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RESOLVENT AND SCATTERING MATRIX AT THE MAXIMUM OF THE POTENTIAL

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Dedicated to Vesselin Petkov on the occasion of his 65th birthday

Abstract. We study the microlocal structure of the resolvent of the semiclassical Schrödinger operator with short range potential at an energy which is a unique non-degenerate global maximum of the potential. We prove that it is a semiclassical Fourier integral operator quantizing the incoming and outgoing Lagrangian submanifolds associated to the fixed hyperbolic point. We then discuss two applications of this result to describing the structure of the spectral function and the scattering matrix of the Schrödinger operator at the critical energy.

1. Introduction.

We consider the semiclassical Schrödinger operator

$$P = P_0 + V, \quad P_0 = -\frac{1}{2}h^2\Delta, \quad 0 < h \ll 1, \tag{1.1}$$

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where $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$, $n > 1$, is a short range potential, *i.e.*, for some $\rho > 1$ and all $\alpha \in \mathbb{N}^n$

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Then P and P_0 admit unique self-adjoint realizations on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$, that we still denote P and P_0. In this paper, we are interested in the microlocal structure of the resolvent and of the spectral measure of P, as well as that of the scattering matrix, at energies which are within $\mathcal{O}(h)$ of a unique non-degenerate global maximum of the potential. More precisely, we show below that they are semiclassical Fourier integral operators (for short h-FIOs). We refer to Appendix A and to the references given therein for a short presentation of the theory of such operators.

The resolvent $\mathcal{R}(E \pm i0)$ can be defined thanks to the limiting absorption principle which states that, for $E > 0$ and when $\alpha > \frac{1}{2}$, the limit

$$\mathcal{R}(E \pm i0) = \lim_{\varepsilon \searrow 0} (P - (E \pm i\varepsilon))^{-1}$$

exists in $\mathcal{B}(L^2_\alpha(\mathbb{R}^n), L^2_{-\alpha}(\mathbb{R}^n))$, where $L^2_\alpha(\mathbb{R}^n) = \{f;\ \langle x\rangle^\alpha f(x) \in L^2(\mathbb{R}^n)\}$. We denote by $d\mathcal{E}_E$ the spectral measure of P. The spectral function e_E is the Schwartz kernel of $\dfrac{d\mathcal{E}_E}{dE}$, and can be represented through the well-known Stone formula

$$\frac{d\mathcal{E}_E}{dE} = \frac{1}{2i\pi}\left(\mathcal{R}(E+i0) - \mathcal{R}(E-i0)\right), \quad E > 0. \tag{1.3}$$

The scattering matrix $\mathcal{S}(E,h)$ is defined by means of the wave operators. We recall that under the assumption (1.2), the wave operators, defined as the strong limits in $L^2(\mathbb{R}^n)$,

$$W_\pm = \operatorname*{s\text{–}lim}_{t\to\pm\infty} e^{-itP/h} e^{itP_0/h} \tag{1.4}$$

exist and are complete. The scattering operator is then defined as $S = W_+^* W_- : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and $\mathcal{S}(E,h) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ is given by

$$S = \int_{\mathbb{R}^+}^{\oplus} F_0(E,h)^{-1} \mathcal{S}(E,h) F_0(E,h)\, dE$$

Here $F_0(E,h)$ denotes the bounded operator from $L^2_\alpha(\mathbb{R}^n)$, $\alpha > 1/2$, to $L^2(\mathbb{S}^{n-1})$ given by

$$\left(F_0(E,h)f\right)(\omega) = (2\pi h)^{-n/2}(2E)^{\frac{n-2}{4}} \int_{\mathbb{R}^n} e^{-i\sqrt{2E}\langle\omega,x\rangle/h} f(x)\, dx, \quad E > 0. \tag{1.5}$$

Notice that most of the results in the literature on the scattering matrix are given for the operator

$$\mathcal{T}(E,h) = \frac{1}{2i\pi}(\mathrm{Id} - \mathcal{S}(E,h)), \tag{1.6}$$

or for the scattering amplitude

$$\mathcal{A}(E,h) = c_0\, K_{\mathcal{T}(E,h)}, \tag{1.7}$$

where we denote $K_{\mathcal{T}(E,h)}$ the Schwartz kernel of the operator $\mathcal{T}(E,h)$ and

$$c_0 = c_0(n,E,h) = -2\pi(2E)^{-(n-1)/4}(2\pi h)^{(n-1)/2}e^{-i(n-3)\pi/4}.$$

The semiclassical behavior of the spectral function for Schrödinger-like operators has been studied extensively. Popov and Shubin [22], Popov [21], and Vainberg [29] have established high energy asymptotics for the spectral function of second order elliptic operators under the assumption that these energies are non-trapping:

Definition 1.1. *The energy $E > 0$ is non-trapping if for every $(x,\xi) \in p^{-1}(E) \subset T^*\mathbb{R}^n$ we have*

$$\lim_{t\to\pm\infty} |\exp(tH_p)(x,\xi)| = \infty.$$

Here $p(x,\xi) = \frac{1}{2}\xi^2 + V(x)$ denotes the principal symbol of P, and

$$H_p = \sum_{j=1}^{n}\left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\right)$$

is its associated Hamiltonian vector field.

Robert and Tamura [27] consider the spectral function for semiclassical Schrödinger operators with short range potentials and establish asymptotic expansions at fixed non-trapping energy, and at non-critical trapping energies in the sense of distributions.

The microlocal structure of the spectral function has also been analyzed. In [30, Theorem XII.5] Vainberg establishes a high energy asymptotic expansion of the spectral function for compactly supported smooth perturbations of the Laplacian assuming that the energy 1 is non-trapping. This asymptotic expansion is expressed in the form of a Maslov canonical operator.

C. Gérard and Martinez [12] have proved that the spectral function for certain long-range Schrödinger operators at non-trapping energies E is a h-FIO associated to the canonical relation $\big(\cup_{t\in\mathbb{R}} \operatorname{graph} \exp(tH_p)|_{p^{-1}(E)}\big)$. Near the diagonal $\{(x,\xi,x,\xi);\ p(x,\xi)=E\}$ they also give the following oscillatory integral representation of the spectral function

$$e_E(x,y,E,h) = \frac{1}{(2\pi h)^n}\int_{\mathbb{S}^{n-1}} e^{i\varphi(x,y,\omega,E)/h} a(x,y,\omega,E)d\omega,$$

where $\varphi \in C^\infty(\mathbb{R}^{2n}\times\mathbb{S}^{n-1})$ is such that

$$\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + V(x) = E, \quad \frac{\partial\varphi}{\partial x}\Big|_{\langle x-y,\omega\rangle=0} = \sqrt{E-V(x)}\omega, \quad \varphi|_{x=y} = 0.$$

In [4] the first author has studied the microlocal structure of the spectral function restricted away from the diagonal in $\mathbb{R}^n\times\mathbb{R}^n$ at trapping energies under the assumption of the absence of resonances near the real axis, as well as at non-trapping energies. In these cases the spectral function is shown to be an h-FIO associated to $\big(\cup_{t\in\mathbb{R}} \operatorname{graph} \exp(tH_p)|_{p^{-1}(E)}\big)$ near a non-trapped trajectory. Under a certain geometric assumption [4] also gives an oscillatory integral representation of the spectral function of the form

$$e_E(x,y,E) = \int e^{iS(x,y,t)/h} a(x,y,t)\,dt,$$

where

$$S(x,y,t) = \int_{l(t,x,y)} \Big(\frac{1}{2}|\xi(s)|^2 + E - V(x(s))\Big)\,ds,$$

is the action over the segment $l(t,x,y)$ of the trajectory which connects x with y at time t and $a \in S^{\frac{n+3}{2}}_{2n+1}(1)$.

The structure of the resolvent in various settings has been studied in [3], [5], and [14]. For compactly supported and short range potentials, the resolvent has been shown to be a h-FIO associated to the Hamiltonian flow relation of the principal symbol of P restricted to the energy surface in [3] and [5]. Hassell and Wunsch have studied in [14] the resolvent on asymptotically conic non-trapped manifolds. This class contains in particular some asymptotically Euclidean spaces after compactification. They prove that the Schwartz kernel of the resolvent is a Legendrian distribution, that is, roughly speaking, a semiclassical Lagrangian distribution where the semiclassical parameter is the distance to the boundary.

The semiclassical behavior of the scattering amplitude has also been of significant interest to researchers in mathematical physics. It is well known that

$\mathcal{A}(E,h)$ satisfies $\mathcal{A}(E,h) \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \mathrm{diag}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}))$. Several authors have proved asymptotic expansions for $\mathcal{A}(E,h)$, showing in particular a direct relation with the underlying classical mechanics.

To describe these results, let us recall that, for $(a,b) \in T^*\mathbb{R}^n \setminus \{0\} = \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, there is a unique bicharacteristic curve (*i.e.* an integral curve of H_p)

$$\gamma_\pm(t,a,b) = (x_\pm(t,a,b), \xi_\pm(t,a,b)), \tag{1.8}$$

such that

$$\begin{aligned} &\lim_{t\to\pm\infty} |x_\pm(t,a,b) - bt - a| = 0 \\ &\lim_{t\to\pm\infty} |\xi_\pm(t,a,b) - b| = 0. \end{aligned} \tag{1.9}$$

Moreover, the mapping

$$\begin{cases} T^*\mathbb{R}^n \setminus \{0\} \longrightarrow & T^*\mathbb{R}^n \\ (a,b) & \gamma_\pm(0,a,b) \end{cases} \tag{1.10}$$

is a C^∞ symplectic diffeomorphism onto its image (see [25, Section XI.2]).

