s_\pm - (Id + R)^{-1} e_\pm, \quad e_\pm = R s_\pm.$$

We claim that $Id + S_0$ has a bounded inverse. Then we obtain that the operator $Id + S_{11}$ in question also has a bounded inverse.

We analyze the behavior of $\int \hat{s}_+(x)\tilde{s}_+(x)\,dx$ and $\int \hat{s}_+(x)\tilde{s}_-(x)\,dx$. As stated above, $|e^{2ik|x-d_\pm|}| = O(d^{1-\mu})$ for some $\mu > 0$. This implies that the L^2 norms of $s_\pm$ and $\tilde{s}_\pm$ obey

$$\|s_\pm\|_2 = O\big(d^{-\mu/2+\rho/2-1/2}\big), \qquad \|\tilde{s}_\pm\|_2 = O\big(d^{-\mu/2+\rho/2+1/2}\big).$$

We also have

$$e_+(x) = u_1(x_2)\big\{e^{ik|x-d_-|}|x - d_-|^{-1/2} + e^{ik|x-d_+|}|x - d_+|^{-1/2}\big\} O\big(d^{-2+\rho}\big) + O\big(d^{-L}\big)$$

by making use of the stationary phase method for the integral with respect to the x_2 variable, and hence it follows that

$$\|e_+\|_2 = O\big(d^{-\mu/2+\rho/2-1/2}\big) O\big(d^{-1+\rho}\big)$$

and similarly for e_-. We can take $\rho > 1/2$ so close to $1/2$ that

$$\int \hat{s}_+(x)\tilde{s}_+(x)\,dx = O\big(d^{-L}\big) + O\big(d^{-\mu}\big)O\big(d^{2\rho-1}\big) = o(1), \quad d \to \infty,$$

and the stationary phase method applied to the integral with respect to the x_2 variable yields

$$\int \hat{s}_+(x)\tilde{s}_-(x)\,dx = -\big(E^{1/2}/2\pi i\big)^{-1/2} e^{ikd} d^{-1/2} + o(1).$$

A similar argument applies to the integrals $\int \hat{s}_-(x)\tilde{s}_+(x)\,dx$ and $\int \hat{s}_-(x)\tilde{s}_-(x)\,dx$. The eigenfunction of S_0 takes the form $c_-\hat{s}_- + c_+\hat{s}_+$ with $|c_-| + |c_+| \neq 0$. Since

$$-2iE^{1/2}c(E)\big(E^{1/2}/2\pi i\big)^{-1/2} = 1,$$

${}^{\mathrm{t}}(c_-, c_+)$ is approximately calculated as an eigenvector of the matrix

$$\begin{pmatrix} o(1) & -e^{ikd}d^{-1/2}f_+ + o(1) \\ -e^{ikd}d^{-1/2}f_- + o(1) & o(1) \end{pmatrix}.$$

When $\zeta = E - i\eta$ satisfies $0 \leqslant \eta < \eta_{\varepsilon d}(E)$, we can take $d_\varepsilon(E) \gg 1$ so large that

$$\big|e^{2ikd}d^{-1}f_-f_+\big| < 1 - \varepsilon/2$$

for $d > d_\varepsilon(E)$ (see (1.6)). This implies that $Id + S_0$ is invertible, and the proof of the lemma is now complete. □

We are now in a position to complete the proof of Lemma 3.5 in question.

Proof of Lemma 3.5. We combine Lemmas 5.4–5.8 to conclude that $Id + T_d(\zeta)$ has a bounded inverse on $L^2(\Sigma_0)$ for $d \gg 1$, provided that $\zeta = E - i\eta \in \overline{D}_{-d}$ satisfies $0 \leqslant \eta < \eta_{\varepsilon d}(E)$. This completes the proof. □

We make only a brief comment on the proof of Lemma 3.2.

Proof of Lemma 3.2. Proposition 5.1 remains true for $\zeta = E + i\eta \in D_{+d}$. Since $\operatorname{Im} k = \operatorname{Im} \zeta^{1/2} > 0$, $|e^{2ikd}/d| \to 0$ as $d \to \infty$. Hence it can be shown that $Id + T_d(\zeta)$ has a bounded inverse on $L^2(\Sigma_0)$ for $d \gg 1$. This proves the lemma. □

5.3. Proof of Proposition 5.1

We end the section by proving Proposition 5.1 which has played a central role in the proof of Lemma 3.5.

Proof of Proposition 5.1. (1) We prove the first statement. By assumption,

$$|x_1| \leqslant 3d/4, \qquad |y_1| \leqslant 3d/4, \qquad |x_2| + |y_2| \geqslant Nd$$

for $N \gg 1$. Let t be on the contour of the line integral in (4.4). Then the following three lemmas enable us to prove the statement in almost the same way as Proposition 4.1.

Lemma 5.9. *One has* $\mathrm{Re}(\zeta r_d(x) r_d(y)/t) > 0$ *for* $\zeta = E - i\eta \in \overline{D}_{-d}$.

Lemma 5.10. *One has* $|\mathrm{Im}\, e^{-i\theta_d(x,y)}| \geqslant c(|x_2| + |y_2|)^{-1}$ *for some* $c > 0$.