On the other hand, if a bicharacteristic curve $(x(t,\rho), \xi(t,\rho)) = \exp(tH_p)(\rho)$ of positive energy satisfies $|x(t,\rho)| \to +\infty$ as $t \to +\infty$, there is $(x_\infty, \xi_\infty) = (x_\infty(\rho), \xi_\infty(\rho)) \in T^*\mathbb{R}^n$ such that

$$\begin{aligned} &\lim_{t\to+\infty} |x(t,\rho) - \xi_\infty t - x_\infty| = 0, \\ &\lim_{t\to+\infty} |\xi(t,\rho) - \xi_\infty| = 0. \end{aligned} \tag{1.11}$$

In that case

$$\begin{aligned} \Theta(\rho) =& \frac{\xi_\infty}{|\xi_\infty|} \in \mathbb{S}^{n-1} \\ Z(\rho) =& x_\infty - \langle x_\infty, \xi_\infty \rangle \frac{\xi_\infty}{|\xi_\infty|^2} \in \Theta^\perp \sim \mathbb{R}^{n-1}, \end{aligned} \tag{1.12}$$

are called the outgoing (asymptotic) direction and outgoing impact factor, respectively.

In particular, for a given $E > 0$, $\alpha \in \mathbb{S}^{n-1}$ and $z \in \alpha^\perp$ (the impact plane), we define

$$\gamma_\pm(t,\alpha,z,E) = (x_\pm(t,\alpha,z,E), \xi_\pm(t,\alpha,z,E)) := \gamma_\pm(t,z,\sqrt{2E}\alpha). \tag{1.13}$$

If for some $(\omega, z_-) \in T^*\mathbb{S}^{n-1}$, we have $|x_-(t, \omega, z_-, E)| \to \infty$ as $t \to +\infty$, we denote by $x_\infty(\omega, z_-, E)$ and $\xi_\infty(\omega, z_-, E)$ the quantities defined through (1.11) for the curve $\gamma_-(t, \omega, z_-, E)$. We also set

$$\begin{cases} \theta = \theta(\omega, z_-, E) = \Theta(\gamma_-(0, \omega, z_-, E)) \\ z_+ = z_+(\omega, z_-, E) = Z(\gamma_-(0, \omega, z_-, E)), \end{cases} \tag{1.14}$$

and we shall say that the trajectory $\gamma_-(t, \omega, z_-, E)$ has initial direction ω and final direction θ, or that it is an (ω, θ)-trajectory.

Definition 1.2. *The outgoing direction $\theta \in \mathbb{S}^{n-1}$ is called regular for the incoming direction $\omega \in \mathbb{S}^{n-1}$, or ω-regular, if $\theta \neq \omega$ and, for all $z' \in \omega^\perp$ with $\xi_\infty(\omega, z', E) = \sqrt{2E}\theta$, the map $\omega^\perp \ni z \mapsto \xi_\infty(\omega, z, E) \in \mathbb{S}^{n-1}$ is non-degenerate at z', i.e. $\widehat{\sigma}(z') \neq 0$ where*

$$\widehat{\sigma}(z') = |\det(\xi_\infty(\omega, z', E), \partial_{z_1}\xi_\infty(\omega, z', E), \dots, \partial_{z_{n-1}}\xi_\infty(\omega, z', E))|.$$

Under the assumption that a certain final direction θ is regular for a given initial direction ω, it has been shown that

$$\mathcal{A}(E, h)(\theta, \omega) = \sum_{j=1}^{l} \hat{\sigma}(\omega, z_j, E)^{-1/2} \exp(ih^{-1}S_j - i\mu_j\pi/2) + \mathcal{O}(h), \tag{1.15}$$

where

$$(z_j)_{j=1}^{l} = \big(\xi_\infty^{-1}(\sqrt{2E}\omega, \cdot, E)\big)(\theta),$$

and

$$S_j = \int_{-\infty}^{\infty} \big(|\xi_-(t, \omega, z_j, E)|^2 - 2E\big) dt - \langle x_\infty(\omega, z_j, E), \sqrt{2E}\theta(\omega, z_j, E)\rangle \tag{1.16}$$

is a modified action along the j-th (ω, θ)-trajectory, and μ_j is the Maslov index of that trajectory. Such a result has been obtained by Vainberg [29], who has studied smooth compactly supported potentials V at energies $E > \sup V$. Guillemin [13] has established a similar asymptotic expansion in the setting of smooth compactly-supported metric perturbations of the Laplacian. Working with some trapping potential perturbations of the Laplacian satisfying (1.2) with $\rho > \max\big(1, \frac{n-1}{2}\big)$, Yajima [32] has proved such an asymptotic expansion in the L^2 sense. For non-trapping short-range ($\rho > 1$) potential perturbations of the Laplacian, Robert and Tamura [28] have proven that (1.15) holds pointwise.

Their result has been extended to the case of trapping energies by Michel [20] under an additional assumption on the distribution of the resonances of P.

First to study the microlocal structure of the scattering amplitude was Protas [23]. He has shown that at non-trapping energies and for fixed initial directions the scattering amplitude is a Maslov canonical operator associated to some natural Lagrangian submanifolds of $T^*\mathbb{S}^{n-1}$. This representation of the scattering amplitude is shown to hold uniformly in an open set containing the final direction and disjoint from the initial direction.

In [3] and [5] the first author has proved, without making the non-degeneracy assumption, that for short-range Schrödinger operators satisfying a polynomial estimate for their resolvent, the scattering amplitude is an h-FIO associated to the scattering relation microlocally near a non-trapped trajectory. The scattering relation for a short range potential at an energy $E > 0$ is defined near a non-trapped trajectory as follows. If $\gamma_0 : t \mapsto \gamma_-(t, \omega_0, z_0, E)$ is non-trapped, there exists an open set $U \subset T^*\mathbb{S}^n$ with $(\omega_0, z_0) \in U$ such that for every $(\omega, z_-) \in U$ the trajectory $t \mapsto \gamma_-(t, z, \omega_0, E)$ is non-trapped. The scattering relation near γ_0 is given by (see Figure 1)

$$\mathcal{SR}(E) = \{(\theta(\omega, z_-, E), -\sqrt{2E}z_+(\omega, z_-, E), \omega, -\sqrt{2E}z_-);\ (\omega, z_-) \in U\}, \tag{1.17}$$

where θ and z_+ are defined in (1.14).

It is also explained in [3] how the expansion (1.15) follows from this result once the non-degeneracy assumption on the initial and final directions is made. The asymptotic expansion obtained is more general than the one given in (1.15) in that it holds microlocally near (ω, θ) trajectories and not only for fixed initial and final directions.

In the context of scattering on a manifold with boundary, Hassell and Wunsch [14] have shown that the scattering matrix at non-trapping energies is a Legendrian-Lagrangian distribution associated to the total sojourn relation. In [31], Vasy has also studied the scattering matrix on asymptotically De Sitter-like spaces (a large class of non-trapped spaces with two asymptotically hyperbolic ends). Under the assumption that the bicharacteristic curves go from one end to the other, he has proved that the scattering matrix is a FIO associated to the natural relation between these two ends.

In this paper we continue the study of the scattering matrix for energies which are within $\mathcal{O}(h)$ of a unique non-degenerate global maximum of the potential. In that setting, in the one-dimensional case, the scattering matrix is a 2 by 2 matrix, and the semiclassical expansion of its coefficient has been given by the third author in [24]. The computations there rely on complex WKB construc-

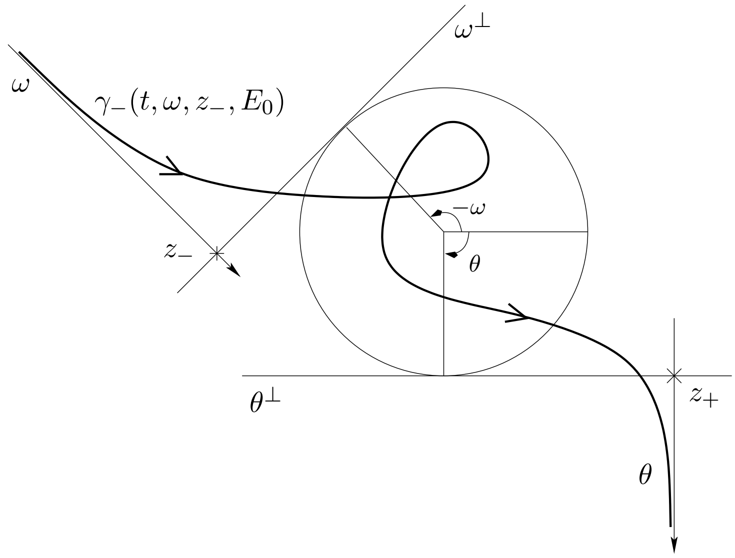

Fig. 1. The scattering relation near a non-trapped trajectory

tions for the generalized eigenfunctions, as well as a microlocal reduction of the operator to a normal form near the maximum point of the potential.

For such a critical energy, we have already studied the scattering amplitude in the n-dimensional case: In [6], we have established the semiclassical expansion of the scattering amplitude. In that paper, we use Robert and Tamura's formula (see (4.6) below) for the scattering amplitude. This formula itself relies on Isozaki and Kitada's construction of a suitable approximation for the wave operators, and, roughly speaking, reduces the problem to that of the description of generalized eigenfunctions in a compact set. To do so, we essentially follow the study in [8], to obtain such a description in a neighborhood of the critical point.

In the present paper we describe the microlocal structure of the spectral function and of the scattering matrix at such energies. More precisely we show that they are h-FIOs associated to quite natural canonical relations. To the contrary of [6], we do not suppose the non-degeneracy assumption, and we state no geometrical assumptions concerning the behavior of the incoming and outgoing stable manifolds at infinity. However the results below are valid in a somewhat smaller region of the phase space. Of course one recovers parts of the results of [6] in that smaller region once the geometric assumptions alluded to above are made.

We are very glad to dedicate this paper to Vesselin Petkov at the occasion of his 65th birthday. His numerous works, in particular in scattering theory and microlocal analysis, have inspired us a lot. We thank him too for his availability and for his judicious advices.

2. Assumptions and main results. We suppose that the potential V is a short-range, C^∞ function on $\mathbb{R}^n$ (see (1.2)), and we make the following additional assumptions:

(A1) V has a non-degenerate global maximum at $x = 0$, with $V(0) = E_0 > 0$. We can always suppose that

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{2} x_j^2 + \mathcal{O}(x^3), \quad x \to 0,$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

(A2) The trapped set at energy E_0 is reduced to $(0,0)$, namely

$$\{(x,\xi) \in p^{-1}(E_0);\ \exp(tH_p)(x,\xi) \not\to \infty \text{ as } t \to \pm\infty\} = \{(0,0)\}.$$

Then, the linearized vector field of H_p at $(0,0)$ is

$$d_{(0,0)}H_p = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2) & 0 \end{pmatrix},$$

and, by the stable/unstable manifold theorem, there exist Lagrangian submanifolds $\Lambda_\pm$ of $T^*\mathbb{R}^n$ (see Figure 2) satisfying

$$\Lambda_\pm = \{(x,\xi) \in T^*\mathbb{R}^n;\ \exp(tH_p)(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\} \subset p^{-1}(E_0).$$

Notice that the assumptions *(A1)* and *(A2)* imply that V has an absolute global maximum at $x = 0$. Indeed, if $\mathcal{L} = \{x \neq 0;\ V(x) \geq E_0\}$ was non empty, the geodesic, for the Agmon distance $(E_0 - V(x))_+^{1/2}dx$, between 0 and $\mathcal{L}$ would be the projection of a trapped bicharacteristic (see [1, Theorem 3.7.7]).

We recall from [15] that if $\rho_\pm \in \Lambda_\pm$ and $(x_\pm(t,\rho_\pm), \xi_\pm(t,\rho_\pm)) = \exp(tH_p)(\rho_\pm)$ is the bicharacteristic starting from $\rho_\pm$, then for some $g_\pm \in C^\infty(\Lambda_\pm; \mathbb{R}^n)$ and $\varepsilon > 0$,

$$x_\pm(t;\rho_\pm) = g_\pm(\rho_\pm)e^{\pm\lambda_1 t} + \mathcal{O}(e^{\pm(\lambda_1+\varepsilon)t}) \text{ as } t \to \mp\infty.$$

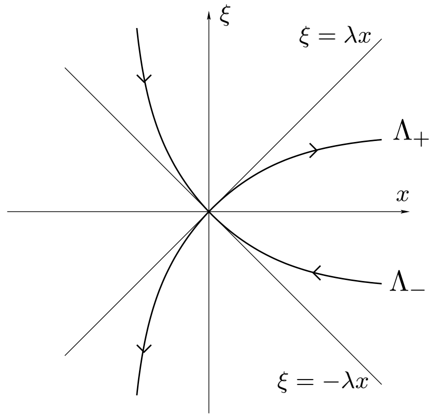

Fig. 2. The incoming Λ_- and outgoing Λ_+ Lagrangian submanifolds

We let

$$\widetilde{\Lambda_+ \times \Lambda_-} = \big\{(\rho_+, \rho_-) \in \Lambda_+ \times \Lambda_-;\ \langle g_+(\rho_+), g_-(\rho_-)\rangle \neq 0\big\},$$

and define $\widetilde{\Lambda_- \times \Lambda_+}$ analogously.

Remark 2.1. The reader may notice that if $\lambda_2 > \lambda_1$, then, by [6, (6.96)], the vectors $g_\pm(\rho)$ are for any ρ collinear with $(1, 0, \dots, 0) \in \mathbb{R}^n$. Therefore, $\widetilde{\Lambda_+ \times \Lambda_-} = \Lambda_+ \setminus \widetilde{\Lambda_+} \times \Lambda_- \setminus \widetilde{\Lambda_-}$, where $\widetilde{\Lambda_\pm} = \{\rho \in \Lambda_\pm;\ g_\pm(\rho) = 0\}$. We recall from [8] that in this case $\dim \widetilde{\Lambda_\pm} = n - 1$.

Our main result is the following

Theorem 2.2. *Assume (A1) and (A2). Then, microlocally near any* $(\rho_+, \rho_-) \in \widetilde{\Lambda_+ \times \Lambda_-}$ *we have*

$$\mathcal{R}(E + i0) \in \mathcal{I}_h^{1 - \frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \big(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_+ \times \Lambda_-{}'\big),$$

and, microlocally near any $(\rho_-, \rho_+) \in \widetilde{\Lambda_- \times \Lambda_+}$,

$$\mathcal{R}(E - i0) \in \mathcal{I}_h^{1 - \frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \big(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_- \times \Lambda_+{}'\big),$$

for $E \in]E_0 - C_0 h, E_0 + C_0 h[$ *with* $C_0 > 0$.

Remark that the symbol of these two h-FIOs can be computed, as well as that of all the operators below. Concerning the spectral function, using Stone's Formula (1.3), we obtain immediately the

Corollary 2.3. *Assume (A1) and (A2). Then the spectral function at energy E satisfies, microlocally near $(\rho_1, \rho_2) \in \widetilde{\Lambda_+ \times \Lambda_-} \cup \widetilde{\Lambda_- \times \Lambda_+}$,*

$$e_E \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \big(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_+ \times \Lambda_-{}' \cup \Lambda_- \times \Lambda_+{}'\big),$$

for $E \in]E_0 - C_0 h, E_0 + C_0 h[$ with $C_0 > 0$.

Now we pass to our result concerning the scattering matrix. We denote by (see Figure 3)

$$\Lambda_+^\infty = \{(\theta, -\sqrt{2E_0} z_+) \in T^*\mathbb{S}^{n-1};\ \gamma_+(0, \theta, z_+, E_0) \in \Lambda_+\},$$
$$\Lambda_-^\infty = \{(\omega, -\sqrt{2E_0} z_-) \in T^*\mathbb{S}^{n-1};\ \gamma_-(0, \omega, z_-, E_0) \in \Lambda_-\}.$$

Notice that $\Lambda_\pm^\infty$ are submanifolds of $T^*\mathbb{S}^{n-1}$ of dimension $n-1$, since the map $(\alpha, z) \mapsto \gamma_\pm(0, \alpha, z, E)$ is a C^∞ diffeomorphism. We set also

$$\widetilde{\Lambda_+^\infty \times \Lambda_-^\infty} = \big\{(\theta, -\sqrt{2E_0} z_+, \omega, -\sqrt{2E_0} z_-) \in \Lambda_+^\infty \times \Lambda_-^\infty; \\ \big\langle g_+(\gamma_+(0, \theta, z_+, E_0)), g_-(\gamma_-(0, \omega, z_-, E_0))\big\rangle \neq 0\big\}.$$

Theorem 2.4. *Assume (A1) and (A2). Then, microlocally near $(\theta, -\sqrt{2E_0} z_+, \omega, -\sqrt{2E_0} z_-)$ in $\widetilde{\Lambda_+^\infty \times \Lambda_-^\infty}$ with $\omega \neq \theta$,*

$$\mathcal{S}(E, h) \in \mathcal{I}_h^{\frac{1}{2}-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \big(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \Lambda_+^\infty \times \Lambda_-^{\infty\prime}\big),$$

for $E \in]E_0 - C_0 h, E_0 + C_0 h[$ with $C_0 > 0$.

For potentials V with compact support, this result can be extended to the case $\omega = \theta$. In fact, for such potentials there exists a nice representation of the scattering matrix which is valid even for $\omega = \theta$ (see [3, Equation (46)]). Starting from this representation, one can follow the proof in Section 4.2.

Notice that, near non-trapped trajectories, our proof here gives the following improvement of [5, Main Theorem] for what concerns the order. The order is here optimal as shown by the results of the paper [28]. Of course, one can obtain analogous results concerning the resolvent or the spectral function (see (4.23)).