Lemma 5.11. *If* $0 < t < \kappa$, *then*

$$\mathrm{Re}\big(\zeta\big(r_d(x)^2 + r_d(y)^2\big)/t\big) \geqslant c\big(|x|^2 + |y|^2\big)/t, \quad c > 0,$$

and if $0 < s < M(|x_2| + |y_2|)$ *for* $t = \kappa + is$, $M \gg 1$ *being fixed arbitrarily, then there exists* $\sigma > 0$ *independent of* N *such that*

$$\mathrm{Re}\big(\zeta\big(r_d(x)^2 + r_d(y)^2\big)/t\big) \geqslant \sigma N \log\big(|x_2| + |y_2|\big).$$

We complete the proof of statement (1), accepting these lemmas as proved. Lemma 5.9 makes it possible for us to decompose $R_\alpha(j_d(x), j_d(y); \zeta)$ into the sum

$$R_\alpha\big(j_d(x), j_d(y); \zeta\big) = (i/4)e^{i\alpha\theta_d(x,y)} H_0\big(kr_d(x,y)\big) + R_{\mathrm{sc},\alpha}\big(j_d(x), j_d(y); \zeta\big)$$

as in (4.9), and Lemmas 5.10 and 5.11 enable us to show in almost the same way as in the proof of Proposition 4.1 that

$$\big|R_{\mathrm{sc},\alpha}\big(j_d(x), j_d(y); \zeta\big)\big| = O\big(\big(|x_2| + |y_2|\big)^{-\sigma N}\big).$$

Thus (1) is obtained.

Proof of Lemma 5.9. We set $w = \zeta r_d(x) r_d(y)/t$. We compute

$$r_d(x)^2 = |x|^2\big(1 + 2i\eta_d(x_2)\big(|x_2|/|x|\big)^2 + O\big(\big((\log d)/d\big)^2\big)\big) \tag{5.12}$$

and similarly for $r_d(y)^2$, where $\eta_d(t)$ obeys $\eta_d(t) = O((\log d)/d)$. Hence we have

$$r_d(x) r_d(y) \sim |x||y|\big\{1 + i\big(\eta_d(x_2)\big(x_2/|x|\big)^2 + \eta_d(y_2)\big(y_2/|y|\big)^2\big)\big\} \tag{5.13}$$

for $d \gg 1$. If $0 < t < \kappa$, then it is easy to see that $\mathrm{Re}\, w > 0$. If $t = \kappa + is$ with $s > 0$, then we have

$$\zeta/t = \big(\kappa^2 + s^2\big)^{-1}\big((E\kappa - \eta s) - i(Es + \eta\kappa)\big), \tag{5.14}$$

and hence $\mathrm{Re}\, w$ behaves like

$$\mathrm{Re}\, w \sim \frac{|x||y|}{\kappa^2 + s^2}\left\{\left(E\left(\eta_d(x_2)\left(\frac{x_2}{|x|}\right)^2 + \eta_d(y_2)\left(\frac{y_2}{|y|}\right)^2\right) - \eta\right)s + E\kappa\right\}$$

for $d \gg 1$. It follows from the definition of $\eta_d(t)$ that

$$\eta_d(x_2) = 5E_0^{-1/2}(\log d)/d \quad \text{or} \quad \eta_d(y_2) = 5E_0^{-1/2}(\log d)/d \tag{5.15}$$

for $|x_2| + |y_2| \geqslant Nd$. Since $0 \leqslant \eta \leqslant 2E_0^{1/2}(\log d)/d$ for $\zeta = E - i\eta \in \overline{D}_{-d}$, we have that $\operatorname{Re} w > 0$ for $t = \kappa + is$ also. □

Proof of Lemma 5.10. The denominator $e^p + e^{-i\theta_d(x,y)}$ of the integrand in (4.13) never vanishes but takes values close to 0 around $p = 0$, provided that $\theta_d(x, y) \sim \pm\pi$. This is the case when $x_2 \gg 1$ and $y_2 \ll -1$ or when $x_2 \ll -1$ and $y_2 \gg 1$. We consider only the former case. We compute $\operatorname{Re}\theta_d(x, y)$ as in the proof of Lemma 4.3. If x_1 and y_1 fulfill the assumption in the proposition and if $x_2 \gg 1$ and $y_2 \ll -1$, then

$$\operatorname{Re}\gamma\big(j_d(x); \omega_+\big) \geqslant \pi/2 + c_1/x_2, \qquad \operatorname{Re}\gamma\big(j_d(y); \omega_+\big) \leqslant 3\pi/2 + c_1/y_2$$

for some $c_1 > 0$, so that $\operatorname{Re}\theta_d(x, y) \geqslant -\pi + c_1(|x_2| + |y_2|)^{-1}$. This, together with (4.14), implies that

$$\big|\operatorname{Im} e^{-i\theta_d(x,y)}\big| \geqslant c\big(|x_2| + |y_2|\big)^{-1}$$

for some $c > 0$. Hence the desired bound is obtained. □

Proof of Lemma 5.11. We set $w = (\zeta/t)(r_d(x)^2 + r_d(y)^2)$. If $0 < t < \kappa$, then it is easy to see that $\operatorname{Re} w > c(|x|^2 + |y|^2)/t$ for some $c > 0$. Assume that $t = \kappa + is$ with $0 < s < M(|x_2| + |y_2|)$. If we take (5.12), (5.14) and (5.15) into account, then a simple computation yields $\operatorname{Re} w > c((\log d)/d)(|x_2| + |y_2|)$ for another $c > 0$. Since $(\log p)/p$ is decreasing for $p \gg 1$, we have

$$\log\big(|x_2| + |y_2|\big)/\big(|x_2| + |y_2|\big) \leqslant \log(Nd)/(Nd) \leqslant (2/N) \times \big((\log d)/d\big)$$

for $|x_2| + |y_2| \geqslant Nd$. This implies that $\operatorname{Re} w \geqslant \sigma N \log(|x_2| + |y_2|)$ for some $\sigma > 0$, and hence the lemma follows at once. □