Theorem 2.5. *Suppose* (1.1), (1.2), $E_0 > 0$ *and, for some* $\alpha > 1/2$, $N \in \mathbb{R}$ *and* $C_0 > 0$,

$$\|\mathcal{R}(E + i0)\|_{\mathcal{B}(L^2_\alpha(\mathbb{R}^n), L^2_{-\alpha}(\mathbb{R}^n))} = \mathcal{O}(h^N), \tag{2.1}$$

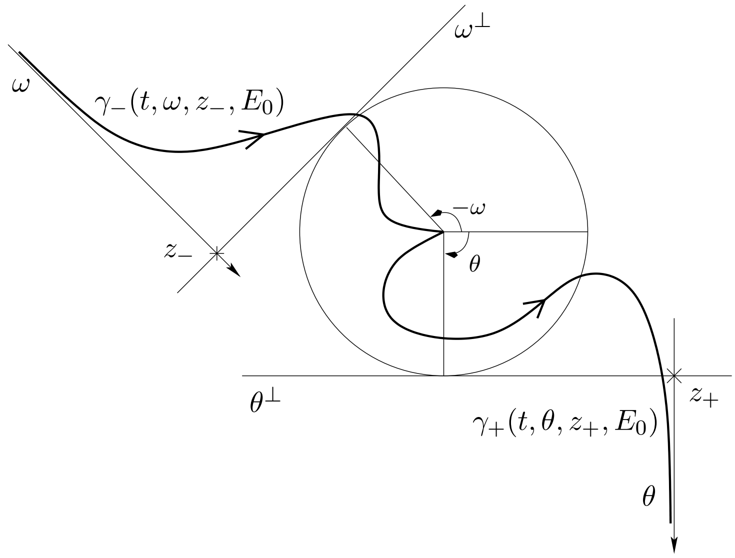

Fig. 3. The scattering relation $\Lambda_+^\infty \times \Lambda_-^\infty$ consists of the points $(\theta, -\sqrt{2E_0}z_+, \omega, -\sqrt{2E_0}z_-)$ related as in this figure.

for $E \in]E_0 - C_0 h, E_0 + C_0 h[$. *If* $(\omega, z_-) \in T^*\mathbb{S}^{n-1}$ *is such that* $\gamma_-(t,\omega,z_-,E_0)$ *is non-trapped, then, microlocally near* $(\theta(\omega,z_-,E_0), -\sqrt{2E_0}z_+(\omega,z_-,E_0), \omega, -\sqrt{2E_0}z_-)$*, provided* $\omega \neq \theta(\omega,z_-,E_0)$ *we have*

$$\mathcal{S}(E,h) \in \mathcal{I}_h^0\big(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}, \mathcal{SR}(E_0)'\big),$$

for $E \in]E_0 - C_0 h, E_0 + C_0 h[$.

For the other non-trapped trajectories, one can see from the proof of Theorem 2.5 that we have the following result.

Theorem 2.6. *Assume* (1.1), (1.2) *and* (2.1). *Let* $(\omega, z_-), (\theta, z_+) \in T^*\mathbb{S}^{n-1}$ *be such that* $\omega \neq \theta$, $\gamma_-(t,\omega,z_-,E_0)$ *or* $\gamma_+(t,\theta,z_+,E_0)$ *is non-trapped and the curves* $\gamma_-(t,\omega,z_-,E_0)$ *and* $\gamma_+(t,\theta,z_+,E_0)$ *do not coincide. Then, microlocally near* $(\theta, -\sqrt{2E_0}z_+, \omega, -\sqrt{2E_0}z_-)$,

$$\mathcal{S}(E,h) = 0,$$

for $E \in]E_0 - C_0 h, E_0 + C_0 h[$.

From the previous results, the reader may notice that, under assumptions *(A1)* and *(A2)*, the scattering matrix can be written, for $E \in]E_0 - C_0 h, E_0 + C_0 h[$

with $C_0 > 0$, as

$$\mathcal{S}(E,h) \in \mathcal{I}_h^{\frac{1}{2}-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}\big(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}, \Lambda_+^\infty \times \Lambda_-^\infty{}'\big) + \mathcal{I}_h^0\big(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}, \mathcal{SR}(E_0)'\big),$$

microlocally near any point $(\theta, -\sqrt{2E_0}z_+, \omega, -\sqrt{2E_0}z_-) \in T^*\mathbb{S}^{n-1}\times T^*\mathbb{S}^{n-1}$ not in $\Lambda_+^\infty \times \Lambda_-^\infty \setminus \widetilde{\Lambda_+^\infty \times \Lambda_-^\infty}$, with $\omega \neq \theta$.

This paper is organized as follows. We prove Theorem 2.2 in Section 3.1, and in Section 3.2, we give the microlocal representations of the resolvent and the spectral function implied by Theorem 2.2. In Section 4.2 we prove Theorem 2.4 using the representation of the scattering amplitude presented in Section 4.1. We sketch the proof of Theorem 2.5 in Section 4.3. We use Theorem 2.4 to deduce an oscillatory integral representation and an integral representation of the scattering amplitude in Section 5. Lastly, in Appendix A we review the notions from semiclassical analysis most relevant to this work.

3. The resolvent as a semiclassical Fourier integral operator.

3.1. Proof of Theorem 2.2. We shall prove that $\mathcal{R}(E+i0) \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}\big(\mathbb{R}^n\times\mathbb{R}^n, \Lambda_+\times\Lambda_-{}'\big)$ microlocally near any $(\rho_+,\rho_-) \in \widetilde{\Lambda_+\times\Lambda_-}$. The proof in the case of the incoming resolvent $\mathcal{R}(E-i0)$ is analogous, and we omit it. We recall the resolvent estimate from [6, Theorem 2.1]

$$\|\mathcal{R}(E\pm i0)\|_{\mathcal{B}(L^2_\alpha, L^2_{-\alpha})} = \mathcal{O}\left(\frac{|\log h|}{h}\right), \text{ for } \alpha > \frac{1}{2}. \tag{3.1}$$

In particular, $K_{\mathcal{R}(E\pm i0)} \in \mathcal{S}_h'(\mathbb{R}^{2n})$ since the above estimate shows that $\mathcal{R}(E\pm i0)$ maps $\mathcal{S}_h(\mathbb{R}^n)$ to $\mathcal{S}_h'(\mathbb{R}^n)$ continuously. Let $\alpha^\pm \in C_0^\infty(T^*\mathbb{R}^n)$ be supported near $\rho_\pm$. We consider

$$\mathcal{I}(E) = \operatorname{Op}(\alpha^+)\mathcal{R}(E+i0)\operatorname{Op}(\alpha^-).$$

Proposition 3.1. *There exist $T_1 > 0$ and $\chi \in C_0^\infty(]0,+\infty[)$ such that*

$$\mathcal{I} = e^{-iT_1(P-E)/h}\operatorname{Op}(\alpha^+_{T_1})\mathcal{J}(E) \left(\frac{i}{h}\int \chi(t)e^{-it(P-E)/h}dt\right) e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-) + R, \tag{3.2}$$

where $\|R\|_{\mathcal{B}(L^2,L^2)} = \mathcal{O}(h^\infty)$, the symbol $\alpha^+_{T_1} \in S(\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$ is given by

$$\operatorname{Op}(\alpha^+_{T_1}) = e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^+)e^{-iT_1(P-E)/h},$$

and $\mathcal{J}(E) \in \mathcal{I}_h^{-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}\big(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_+ \times \Lambda_-'\big)$ *is given by* [8, Theorem 2.6] *and* [8, Remark 2.7].

It is possible to show a better estimate for the remainder term R. In fact, we have

$$\|\langle (x,hD)\rangle^N R \langle (x,hD)\rangle^N\|_{\mathcal{B}(L^2,L^2)} = \mathcal{O}(h^\infty),$$

for any $N \in \mathbb{R}$.

P r o o f. Since $(\rho_+,\rho_-) \in \widetilde{\Lambda_+ \times \Lambda_-}$, one can find $T_1 > 0$ such that $\rho_1 = \exp(-T_1 H_p)(\rho_+)$ belongs to $\Lambda_+ \setminus \widetilde{\Lambda_+}(\rho_-)$ and is as close as needed to $(0,0)$. We have

$$\begin{aligned} \operatorname{Op}(\alpha^+)\mathcal{R}(E+i0)\operatorname{Op}(\alpha^-) &= e^{-iT_1(P-E)/h}\operatorname{Op}(\alpha^+_{T_1})e^{iT_1(P-E)/h}\mathcal{R}(E+i0)\operatorname{Op}(\alpha^-) \\ &= e^{-iT_1(P-E)/h}\operatorname{Op}(\alpha^+_{T_1})\mathcal{R}(E+i0)e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-). \end{aligned} \tag{3.3}$$

We denote $\mathcal{K} = \mathcal{R}(E+i0)e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-)$. First we observe that

$$(P-E)\mathcal{K} = e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-) = 0 \text{ microlocally near } (0,0),$$

and we want to apply the results of [8] in order to compute $\mathcal{K}$ microlocally near $(0,0)$. Here, and in what follows, we say that an operator A is microlocally 0 near $V \subset T^*\mathbb{R}^n$ (respectively $\rho \in T^*\mathbb{R}^n$) when there exists $\beta \in S(1)$ with $\beta = 1$ in a neighborhood of V (respectively ρ) such that

$$\|\operatorname{Op}(\beta)A\|_{\mathcal{B}(L^2,L^2)} = \mathcal{O}(h^\infty).$$

To that end, we need to know $\mathcal{K}$ microlocally near $\mathcal{S} = \{(x,\xi) \in \Lambda_-;\ |x| = \varepsilon\}$ for some given $\varepsilon > 0$ small enough.

We choose $R > 0$ such that $e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-)$ is microlocally 0 outside of $B(0,R)$. One can easily see that there exist $T > 0$ and a neighborhood U of $\mathcal{S}$ in $T^*\mathbb{R}^n$, such that

$$\forall \rho \in U,\ \forall t \geq T, \quad \exp(-tH_p)(\rho) \notin B(0,R) \times \mathbb{R}^n.$$

Now we have

$$\mathcal{K} = \frac{i}{h}\int_0^T e^{-it(P-E)/h}e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-)\,dt + e^{-iT(P-E)/h}\mathcal{K},$$

and we claim that the second term of the right hand side vanishes microlocally in U. Indeed, as in [6, Section 5], one can show that $e^{iT(P-E)/h}\mathcal{K}$ is microlocally

0 in some incoming region $\Gamma_-(R_0,\sigma,d)$, where we use the standard notation

$$(3.4)\quad \Gamma_\pm(R,d,\sigma) = \big\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n;\ |x|>R,\ d^{-1}<|\xi|<d,\ \pm\cos(x,\xi)>\pm\sigma\big\},$$

for incoming and outgoing regions. Moreover we have

$$(P-E)e^{-iT(P-E)/h}\mathcal{K} = e^{-iT(P-E)/h}e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-)=0,$$

microlocally in $\cup_{t\geq 0}\exp(-tH_p)U$, and the claim follows by a usual propagation of singularities argument.