(2) We proceed to the second statement. We assume that $|x_2| + |y_2| \leqslant Nd$ for $N \gg 1$ fixed above. The kernel $R_\alpha(x, y; \zeta)$ is represented by the line integral (4.4) even for $\zeta = E - i\eta \in \overline{D}_{-d}$. However, the integral representation (2.3) for $I_\nu(\zeta|x||y|/t)$ with $\nu = |l - \alpha|$ does not make sense any longer. In fact,

$$\operatorname{Re}\big(\zeta|x||y|/t\big) \sim -\eta|x||y|/s < 0$$

for $t = \kappa + is$ with $s \gg 1$. For this reason, we make use of the different representation formula for $I_\nu(\zeta|x||y|/t)$ when $\operatorname{Re}(\zeta|x||y|/t) < 0$. The proof of the statement is divided into four steps.

(i) We begin by decomposing $R_\alpha(j_d(x), j_d(y); \zeta)$ into the sum of three terms. To do this, we take κ as

$$\kappa = M^2 \log d, \quad M \gg 1,$$

in the line integral (4.4), so that e^t is at most of polynomial growth $|e^t| = O(d^{M^2})$ as $d \to \infty$ on the contour $(0, \kappa) \cup (\kappa + i0, \kappa + i\infty)$. We set $\chi_{M0}(t) = \chi_0(s/Md)$ and $\chi_{M\infty}(t) = \chi_\infty(s/Md)$ for $s = \operatorname{Im} t \geqslant 0$ and decompose $R_\alpha(x, y; \zeta)$ into the sum

$$R_\alpha(x, y; \zeta) = R(x, y; \zeta) + R_\infty(x, y; \zeta),$$

where

$$R = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty}\chi_{M0}(t)\exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right)I_\nu\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t},$$

$$R_\infty = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty}\chi_{M\infty}(t)\exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right)I_\nu\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t}$$

with $\nu = |l-\alpha|$, and ψ is defined by $\psi = \theta - \omega$ for $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$. We note that the choice of M depends on N and that $\mathrm{Re}(\zeta|x||y|/t) > 0$ for $0 < s < 2Md$, which is seen from (5.14). Hence, by formula (2.3), the first term $R(x, y; \zeta)$ on the right side is further decomposed into the sum of two terms

$$R(x, y; \zeta) = R_{\mathrm{fr}}(x, y; \zeta) + R_{\mathrm{sc}}(x, y; \zeta)$$

after calculating the series $\sum_l e^{il\psi} I_\nu(\zeta|x||y|/t)$ as in the proof of Proposition 4.1, where

$$R_{\mathrm{fr}}(x, y; \zeta) = \frac{e^{i\alpha\psi}}{4\pi}\int_0^{\kappa+i\infty}\chi_{M0}(t)\exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right)\frac{dt}{t},$$

$$R_{\mathrm{sc}}(x, y; \zeta) = \frac{1}{4\pi}\int_0^{\kappa+i\infty}\chi_{M0}(t)\exp\left(\frac{t}{2} - \frac{\zeta(|x|^2+|y|^2)}{2t}\right)I_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t}, \psi\right)\frac{dt}{t}$$

and $I_{\mathrm{sc}}(w, \psi)$ is defined by (4.8).

We make a similar decomposition for $R_\alpha(j_d(x), j_d(y); \zeta)$. Since $0 \leqslant \eta \leqslant 2E_0^{1/2}(\log d)/d$ for $\zeta = E - i\eta \in \overline{D}_{-d}$, we can take M so large that

$$\mathrm{Re}\, w = \mathrm{Re}\big(\zeta r_d(x) r_d(y)/t\big) \sim \big(\kappa^2 + s^2\big)^{-1}(E\kappa - \eta s)|x||y| > 0 \tag{5.16}$$

for $0 < s < 2Md$. Thus the integral representation (2.3) still makes sense for w as above, and we have

$$R_\alpha\big(j_d(x), j_d(y); \zeta\big) = G_{\mathrm{fr}}(x, y; \zeta) + G_{\mathrm{sc}}(x, y; \zeta) + G_\infty(x, y; \zeta),$$

where $G_{\mathrm{fr}}(x, y; \zeta) = R_{\mathrm{fr}}(j_d(x), j_d(y); \zeta)$ and similarly for G_{sc} and G_∞. If we use the new notation

$$p_d(x) = r_d(x)^2 + r_d(y)^2, \qquad q_d(x, y) = r_d(x) r_d(y),$$

then these three terms have the following representations:

$$G_{\mathrm{fr}} = \frac{e^{i\alpha\theta_d(x,y)}}{4\pi} \int_0^{\kappa+i\infty} \chi_{M0}(t) \exp\left(\frac{t}{2} - \frac{\zeta r_d(x,y)^2}{2t}\right) \frac{dt}{t},$$