Thus we have, with the notation of [8, Section 2], microlocally near ρ_1,

$$\mathcal{R}(E+i0)e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-) = \mathcal{J}(E)\left(\frac{i}{h}\int_0^T e^{-it(P-E)/h}\,dt\right)e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-).$$

Finally, we notice that there exists $\delta>0$ such that, for any $\chi\in C_0^\infty(]0,T[)$ with $\chi=1$ on $[\delta,T-\delta]$, we have, microlocally near ρ_1,

$$\mathcal{R}(E+i0)e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-) = \mathcal{J}(E)\left(\frac{i}{h}\int \chi(t)e^{-it(P-E)/h}\,dt\right)e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-).$$

Indeed, by Egorov's theorem, $e^{-it(P-E)/h}e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-)$ is microlocally 0 in U for $t<\delta$ and $t>T-\delta$, provided δ is small enough. The proposition then follows directly from (3.3) with a remainder term $R=\mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2,L^2)$. □

Now it remains to show that all operators above compose as h-FIOs. We shall use several lemmas and we begin with the well-known approximation of the quantum propagator.

Lemma 3.2. *For any $t\in\mathbb{R}$, $e^{-it(P-E)/h}$ is a h-FIO of order 0 associated to the canonical relation*

$$\Lambda_t=\{(x,\xi,y,\eta)\in T^*\mathbb{R}^n\times T^*\mathbb{R}^n;\ (x,\xi)=\exp(tH_p)(y,\eta)\},$$

uniformly for t in a compact.

Proof. For t small enough, it is well-known that one can write the kernel K of the operator $e^{-it(P-E)/h}$ as

$$K=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}e^{-i(\varphi(t,x,\theta)-y\cdot\theta+tE)/h}a(t,x,\theta;h)\,d\theta,$$

modulo an operator $\mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2, L^2)$ uniformly for t in a compact. See *e.g.* Proposition IV-30 in Robert's book [26] or Theorem 10.9 in the book of Evans and Zworski [11]. Here φ is a non-degenerate phase function, which satisfies the eikonal equation

$$\varphi'_t + p(x, \varphi'_x) = 0, \tag{3.5}$$

and (see Proposition IV-14 *i)* of [26])

$$(x, \varphi'_x) = \exp(tH_p)(\varphi'_\theta, \theta). \tag{3.6}$$

This gives the lemma for t small enough. For other values of t, Robert uses the following trick. For some $k \in \mathbb{N}$ large enough, one can write

$$e^{-it(P-E)/h} = \prod_{j=1}^{k} e^{-it(P-E)/kh}.$$

It is then easy to see that these operators compose as h-FIOs, and that the result is associated to Λ_t and of order 0. $\square$

Lemma 3.3. *Let $\alpha \in C_0^\infty(T^*\mathbb{R}^n)$ be such that $H_p(x,\xi) \neq 0$ for all $(x,\xi) \in \operatorname{supp} \alpha \cap p^{-1}(E_0)$. There exists $\delta > 0$ such that, for any $\chi \in C_0^\infty(]0, \delta[)$, the operator $\mathcal{L} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by*

$$\mathcal{L} = \frac{i}{h} \int \chi(t) e^{-it(P-E)/h} dt \operatorname{Op}(\alpha),$$

is a h-FIO with compactly supported symbol of order $1/2$ associated to the canonical relation $\Lambda_{\alpha,\chi}(E_0)$ given by

$$\begin{aligned} \Lambda_{\alpha,\chi}(E_0) = \big\{ &(x,\xi,y,\eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n;\ p(y,\eta) = E_0, \\ &(y,\eta) \in \operatorname{supp}(\alpha) + B(0,\varepsilon), \textit{and } \exists t \in \operatorname{supp} \chi +]-\varepsilon, \varepsilon[, \\ &\qquad (x,\xi) = \exp(tH_p)(y,\eta) \big\}, \end{aligned}$$

for any $\varepsilon > 0$.

Remark 3.4. Note that $\Lambda_{\alpha,\chi}(E_0)$ is not a closed Lagragian submanifold. Nevertheless, this is not important here since the support of the symbol of the h-FIO does not reach the boundary of $\Lambda_{\alpha,\chi}(E_0)$ for any $\varepsilon > 0$. In particular, the parameter ε plays no role. It would be natural to write that the canonical relation of this h-FIO is $\Lambda(E_0)$ given by

$$\Lambda(E_0) = \{(x,\xi,y,\eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n; \\ p(x,\xi) = E_0,\ \exists t \in \mathbb{R},\ (x,\xi) = \exp(tH_p)(y,\eta)\}.$$

However, since the Hamiltonian flow vanishes at $(0,0)$, $\Lambda(E_0)$ is not a manifold. Of course, in the non trapping case, there is not such difficulty and $\Lambda_{\alpha,\chi}(E_0)$ can be replaced by $\Lambda(E_0)$.

Proof. As in the proof of Lemma 3.2, we have, modulo an operator $\mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2, L^2)$,

$$K_{\mathcal{L}} = \frac{i}{(2\pi)^n h^{n+1}} \iint \chi(t) e^{i(\varphi(t,x,\theta)-y\cdot\theta+tE_0)/h} e^{itE_1} b(t,x,y,\theta;h)\, dt\, d\theta,$$

and we consider (t,θ) as phase variables. Such a formula can be obtained by usual WKB construction (see *e.g.* Théorème 2 of [7]). Here, $e^{itE_1}\chi(t)b(t,x,y,\theta,h) \sim \sum_j b_j(t,x,y,\theta)h^j$ is a classical symbol of order 0 and has compact support in t,x,y,θ with $\Pi_{y,\theta}\operatorname{supp}(e^{itE_1}\chi b) \subset \operatorname{supp}(\alpha)$. We have to show that the function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$\Phi(x,y,(t,\theta)) = \varphi(t,x,\theta) - y\cdot\theta + tE_0,$$

is a non-degenerate phase function. We denote by

$$\begin{aligned} C_\Phi &= \{(x,y,t,\theta) \in \operatorname{supp}(\chi b);\ \Phi'_t(t,\theta,x,y) = 0,\ \Phi'_\theta(t,\theta,x,y) = 0\} \\ &= \{(x,y,t,\theta) \in \operatorname{supp}(\chi b);\ \varphi'_t + E_0 = 0,\ \varphi'_\theta = y\}, \end{aligned}$$

the critical set of the phase Φ intersected with the support of the symbol. We have to show that at any point (x,y,t,θ) of C_Φ, the matrix

$$\begin{pmatrix} d\Phi'_t(x,y,t,\theta) \\ d\Phi'_\theta(x,y,t,\theta) \end{pmatrix} = \begin{pmatrix} \varphi''_{t,t} & \varphi''_{t,\theta} & \varphi''_{t,x} & 0 \\ \varphi''_{\theta,t} & \varphi''_{\theta,\theta} & \varphi''_{\theta,x} & -\mathrm{Id} \end{pmatrix},$$

is of maximal rank. The bottom n rows are clearly independent and it is enough to prove that the first line does not vanish on the compact C_Φ. Assume that the first line vanishes at some point of C_Φ. At this point, $(y,\theta) \in \operatorname{supp}(\alpha)$, $\varphi'_t + E_0 = 0$ and $\varphi'_\theta = y$. Differentiating (3.6) with respect to t, we obtain

$$(0, \varphi''_{t,x}) = H_p\big(\exp(tH_p)(\varphi'_\theta,\theta)\big) + d_{(\varphi'_\theta,\theta)} \exp(tH_p)(\varphi''_{t,\theta}, 0),$$

and then

$$H_p\big(\exp(tH_p)(y,\theta)\big) = (0,0).$$

Since $\big(d_{(x,\xi)}\exp(tH_p)\big)(H_p(x,\xi)) = H_p\big(\exp(tH_p)(x,\xi)\big)$, we deduce

$$H_p(y,\theta) = (0,0). \tag{3.7}$$

Moreover, from $\varphi'_\theta = y$, (3.6), the eikonal equation (3.5) and $\varphi'_t + E_0 = 0$ we have

$$p(y,\theta) = p(\varphi'_\theta,\theta) = p(x,\varphi'_x) = -\varphi'_t = E_0.$$

But since $(y,\theta) \in \operatorname{supp}(\alpha)$ and H_p does not vanish on $\operatorname{supp}\alpha \cap p^{-1}(E_0)$, this contradicts (3.7). Therefore, Φ is a non-degenerate phase function and $\mathcal{L}$ is an h-FIO with compactly supported symbol associated to

$$\begin{aligned}\Lambda_\Phi &= \{(x,\Phi'_x(x,y,t,\theta),y,-\Phi'_y(x,y,t,\theta));\ (x,y,t,\theta)\in C_\Phi\}\\ &= \{(x,\varphi'_x(t,x,\theta),y,\theta);\ \varphi'_t + E_0 = 0,\ \varphi'_\theta = y,\\ &\qquad\qquad (x,y,t,\theta)\in \operatorname{supp}(\chi b)\} \Subset \Lambda_{\alpha,\chi}(E_0),\end{aligned}$$

thanks to the equations (3.5) and (3.7). From Definition A.4, we obtain that the order of this h-FIO is $1/2$. □

We are now able to prove the following

Lemma 3.5. *Let $\alpha \in C_0^\infty(T^*\mathbb{R}^n)$ be such that $H_p(x,\xi) \neq 0$ for all $(x,\xi) \in \operatorname{supp}\alpha \cap p^{-1}(E_0)$. For any $\chi \in C_0^\infty(]0,+\infty[)$, the operator $\mathcal{L} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by*

$$\mathcal{L} = \frac{i}{h}\int \chi(t) e^{-it(P-E)/h} dt\, \operatorname{Op}(\alpha),$$

is a h-FIO with compactly supported symbol of order $1/2$ associated with the canonical relation $\Lambda_{\alpha,\chi}(E_0)$ given by

$$\begin{aligned}\Lambda_{\alpha,\chi}(E_0) = \{&(x,\xi,y,\eta)\in T^*\mathbb{R}^n\times T^*\mathbb{R}^n;\ p(y,\eta)=E_0,\\ &(y,\eta)\in \operatorname{supp}(\alpha)+B(0,\varepsilon), \text{ and } \exists t\in \operatorname{supp}\chi +]-\varepsilon,\varepsilon[,\\ &\qquad (x,\xi) = \exp(tH_p)(y,\eta)\},\end{aligned}$$

for any $\varepsilon > 0$.

Remark 3.4 still applies here and one can, formally, replace $\Lambda_{\alpha,\chi}(E_0)$ by $\Lambda(E_0)$.

P r o o f. For $\delta > 0$ small enough so that Lemma 3.3 applies, we can find $\widetilde{\chi} \in C_0^\infty(]0, \delta[)$ so that, for some $\nu > 0$,

$$\sum_{k \in \mathbb{N}} \widetilde{\chi}(y - \nu k) = 1.$$

We have, for some $N \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{L} &= \frac{i}{h} \sum_{k \in \mathbb{N}} \int \chi(t) \widetilde{\chi}(t - \nu k) e^{-it(P-E)/h} dt \operatorname{Op}(\alpha) \\
&= \frac{i}{h} \sum_{k=0}^{N} \int \chi(t) \widetilde{\chi}(t - \nu k) e^{-it(P-E)/h} dt \operatorname{Op}(\alpha) \\
&= \frac{i}{h} \sum_{k=0}^{N} e^{-i\nu k(P-E)/h} \circ \int \chi(t + \nu k) \widetilde{\chi}(t) e^{-it(P-E)/h} dt \operatorname{Op}(\alpha).
\end{aligned}$$

Using that the operator in Lemma 3.3 is a h-FIO with compactly supported symbol and the Egorov theorem, we can find $\beta, \gamma \in C_0^\infty(T^*\mathbb{R}^n)$ such that

$$\mathcal{L} = \frac{i}{h} \sum_{k=0}^{N} \operatorname{Op}(\beta) e^{-i\nu k(P-E)/h} \operatorname{Op}(\gamma) \circ \int \chi(t + \nu k) \widetilde{\chi}(t) e^{-it(P-E)/h} dt \operatorname{Op}(\alpha) + R,$$

where $R = \mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2, L^2)$. From Lemma 3.3,

$$\operatorname{Op}(\beta) e^{-i\nu k(P-E)/h} \operatorname{Op}(\gamma) \in \mathcal{I}_h^0\big(\mathbb{R}^n \times \mathbb{R}^n, {\Lambda_k}'\big)$$

with compactly supported symbol.

To finish the proof, it is enough to compose this operator with the h-FIOs described in Lemma 3.2. Since Λ_k is given by a canonical transformation, $\Lambda_k \times \Lambda_{\alpha, \chi(t+\nu k)\widetilde{\chi}(t)}(E_0)$ intersects $T^*\mathbb{R}^n \times \operatorname{diag}(T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \times T^*\mathbb{R}^n$ transversely (cleanly with excess 0). Then, using Theorem A.7, they compose as h-FIOs with compactly supported symbol of order 1/2 with canonical relation

$$\Lambda_k \circ \Lambda_{\alpha, \chi(t+\nu k)\widetilde{\chi}(t)}(E_0) = \Lambda_{\alpha, \chi(t)\widetilde{\chi}(t-\nu k)}(E_0).$$

Summing over k, we obtain the lemma. □

P r o o f o f T h e o r e m 2.2. From Proposition 3.1, to calculate $\mathcal{I}(E)$, it is enough to compose the h-FIOs which appear in (3.2). We will use Theorem

A.7 for that. As in the end of the proof of Lemma 3.5, we have from Lemma 3.2 and Lemma 3.5,

$$(3.8)\qquad \Big(\frac{i}{h}\int \chi(t)e^{-it(P-E)/h}dt\Big)\, e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-) \in \mathcal{I}_h^{\frac{1}{2}}\big(\mathbb{R}^n\times\mathbb{R}^n, \Lambda_{\alpha\circ\exp(T_1H_p),\chi}(E_0)'\big),$$

with compactly supported symbol.

We recall that, from [8, Remark 2.7],

$$(3.9)\qquad \mathcal{J}(E)\in \mathcal{I}_h^{-\frac{\sum_{j=1}^n\lambda_j}{2\lambda_1}}\big(\mathbb{R}^n\times\mathbb{R}^n, \Lambda_+\times{\Lambda_-}'\big),$$

with compactly supported symbol. The manifold $(\Lambda_+\times\Lambda_-)\times\Lambda_{\alpha\circ\exp(T_1H_p),\chi}(E_0)$ intersects $T^*\mathbb{R}^n\times\operatorname{diag}(T^*\mathbb{R}^n\times T^*\mathbb{R}^n)\times T^*\mathbb{R}^n$ cleanly with excess 1 and

$$(\Lambda_+\times\Lambda_-)\circ\Lambda_{\alpha\circ\exp(T_1H_p),\chi}(E_0)\subset\Lambda_+\times\Lambda_-.$$

Then, the composition rules for the h-FIOs in (3.8) and (3.9) implies that

$$(3.10)\qquad \mathcal{J}(E)\Big(\frac{i}{h}\int \chi(t)e^{-it(P-E)/h}dt\Big)\, e^{iT_1(P-E)/h}\operatorname{Op}(\alpha^-) \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n\lambda_j}{2\lambda_1}}\big(\mathbb{R}^n\times\mathbb{R}^n, \Lambda_+\times{\Lambda_-}'\big),$$

with compactly supported symbol.

Finally, from Lemma 3.2,

$$(3.11)\qquad e^{-iT_1(P-E)/h}\operatorname{Op}(\alpha_{T_1}^+)\in\mathcal{I}_h^0\big(\mathbb{R}^n\times\mathbb{R}^n,{\Lambda_{T_1}}'\big),$$

with a compactly supported symbol. Since Λ_{T_1} is given by a canonical transformation, the intersection between $\Lambda_{T_1}\times(\Lambda_+\times\Lambda_-)$ and $T^*\mathbb{R}^n\times\operatorname{diag}(T^*\mathbb{R}^n\times T^*\mathbb{R}^n)\times T^*\mathbb{R}^n$ is clean with excess 0. Moreover

$$\Lambda_{T_1}\circ(\Lambda_+\times\Lambda_-)\subset\Lambda_+\times\Lambda_-.$$

Then, (3.2) and the composition of the h-FIOs appearing in (3.10) and (3.11) gives

$$(3.12)\qquad \operatorname{Op}(\alpha^+)\mathcal{R}(E+i0)\operatorname{Op}(\alpha^-)\in\mathcal{I}_h^{1-\frac{\sum_{j=1}^n\lambda_j}{2\lambda_1}}\big(\mathbb{R}^n\times\mathbb{R}^n,\Lambda_+\times{\Lambda_-}'\big).$$

□

3.2. Microlocal representation of the spectral function. We give here the representation of the spectral function as an oscillatory integral operator microlocally near any point $(\rho_+, \rho_-) \in \widetilde{\Lambda_+ \times \Lambda_-}$. The oscillatory integral representation near points in $\widetilde{\Lambda_- \times \Lambda_+}$ is analogous.

Theorem 3.6. *Let $(\rho_+, \rho_-) \in \widetilde{\Lambda_+ \times \Lambda_-}$. Then there exist $m \in \mathbb{N}$, a non-degenerate phase function $\Psi \in C^\infty(\mathbb{R}^{2n+m})$ and a symbol $b \in S_{2n+m}^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}+\frac{n}{2}+\frac{m}{2}}(1)$ such that, microlocally near (ρ_+, ρ_-),*

$$e_E(x, y; h) = \int_{\mathbb{R}^m} e^{i\Psi(x,y,\tau)/h} b(x, y, \tau; h)\, d\tau.$$

Furthermore, if $(\rho_+, \rho_-) \in \widetilde{\Lambda_+ \times \Lambda_-}$ and the projections $\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ are diffeomorphisms when restricted to some neighborhood of $\rho_\pm$ in $\Lambda_\pm$, then there exists a symbol $b \in S_{2n}^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}+\frac{n}{2}}(1)$ such that, microlocally near (ρ_+, ρ_-),

$$e_E(x, y; h) = e^{i(S_+(x)+S_-(y))/h} b(x, y; h),$$

where

$$S_\pm(z) = \int_{\gamma_\pm(z)} \frac{1}{2}|\xi_\pm(t)|^2 + E_0 - V(x_\pm(t))\, dt,$$

are the actions over the Hamiltonian half-trajectories $\gamma_\pm(z) = (x_\pm, \xi_\pm)$ which start at $\pi_{|\Lambda_\pm}^{-1}(z)$ and approach $(0, 0)$ as $t \to \mp\infty$.