$$G_{\mathrm{sc}} = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \chi_{M0}(t) \exp\left(\frac{t}{2} - \frac{\zeta p_d(x,y)}{2t}\right) I_{\mathrm{sc}}\left(\frac{\zeta q_d(x,y)}{t}, \theta_d(x,y)\right) \frac{dt}{t},$$

$$G_\infty = \frac{1}{4\pi} \sum_l e^{il\theta_d(x,y)} \int_0^{\kappa+i\infty} \chi_{M\infty}(t) \exp\left(\frac{t}{2} - \frac{\zeta p_d(x,y)}{2t}\right) I_\nu\left(\frac{\zeta q_d(x,y)}{t}\right) \frac{dt}{t}.$$

Statement (2) is obtained by showing that:

$$G_\infty = O\big(d^{-N}\big), \tag{5.17}$$

$$G_{\mathrm{fr}} = (i/4)e^{i\alpha\theta_d(x,y)} H_0\big(kr_d(x,y)\big) + O\big(d^{-N}\big), \tag{5.18}$$

$$G_{\mathrm{sc}} = c(E)e^{ik(r_d(x)+r_d(y))} q_d(x,y)^{-1/2}(f_\alpha + e_N) + O\big(d^{-N}\big), \tag{5.19}$$

where $f_\alpha = f_\alpha(-\omega \to \theta; E)$ with $\theta = x/|x|$ and $\omega = y/|y|$, and $e_N = e_N(x, y; \zeta)$ satisfies the estimate in the proposition.

(ii) To prove (5.17), we employ the formula

$$I_\mu(w) = \frac{e^{-i\mu\pi/2}}{\pi} \left\{ \int_0^\pi \cos(\mu\rho - iw\sin\rho)\, d\rho - \sin(\mu\pi) \int_0^\infty e^{-iw\sinh p - \mu p}\, dp \right\}$$

for $\operatorname{Im} w \leqslant 0$, which follows as an immediate consequence of the relation $I_\mu(w) = e^{-i\mu\pi/2} J_\mu(iw)$ [22, p. 176]. We note that $\operatorname{Im}(\zeta q_d(x,y)/t) < 0$ for $t = \kappa + is$ with $s > Md$, $M \gg 1$, which is seen from (5.13) and (5.14). We insert $I_\nu(\zeta q_d(x,y)/t)$ into the integral representation for $G_\infty(x, y; \zeta)$ and evaluate the resulting integral by partial integration for each l with $|l| < d$. If $M \gg 1$, then

$$\big|\partial_t\big(t - \zeta p_d(x,y)/t \pm \big(\zeta q_d(x,y)/t\big)\sin\rho\big)\big| > c > 0,$$

$$\big|\partial_t\big(t - \zeta p_d(x,y)/t - 2i\big(\zeta q_d(x,y)/t\big)\sinh p\big)\big| > c > 0$$

for $t = \kappa + is$ with $s > Md$ uniformly in ρ, $0 < \rho < \pi$, and in p, $0 < p < 1$. If $p > 1$, then we use $|\partial_t(t - \zeta p_d(x,y)/t)| > c > 0$ and

$$\partial_t e^{-i(\zeta q_d(x,y)/t)\sinh p} = -t^{-1}(\sinh p/\cosh p)\partial_p e^{-i(\zeta q_d(x,y)/t)\sinh p}.$$

We take into account these relations to repeat the integration by parts. Since $\operatorname{Im}\theta_d(x,y) = O((\log d)/d)$ as is seen from (4.14), the sum of the integrals with $|l| < d$ obeys $O(d^{-N})$. To see that the sum over l with $|l| > d$ is of order $O(d^{-N})$, we make use of the other representation formula

$$I_\mu(w) = \frac{(w/2)^\mu}{\Gamma(\mu+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{-w\rho}\bigl(1-\rho^2\bigr)^{\mu-1/2}\, d\rho \tag{5.20}$$

for $I_\mu(w)$ with $\mu \geqslant 0$ [22, p. 172]. Since $|x|+|y| = O(d)$, we have $|\zeta q_d(x,y)/t| = M^{-1}O(d)$ for $s = \operatorname{Im} t > Md$ and

$$\bigl|e^{-w\rho}\bigr| = O\bigl(e^{|\operatorname{Re}(\zeta q_d(x,y)/t)|}\bigr) = O\bigl(e^d\bigr), \quad |\rho| < 1,$$

for $w = \zeta q_d(x,y)/t$. Since $\Gamma(\mu)$ behaves like $\Gamma(\mu) \sim (2\pi)^{1/2} e^{-\mu}\mu^{\mu-(1/2)}$ for $\mu \gg 1$ by the Stirling formula, we can take $M \gg 1$ so large that

$$\bigl|e^{il\theta_d(x,y)} w^\nu/\Gamma(\nu)\bigr| \leqslant (1/2)^{|l|}, \quad |l| > d.$$

Hence the sum of integrals with l with $|l| > d$ also obeys $O(d^{-N})$, and (5.17) is proved.

(iii) (5.18) is easy to prove. By (4.5), we have

$$G_{\mathrm{fr}}(x,y;\zeta) = (i/4)e^{i\alpha\theta_d(x,y)} H_0\bigl(kr_d(x,y)\bigr) + G(x,y;\zeta),$$

where

$$G = \frac{e^{i\alpha\theta_d(x,y)}}{4\pi} \int_0^{\kappa+i\infty} \chi_{M\infty}(t) \exp\biggl(\frac{t}{2} - \frac{\zeta r_d(x,y)^2}{2t}\biggr)\frac{dt}{t}.$$

Since $|\partial_t(t - \zeta r_d(x,y)^2/t)| > c > 0$ for $s > Md$, we have $G(x,y;\zeta) = O(d^{-N})$ by partial integration, and hence (5.18) is established.

(iv) The proof of (5.19) uses the stationary phase method. By (4.8), we have

$$I_{\mathrm{sc}}\bigl(\zeta q_d(x,y)/t, \theta_d(x,y)\bigr) = -C_\alpha e^{i[\alpha](\theta_d(x,y)+\pi)} L_{\mathrm{sc}}(t,x,y;\zeta),$$

where $C_\alpha = \sin(\alpha\pi)/\pi$ and

$$L_{\mathrm{sc}}(t,x,y;\zeta) = \int_{-\infty}^{\infty} e^{i(i\zeta q_d(x,y)/t)\cosh p} \frac{e^{(1-\beta)p}}{e^p + e^{-i\theta_d(x,y)}}\, dp$$

with $\beta = \alpha - [\alpha]$, $0 < \beta < 1$. By (5.16), $\operatorname{Re}(\zeta q_d(x,y)/t) > 0$. Since $d/4 \leqslant |x_1|, |y_1| \leqslant 3d/4$ and $|x_2| + |y_2| \leqslant Nd$ by assumption, $\theta_d(x,y)$ stays away from $\pm\pi$ uniformly in x, y and $\zeta \in D_{-d}$, so that $I_{\mathrm{sc}}(\zeta q_d(x,y)/t, \theta_d(x,y))$ is bounded uniformly in x, y and ζ as above. If $0 < t < \kappa$, then

$$\bigl|\exp\bigl(t/2 - \zeta p_d(x,y)/2t\bigr)\bigr| \leqslant \exp\bigl(-cd^2/t\bigr), \quad c > 0,$$

and if $0 < s < d/M$ for $t = \kappa + is$, then it follows from (5.14) that $\operatorname{Re}(\zeta/t)$ behaves like $\operatorname{Re}(\zeta/t) \sim M^4(\log d)/d^2$ for $d \gg 1$. Hence we have

$$\bigl|\exp\bigl(t/2 - \zeta p_d(x,y)/2t\bigr)\bigr| = O\bigl(\exp\bigl(\bigl(M^2 - cM^4\bigr)\log d\bigr)\bigr) = O\bigl(d^{-N-1}\bigr)$$

for $M \gg 1$. Thus the integral over the intervals $(0,\kappa)$ and $(\kappa + i0, \kappa + id/M)$ is negligible. We assume that $d/M < s < 2Md$. We apply the stationary phase method [13, Theorem 7.7.5] to the integral $L_{\mathrm{sc}}(t,x,y;\zeta)$ above. The stationary point is given by $p = 0$, and $I_{\mathrm{sc}}(\zeta q_d(x,y)/t, \theta_d(x,y))$ is seen to take the asymptotic form

$$I_{\mathrm{sc}} = e^{-\zeta q_d(x,y)/t}\big(b_0(t,x,y;\zeta) + b_L(t,x,y;\zeta)\big) + O\big(d^{-L}\big)$$

for any $L \gg 1$, where

$$b_0 = -C_\alpha (2\pi)^{1/2} e^{i[\alpha](\theta_d(x,y)+\pi)}\big(\zeta q_d(x,y)/t\big)^{-1/2}\big(1 + e^{-i\theta_d(x,y)}\big)^{-1}$$

and b_L obeys $|\partial_x^n \partial_y^m \partial_t^j b_L| = O(d^{-3/2-|n|-|m|-j})$. The phase term is calculated as

$$t/2 - \zeta\big(p_d(x,y) + 2q_d(x,y)\big)/2t = t/2 - \zeta\big(r_d(x) + r_d(y)\big)^2/2t.$$

If $s = \operatorname{Im} t$ satisfies $d/M < s < 2d/M$ or $Md < s < 2Md$, then

$$\big|\partial_t\big(t/2 - \zeta\big(r_d(x) + r_d(y)\big)^2/2t\big)\big| > c > 0$$

for $0 < \operatorname{Re} t < \kappa$. Hence we deform the contour to the imaginary axis by analyticity and repeat integration by parts to obtain that the leading term comes from the integral

$$a_0(x,y;\zeta) = \int_0^\infty \chi_M(s)\exp\left(i\left(\frac{s}{2} + \frac{\zeta(r_d(x)+r_d(y))^2}{2s}\right)\right) b_0(is,x,y;\zeta)\frac{ds}{s},$$

where $\chi_M(s) = \chi_\infty(2Ms/d)\chi_0(2s/Md)$. We now set

$$\lambda_d(x,y) = \operatorname{Re}\big(k\big(r_d(x) + r_d(y)\big)\big), \qquad \mu_d(x,y) = \operatorname{Im}\big(k\big(r_d(x) + r_d(y)\big)\big)$$