Proof. The first part of the theorem follows from [2, Theorem 1] and Theorem 2.2. Assume now that $\pi_{|\Lambda_\pm}$ is a diffeomorphism in a neighborhood of $\rho_\pm$. We will now show that

$$\Lambda_\pm = \big\{(z, \pm\partial_z S_\pm(z));\ z \text{ near } \pi(\rho_\pm)\big\}, \tag{3.13}$$

locally near $\rho_\pm$. We only prove (3.13) for Λ_+ since the manifold Λ_- can be treated by the same way. Let

$$(x_+(t, z), \xi_+(t, z)) = \exp(t\mathrm{H}_p)\big(\pi_{|\Lambda_+}^{-1}(z)\big).$$

From the definition of the Hamiltonian vector field, we have

$$\begin{aligned}
\partial_t\big(\xi_+(t, z)\partial_z(x_+(t, z))\big) &= \xi_+(t, z)\partial_z(\xi_+(t, z)) - (\partial_x V)(x_+(t, z))\partial_z(x_+(t, z)) \\
&= \frac{1}{2}\partial_z\big(|\xi_+(t, z)|^2\big) - \partial_z\big(V(x_+(t, z))\big) \\
&= \partial_z\Big(\frac{1}{2}|\xi_+(t, z)|^2 + E_0 - V(x_+(t, z))\Big). \qquad (3.14)
\end{aligned}$$

Moreover, as $t \to -\infty$, we have $\xi_+(t,z) \to 0$ and

$$\partial_z(x_+(t,z)) = d\Pi_x \circ d\exp(tH_p)\big(\partial_z x_+(0,z), \partial_z \xi_+(0,z)\big) \longrightarrow 0,$$

since $(x_+(0,z), \xi_+(0,z)) \in \Lambda_+$ for all z and 0 is a unstable node of H_p restricted to Λ_+. Using $x_+(0,z) = z$, we obtain

$$\begin{aligned}\partial_z S_\pm(z) = &\int_{-\infty}^0 \partial_z\Big(\frac{1}{2}|\xi_+(t,z)|^2 + E_0 - V(x_+(t,z))\Big)ds \\ &= \xi_+(0,z)\partial_z(x_+(0,z)) = \xi_+(0,z).\end{aligned}$$

Since $\Lambda_+ = \{(z, \xi_+(0,z));\ z \text{ near } \pi(\rho_\pm)\}$ locally near ρ_+, we get (3.13). Then the second part of the theorem follows again from [2, Theorem 1] and Theorem 2.2. $\square$

Remark 3.7. From [8, Section 2.2] we have that there exists a neighborhood $\Omega \subset T^*\mathbb{R}^n$ of $(0,0)$ such that the projection $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ restricted to $\Omega \cap \Lambda_\pm$ is a diffeomorphism.

4. The scattering matrix.

4.1. Representation of the scattering matrix. Here we review the representation of the short range scattering matrix which we shall use in the proof of Theorem 2.4. The construction is close to the one used by Robert and Tamura [28] and constitutes a semiclassical adaptation of the representation of the short range amplitude originally established by Isozaki and Kitada [18]. Their starting point is a set of WKB parametrices for the wave operators given in (1.4).

For $R_0 \gg 0$, $1 < d_4 < d_3 < d_2 < d_1 < d_0$, and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 < 1$ Robert and Tamura construct phase functions $\Phi_\pm$ and symbols $(a_{\pm j})_{j=0}^\infty$ and $(b_{\pm j})_{j=0}^\infty$ such that:

i) $\Phi_\pm \in C^\infty(T^*\mathbb{R}^n)$ solve the eikonal equation

$$\frac{1}{2}|\nabla_x \Phi_\pm(x,\xi)|^2 + V(x) = \frac{1}{2}\xi^2 \tag{4.1}$$

for $(x,\xi) \in \Gamma_\pm(R_0, d_0, \pm\sigma_0)$ respectively (see (3.4) for the definition of these sets).

ii) Let $A_m(\Omega)$ be the class of symbols a such that $(x,\xi) \mapsto a(x,\xi;h)$ belongs to $C^\infty(\Omega)$ and, for any $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$ and $L > 0$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha,\beta}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{-L},$$

for all $(x, \xi) \in \Omega$. We have, from Proposition 2.4 of [17],

$$\Phi_{\pm}(x,\xi) - \langle x, \xi \rangle \in A_{1-\rho}\left(\Gamma_{\pm}(R_0, d_0, \pm\sigma_0)\right). \tag{4.2}$$

iii) For all $(x, \xi) \in T^*\mathbb{R}^n$

$$\left| \frac{\partial^2 \Phi_{\pm}}{\partial_{x_j} \partial_{\xi_k}}(x,\xi) - \delta_{jk} \right| < \varepsilon(R_0),$$

where δ_{jk} is the Kronecker delta and $\varepsilon(R_0) \to 0$ as $R_0 \to \infty$.

iv) $(a_{\pm j})_j$ and $(b_{\pm j})_j$ are determined inductively as solutions to certain transport equations and satisfy

$$a_{\pm j} \in A_{-j}(\Gamma_{\pm}(3R_0, d_1, \pm\sigma_1)), \quad \operatorname{supp} a_{\pm j} \subset \Gamma_{\pm}(3R_0, d_1, \pm\sigma_1),$$

$$b_{\pm j} \in A_{-j}(\Gamma_{\pm}(5R_0, d_3, \pm\sigma_4)), \quad \operatorname{supp} b_{\pm j} \subset \Gamma_{\pm}(5R_0, d_3, \pm\sigma_4).$$

Using the Borel process, we can find two symbols $a_{\pm} \in A_0(\Gamma_{\pm}(3R_0, d_1, \pm\sigma_1))$ and $b_{\pm} \in A_0(\Gamma_{\pm}(5R_0, d_3, \pm\sigma_4))$ such that $a_{\pm} \sim \sum_{j=0}^{\infty} h^j a_{\pm j}$ and $b_{\pm} \sim \sum_{j=0}^{\infty} h^j b_{\pm j}$.

For a symbol c and a phase function φ, we denote by $I_h(c, \varphi)$ the oscillatory integral

$$I_h(c,\varphi) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{i(\varphi(x,\xi) - \langle y, \xi \rangle)/h} c(x, \xi; h)\, d\xi$$

and let

$$K_{\pm a}(h) = P(h) I_h(a_{\pm}, \Phi_{\pm}) - I_h(a_{\pm}, \Phi_{\pm}) P_0(h)$$
$$K_{\pm b}(h) = P(h) I_h(b_{\pm}, \Phi_{\pm}) - I_h(b_{\pm}, \Phi_{\pm}) P_0(h).$$

The scattering matrix, or more precisely the operator $\mathcal{T}(E,h)$ is then given for $E \in \left] \frac{2}{d_4^2}, \frac{d_4^2}{2} \right[$ by (see [18, Theorem 3.3])

$$\mathcal{T}(E,h) = T_{+1}(E,h) + T_{-1}(E,h) - T_2(E,h), \tag{4.3}$$

where

$$T_{\pm 1}(E,h) = F_0(E,h) I_h(a_{\pm}, \Phi_{\pm})^* K_{\pm b}(h) F_0^{-1}(E,h)$$

and, with $F_0(E,h)$ given in (1.5),

$$T_2(E,h) = F_0(E,h) K_{+a}^*(h) \mathcal{R}(E + i0, h) \left(K_{+b}(h) + K_{-b}(h)\right) F_0^*(E,h). \tag{4.4}$$

4.2. Proof of Theorem 2.4. Since $\mathcal{S}(E,h)$ is a unitary operator on $L^2(\mathbb{S}^{n-1})$, we have, by [5, Lemma 1], that its kernel $K_{\mathcal{S}(E,h)} \in \mathcal{S}'_h(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ and therefore $K_{\mathcal{T}(E,h)} \in \mathcal{S}'_h(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$.

Since we are working away from the diagonal in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ we can use integration by parts, as in [28] and [20], to obtain

$$K_{T_{\pm 1}(E,h)} = \mathcal{O}_{C^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\setminus \operatorname{diag}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}))}(h^\infty).$$

Therefore

$$WF_h^f\big(K_{T_{\pm 1}(E,h)}{}_{|_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\setminus \operatorname{diag}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}}\big) = \emptyset. \tag{4.5}$$

We now observe that the proof of [28, Lemma 2.1] depends only on the estimate (3.1) and the support properties of the symbols $a_\pm$ and $b_\pm$, and by the same method of proof, we obtain the following strengthened version of [28, Lemma 2.1].

Lemma 4.1. *Suppose (A1) and (A2). For $\gamma \gg 1$,*

i) $\|K^*_{+a}(h)\mathcal{R}(E+i0)K_{+b}(h)\|_{\mathcal{B}(L^2_{-\gamma},L^2_\gamma)} = \mathcal{O}(h^\infty)$,

ii) $\|K^*_{+a}(h)\mathcal{R}(E+i0)\,(1-\chi_b)\,K_{+b}(h)\|_{\mathcal{B}(L^2_{-\gamma},L^2_\gamma)} = \mathcal{O}(h^\infty)$,

iii) $\|\left((1-\chi_a)\,K_{+a}(h)\right)^*\mathcal{R}(E+i0)\chi_b K_{-b}(h)\|_{\mathcal{B}(L^2_{-\gamma},L^2_\gamma)} = \mathcal{O}(h^\infty)$.