for $k = \zeta^{1/2}$. Then $\lambda_d(x,y)$ behaves like $\lambda_d(x,y) \sim d$ and $\mu_d(x,y)$ obeys $\mu_d(x,y) = O(\log d)$. If we make a change of variable $s = \lambda_d(x,y)\tau$, then $a_0(x,y;\zeta)$ takes the form

$$a_0 = \int_0^\infty \exp\left(i\lambda_d(x,y)\left(\frac{\tau}{2} + \frac{1}{2\tau}\right)\right) e^{i\sigma_d(\tau,x,y)}\tilde{\chi}_M(\tau) b_0\big(i\lambda_d(x,y)\tau, x, y;\zeta\big)\frac{d\tau}{\tau}$$

where $\tilde{\chi}_M(\tau) = \chi_M(\lambda_d(x,y)\tau)$ and

$$\sigma_d(\tau,x,y) = \frac{\zeta(r_d(x)+r_d(y))^2 - \lambda_d(x,y)^2}{2\lambda_d(x,y)\tau} = \frac{i\mu_d(x,y)}{\tau} + O\big((\log d)^2/d\big).$$

We apply the stationary phase method to the above integral with $\tau = 1$ as a stationary point to derive the asymptotic form of $a_0(x,y;\zeta)$. We have

$$\lambda_d(x,y) + \sigma_d(1,x,y) = k\big(r_d(x) + r_d(y)\big) + O\big((\log d)^2/d\big)$$

and $b_0(i\lambda_d(x,y)\tau, x, y; \zeta)$ takes the value

$$b_0 = -C_\alpha(2\pi i)^{1/2} e^{i[\alpha](\theta_d(x,y)+\pi)} \big(\zeta q_d(x,y)\big)^{-1/2} \lambda_d(x,y)^{1/2} \big(1 + e^{-i\theta_d(x,y)}\big)^{-1}$$

at $\tau = 1$. We also have $\zeta = E + O((\log d)/d)$ and

$$\theta_d(x,y) = \psi + O\big((\log d)/d\big) = \theta - \omega + O\big((\log d)/d\big).$$

We recall that the amplitude $f_\alpha(\omega \to \theta; E)$ is defined by (1.4) and the constant $c(E)$ is defined by (5.1). We take into account the contribution $(\lambda_d(x,y)/2\pi i)^{-1/2}$ from the Hessian at the stationary point $\tau = 1$ and compute

$$\begin{aligned}
&-C_\alpha(2\pi i) e^{i[\alpha](\psi+\pi)} E^{-1/2} \big(e^{i\psi}/\big(1+e^{i\psi}\big)\big) \\
&\quad = 4\pi c(E)(2/\pi)^{1/2} e^{i\pi/4} E^{-1/4} \sin(\alpha\pi) e^{i[\alpha](\psi+\pi)} \big(e^{i(\psi+\pi)}/\big(1-e^{i(\psi+\pi)}\big)\big) \\
&\quad = 4\pi c(E) f_\alpha(-\omega \to \theta; E).
\end{aligned}$$

Since $\sigma_d(1,x,y)$ satisfies

$$\big|\partial_x^n \partial_y^m \big(e^{-i\sigma_d} \partial_\tau^j e^{i\sigma_d}\big)\big| = O\big((\log d)^j d^{-|n|-|m|}\big),$$

we see that $a_0(x,y;\zeta)$ takes the asymptotic form

$$a_0 = 4\pi c(E) e^{ik(r_d(x)+r_d(y))} q_d(x,y)^{-1/2} \big(f_\alpha(-\omega \to \theta; E) + e_N\big) + O\big(d^{-N}\big),$$

where $e_N(x,y;\zeta)$ satisfies the remainder estimate in the proposition. A similar argument applies to the integral associated with $b_L(t,x,y;\zeta)$. It takes the form

$$4\pi c(E) e^{ik(r_d(x)+r_d(y))} q_d(x,y)^{-1/2} O\big(d^{-1}\big) + O\big(d^{-N}\big)$$

and is regarded as a remainder term. Thus (5.19) is established.

(3) Finally we make only a brief comment on the asymptotic form of derivatives such as $\partial R_\alpha(j_d(x), j_d(y); \zeta)/\partial x_j$. If we take a careful look at the proof of statements (1) and (2), then we see that the asymptotic forms obtained in (1) and (2) remain true in the C^1 topology. We skip the details. The proof of the proposition is now complete. □

6. Proof of Lemma 3.4

The last section is devoted to proving Lemma 3.4. The proof is based on the following proposition.

Proposition 6.1. *Let D_d be defined by* (3.5) *and let $R_\alpha(x,y;\zeta)$ be the kernel of the resolvent $R(\zeta; P_\alpha)$ with $\zeta \in D_d$. If*

$$d/c < |x_1| < cd, \qquad |x_2| > d/c, \qquad |y_1| < cd, \qquad |y_2| < c \tag{6.1}$$

for some $c > 1$ or if

$$|x_1| < cd, \qquad |x_2| < c, \qquad d/c < |y_1| < cd, \qquad |y_2| > d/c,$$

then

$$R_\alpha\big(j_d(x), j_d(y); \zeta\big) = O\big(\big(|x_2| + |y_2|\big)^{-L}\big), \qquad |x_2| + |y_2| \gg 1,$$

for any $L \gg 1$, where the order estimate depends on d but is uniform in ζ. The derivative $\partial R_\alpha(j_d(x), j_d(y); \zeta)/\partial\zeta$ also obeys a similar bound.

Proof of Proposition 6.1. The proof uses formula (5.20) to evaluate the kernel. We consider only the case when x and y fulfill (6.1). In this case, we have that $R_\alpha(j_d(x), j_d(y); \zeta)$ equals $R_\alpha(j_d(x), y; \zeta)$. The dependence on d does not matter in the proof of the proposition. We only look at the dependence on x_2 with $|x_2| \gg 1$. We write $y = (|y|\cos\omega, |y|\sin\omega)$ and set $\theta_d(x) = \gamma(j_d(x); \omega_+)$. Then we can represent $R_\alpha(j_d(x), y; \zeta)$ as

$$R_\alpha = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \exp\left(\frac{t}{2} - \frac{\zeta r_d(x)^2}{2t}\right) \exp\left(-\frac{\zeta |y|^2}{2t}\right) S(t, x, y; \zeta) \frac{dt}{t}$$

by the line integral (4.4), where

$$S(t, x, y; \zeta) = \sum_l e^{il(\theta_d(x)-\omega)} I_\nu\left(\frac{\zeta r_d(x)|y|}{t}\right), \quad \nu = |l - \alpha|.$$

Since $\operatorname{Im}\theta_d(x) = O((\log d)/d)$ uniformly in x with $|x_2| \gg 1$, we make use of (5.20) to obtain that $S(t, x, y; \zeta)$ has the following properties:

$$\big|S(t, x, y; \zeta)\big| \leqslant \exp\big(c|x_2|/|t|\big), \tag{6.2}$$

$$\big|\partial_t^m S(t, x, y; \zeta)\big| \leqslant |t|^{-m} \exp\big(\sigma_m |x_2|/|t|\big), \tag{6.3}$$

where $c > 0$ and $\sigma_m > 0$ depend on d but are independent of $\zeta \in D_d$ and x_2. If $\zeta = E + i\eta \in D_d$, then

$$\zeta/t = \big(\kappa^2 + s^2\big)^{-1}\big((E\kappa + \eta s) - i(Es - \eta\kappa)\big)$$

for $t = \kappa + is$ on the contour of the line integral above. Since $|\eta| \leqslant 2E_0^{1/2}(\log d)/d$ for $\zeta \in D_d$ and since $r_d(x)^2$ behaves like

$$r_d(x)^2 \sim \big(1 - O\big((\log d)^2/d^2\big)\big)x_2^2 + 2i\eta_{0d}x_2^2, \quad |x_2| \gg 1,$$

with $\eta_{0d} = 5E_0^{-1/2}(\log d)/d$, we have

$$\operatorname{Re}\big(\zeta r_d(x)^2/t\big) \geqslant c_1|x_2|^2/|t| \tag{6.4}$$

for some $c_1 > 0$. We divide the line integral into the sum of two parts by use of the cut-off functions $\chi_{M0}(t) = \chi_0(s/M|x_2|)$ and $\chi_{M\infty}(t) = \chi_\infty(s/M|x_2|)$. We can take $M \gg 1$ so large that $|\partial_t(t - \zeta r_d(x)^2/t)| > c_2 > 0$ for $s > M|x_2|/2$. This, together with (6.3), enables us to repeat the partial integration, and we can obtain the bound $O(|x_2|^{-L})$ for the line integral cut off by $\chi_{M\infty}(t)$. On the other hand, we see from (6.2) and (6.4) that the line integral cut off by $\chi_{M0}(t)$ also obeys the desired bound. A similar argument applies to the derivative $\partial R_\alpha(j_d(x), j_d(y); \zeta)/\partial\zeta$. Thus the proof is complete. □

Lemma 6.1. *Let*

$$\Sigma = \{x\colon d/c < |x_1| < cd\}, \qquad \Omega = \{x\colon |x_1| < cd,\ |x_2| < c\}$$

for some $c > 1$. Then the resolvent $R(\zeta; \tilde{P}_{\alpha d})$ of the closed operator $\tilde{P}_{\alpha d} = J_d P_\alpha J_d^{-1}$ is analytic as a function of $\zeta \in D_d$ with values in bounded operators from $L^2(\Sigma)$ to $L^2(\Omega)$ or from $L^2(\Omega)$ to $L^2(\Sigma)$.

Proof. We know that $R(\zeta; P_\alpha) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ is well defined. If we set $\Sigma' = \{x \in \Sigma\colon |x_2| < d/c\}$, then

$$R(\zeta; \tilde{P}_{\alpha d}) = R(\zeta; P_\alpha) : L^2(\Sigma') \to L^2(\Omega), \quad L^2(\Omega) \to L^2(\Sigma')$$

is bounded. Hence the lemma is obtained as an immediate consequence of Proposition 6.1. □

Proof of Lemma 3.4.. The operator $H_{\pm d}$ has the solenoidal field $2\pi\alpha_\pm\delta(x - d_\pm)$ with $d_\pm = (\pm d/2, 0)$ as a center. Since the relations

$$\Sigma_0 = \{7d/16 < x_1 + d/2 < 9d/16\} = \{-9d/16 < x_1 - d/2 < -7d/16\}$$

and $\Omega_0 \subset \{|x_1 \pm d/2| < 3d/2,\ |x_2| < r_0\}$ hold true for Σ_0 and Ω_0, the lemma is obtained by applying Lemma 6.1 to

$$R(\zeta; K_{\pm d}) = \exp(ig_{\mp d})\tilde{R}_{\pm d}(\zeta)\exp(-ig_{\mp d})$$

with $\tilde{R}_{\pm d}(\zeta) = J_d R(\zeta; H_{\pm d}) J_d^{-1}$. □

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Appendix A

We shall prove that the resolvent of the magnetic Schrödinger operator with two solenoidal fields has the meromorphic continuation over the lower-half plane as a function of the spectral parameter ζ with values in operators from L^2_{comp} to L^2_{loc}. The resolvent is shown to be meromorphically continued over the complex plane $\boldsymbol{C} \setminus (-\infty, 0]$ slit along the negative real axis across the positive real axis where the spectrum of the operator is located. The argument here extends to the case of several solenoidal fields without any essential changes.

We consider the operator

$$H = H(\Psi) = (-i\nabla - \Psi)^2$$

with two solenoidal fields $2\pi\alpha_+\delta(x - p_+)$ and $2\pi\alpha_-\delta(x - p_-)$, where

$$\Psi(x) = \Phi_+(x) + \Phi_-(x) = \alpha_+\Phi(x - p_+) + \alpha_-\Phi(x - p_-).$$

The operator H becomes self-adjoint on $L^2 = L^2(\boldsymbol{R}^2)$ under the boundary condition $\lim_{|x-p_\pm|\to 0} |u(x)| < \infty$ at both the centers $p_+ = (p, 0)$ and $p_- = (-p, 0)$ for $p > 0$. We cover the whole space with the three regions

$$X_\pm = \{x\colon\ |x_1 \mp p| < 3p/2,\ |x_2| < 3p/2\}, \qquad X_0 = \boldsymbol{R}^2 \setminus \{x\colon\ |x_1| < 2p,\ |x_2| < p\}$$

and approximate H by operators with one solenoidal field over each region. To do this, we define the three operators $P_\pm = H(\Phi_\pm)$ and $P = H(\Phi_0)$, where $\Phi_0(x) = \alpha_0\Phi(x)$ with $\alpha_0 = \alpha_+ + \alpha_-$. These auxiliary operators become self-adjoint by imposing the boundary condition as in (1.3) at the center of the solenoidal field. We can construct real smooth bounded functions $g_\pm(x)$ and $g_0(x)$ such that $H = e^{ig_\pm}P_\pm e^{-ig_\pm}$ over $X_\pm$ and $H = e^{ig_0}P_0e^{-ig_0}$ over X_0. We set

$$\hat{P}_\pm = e^{ig_\pm}P_\pm e^{-ig_\pm}, \qquad \hat{P}_0 = e^{ig_0}P_0e^{-ig_0}$$

and introduce a smooth nonnegative partition of unity $\{u_+, u_-, u\}$ such that $u_\pm$ and u have support in $X_\pm$ and X_0, respectively. Then we define the bounded operator

$$G(\zeta) = u_+R(\zeta; \hat{P}_+) + u_-R(\zeta; \hat{P}_-) + u_0R(\zeta; \hat{P}_0) : L^2 \to L^2$$

for $\zeta \in \boldsymbol{C}_+ = \{\zeta \in \boldsymbol{C}\colon\ \operatorname{Im}\zeta > 0\}$. This operator satisfies $(H - \zeta)G(\zeta) = Id + Q(\zeta)$, where

$$Q(\zeta) = [\hat{P}_+, u_+]R(\zeta; \hat{P}_+) + [\hat{P}_-, u_-]R(\zeta; \hat{P}_-) + [\hat{P}_0, u_0]R(\zeta; \hat{P}_0).$$

The commutators above vanish outside the region $X = \{x\colon\ |x_1| < 3p,\ |x_2| < 3p\}$. For the Hamiltonians $P_\pm$ and P_0 with one solenoidal field, the resolvents $R(\zeta; \hat{P}_\pm)$ and $R(\zeta; \hat{P}_0)$ are continued as analytic functions of ζ with values in operators from L^2_{comp} to L^2_{loc} over the lower-half plane across the positive real axis. If we consider $Q(\zeta)$ as an operator from $L^2(X)$ into itself, then $Q(\zeta)$ turns out to be an analytic function of $\zeta \in \boldsymbol{C} \setminus (-\infty, 0]$ with values in compact operators. Hence it follows from the analytic perturbation theory of Fredholm that the inverse $(Id + Q(\zeta))^{-1} : L^2(X) \to L^2(X)$ has the meromorphic continuation over $\boldsymbol{C} \setminus (-\infty, 0]$. Thus $R(\zeta; H)$ is represented as

$$R(\zeta; H) = G(\zeta) - G(\zeta)Q(\zeta)\big(Id + Q(\zeta)\big)^{-1}$$

and is defined as a meromorphic function over $\boldsymbol{C} \setminus (-\infty, 0]$ with values in operators from $L^2_{\mathrm{comp}}(X)$ to L^2_{loc}. Once this is established, we can show in almost the same way as in the proof of Theorem 1.1 (Step 5) that $R(\zeta; H)$ becomes a meromorphic function over $\boldsymbol{C} \setminus (-\infty, 0]$ with values in operators from L^2_{comp} to L^2_{loc}.

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