From (4.5), Lemma 4.1, and [5, Equation (10)] we then conclude, as in [28, Corollary, page 168], that

$$WF_h^f(\chi(K_{\mathcal{S}(E,h)} - c_1\,G)) = \emptyset, \tag{4.6}$$

for every $\chi \in C_0^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \operatorname{diag}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}))$, where

(4.7)

$$G(\theta,\omega;E,h) = \langle \mathcal{R}(E+i0)e^{i\Phi_-(y,\sqrt{2E_0}\omega)/h}g_-(y,\omega;h),\ e^{i\Phi_+(x,\sqrt{2E_0}\theta)/h}g_+(x,\theta;h)\rangle,$$

$$g_+(x,\theta;h) = e^{-i\Phi_+(x,\sqrt{2E_0}\theta)/h}[\chi_a,P_0(h)]a_+\big(x,\sqrt{2E}\theta;h\big)e^{i\Phi_+(x,\sqrt{2E}\theta)/h},$$

$$g_-(y,\omega;h) = e^{-i\Phi_-(y,\sqrt{2E_0}\omega)/h}[\chi_b,P_0(h)]b_-\big(y,\sqrt{2E}\omega;h\big)e^{i\Phi_-(y,\sqrt{2E}\omega)/h},$$

and

$$c_1 = c_1(n,E,h) = -2i\pi(2E)^{\frac{n}{2}-1}(2\pi h)^{-n}.$$

Here $\chi_a(x)$ and $\chi_b(y)$ are $C_0^\infty(\mathbb{R}^n)$ functions with value 1 in a large disc. In particular, the symbols $g_+(x,\theta;h)$, $g_-(y,\omega;h) \in S^{-1}(1)$ have compact support (uniformly with respect to h). Notice that we have used the fact that $E - E_0 = E_1 h$.

From (4.7), one can see that $G(\theta, \omega; E, h)$ is the kernel of the operator

$$\mathcal{G} = \mathcal{M}_+^* \mathcal{R}(E + i0)\mathcal{M}_-, \tag{4.8}$$

where $\mathcal{M}_\pm : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)$ are given by

$$K_{\mathcal{M}_+}(x, \theta; h) = e^{i\Phi_+(x, \sqrt{2E_0}\theta)/h} g_+(x, \theta; h),$$
$$K_{\mathcal{M}_-}(x, \omega; h) = e^{i\Phi_-(y, \sqrt{2E_0}\omega)/h} g_-(y, \omega; h).$$

The operator $\mathcal{M}_+$ can be view as an h-FIO

$$\mathcal{M}_+ \in \mathcal{I}_h^{-\frac{2n+3}{4}} \big(\mathbb{R}^n \times \mathbb{S}^{n-1}, {C_+}'\big), \tag{4.9}$$

with compactly supported symbol (and no phase variable). The canonical relation C_+ is given by

$$\begin{aligned} C_+ = \big\{ (x, \xi, \theta, \sqrt{2E_0} z_+);\ & \xi = \partial_x \Phi_+(x, \sqrt{2E_0}\theta), \\ \sqrt{2E_0} z_+ = -\partial_\theta \Phi_+(x, \sqrt{2E_0}\theta),\ & (x, \theta) \in \operatorname{supp}(g_+) + B(0, \varepsilon) \big\}, \end{aligned} \tag{4.10}$$

for any $\varepsilon > 0$ (see Remark 3.4). Notice that ∂_θ denotes the derivative on $\mathbb{S}^{n-1}$. Now we calculate more precisely C_+.

Lemma 4.2. *We have*

$$\begin{aligned} C_+ = \big\{ (x, \xi, \theta, -\sqrt{2E_0} z_+);\ \exists t \in \mathbb{R},\ (x, \xi) = \gamma_+(t, z_+, \theta, E_0), \\ (x, \theta) \in \operatorname{supp}(g_+) + B(0, \varepsilon) \big\}, \end{aligned}$$

where $\gamma_+(t, z, \alpha, E)$ *is defined in* (1.13).

P r o o f. We set

$$\Psi_+(x, \theta) = \Phi_+(x, \sqrt{2E_0}\theta).$$

Let x be such that $(x, \sqrt{2E_0}\theta) \in \Gamma_+(3R_0, d_1, \sigma_1)$. We denote

$$(y(t, x, \theta), \eta(t, x, \theta)) = \exp(tH_p)(x, \partial_x \Psi_+(x, \theta)). \tag{4.11}$$

Remark that $(y(t, x, \theta), \sqrt{2E_0}\theta)$ stays in $\Gamma_\pm(R_0, d_0, \sigma_0)$ for all $t \geq 0$ and then the following limits exist

$$\begin{cases} \lim_{t \to +\infty} \eta(t, x, \theta) = \eta_\infty \in \sqrt{2E_0}\mathbb{S}^{n-1} \\ \lim_{t \to +\infty} y(t, x, \theta) - t\eta_\infty = y_\infty. \end{cases} \tag{4.12}$$

By (4.1) we have

$$\frac{1}{2}|\partial_x \Psi_\pm(x,\theta)|^2 + V(x) = \frac{1}{2}E_0. \tag{4.13}$$

Differentiating with respect to x we obtain

$$(\partial^2_{x,x}\Psi_+)(x,\theta)(\partial_x\Psi_+)(x,\theta) + (\partial_x V)(x) = 0.$$

Therefore, the Hamiltonian flow H_p is tangent to $\{(x, \partial_x\Psi_+(x,\theta));\ x \in \mathbb{R}^n\}$ and then

$$\eta(t,x,\theta) = (\partial_x\Psi_+)(y(t,x,\theta),\theta), \tag{4.14}$$

for all $t \geq 0$. In particular, from (4.2),

$$\begin{aligned} \eta_\infty = & \lim_{t\to+\infty}\eta(t,x,\theta) = \lim_{t\to+\infty}\sqrt{2E_0}\theta + \mathcal{O}\big(|y(t,x,\theta)|^{-\rho}\big) \\ = & \sqrt{2E_0}\theta. \end{aligned} \tag{4.15}$$

On the other hand, differentiating (4.1) with respect to θ, we get

$$(\partial^2_{x,\theta}\Psi_+)(x,\theta)(\partial_x\Psi_+)(x,\theta) = 0. \tag{4.16}$$

Using (4.14) and (4.16), we obtain

$$\begin{aligned} \partial_t(\partial_\theta\Psi_+)(y(t,x,\theta),\theta) &= (\partial^2_{x,\theta}\Psi_+)(y(t,x,\theta),\theta)\partial_t y(t,x,\theta) \\ &= (\partial^2_{x,\theta}\Psi_+)(y(t,x,\theta),\theta)\eta(t,x,\theta) \\ &= (\partial^2_{x,\theta}\Psi_+)(y(t,x,\theta),\theta)(\partial_x\Psi_+)(y(t,x,\theta),\theta) \\ &= 0. \end{aligned}$$

Now (4.2) and (4.12) yield

$$\begin{aligned} (\partial_\theta\Psi_+)(x,\theta) &= \lim_{t\to+\infty}(\partial_\theta\Psi_+)(y(t,x,\theta),\theta) \\ &= \lim_{t\to+\infty}\sqrt{2E_0}\Pi_{\theta^\perp}y(t,x,\theta) + \mathcal{O}\big(|y(t,x,\theta)|^{1-\rho}\big) \\ &= \sqrt{2E_0}\Pi_{\theta^\perp}y_\infty, \end{aligned} \tag{4.17}$$

where $\Pi_{\theta^\perp}$ is the orthogonal projection on the hyperplane orthogonal to θ:

$$\Pi_{\theta^\perp}x = x - \langle x,\theta\rangle.$$

Finally, let $(x, \xi, \theta, -\sqrt{2E_0} z_+) \in C_+$. The asymptotic momentum and position (4.12) of the Hamiltonian curve (4.11) were calculated in (4.15) and (4.17). Then, there exist $t \in \mathbb{R}$ such that

$$(x, \nabla_x \Phi_+(x, \sqrt{2E_0}\theta)) = \gamma_+(t, \theta, \Pi_{\theta^\perp} y_\infty, E_0),$$

and, from (4.10), we conclude

$$(x, \xi) = \gamma_+(t, \theta, z_+, E_0). \qquad \square$$

The same way,

$$\mathcal{M}_- \in \mathcal{I}_h^{-\frac{2n+3}{4}}\big(\mathbb{R}^n \times \mathbb{S}^{n-1}, C_-{}'\big), \tag{4.18}$$

with compactly supported symbol (and no phase variable). The canonical relation C_- is given by

$$C_- = \big\{(y, \eta, \omega, -\sqrt{2E_0} z_-);\ \exists t \in \mathbb{R},\ (y, \eta) = \gamma_-(t, z_-, \omega, E_0), \\ (y, \omega) \in \operatorname{supp}(g_-) + B(0, \varepsilon)\big\}.$$

Let now $(\theta^0, z_+^0, \omega^0, z_-^0) \in \widetilde{\Lambda_+^\infty \times \Lambda_-^\infty}$ be as in Theorem 2.4. The reader may notice that we use here shorter notation. Let $\beta_\pm \in C_0^\infty(T^*\mathbb{S}^{n-1})$ with β_+ (resp. β_-) be supported in a small neighborhood of (θ^0, z_+^0) (resp. (ω^0, z_-^0)) and equal to 1 near (θ^0, z_+^0) (resp. (ω^0, z_-^0)). From (4.6) and (4.8), we have

$$\operatorname{Op}(\beta_+)\mathcal{S}(E, h)\operatorname{Op}(\beta_-) = c_1 \operatorname{Op}(\beta_+)\mathcal{M}_+^*\mathcal{R}(E + i0)\mathcal{M}_- \operatorname{Op}(\beta_-) + R,$$

where $R = \mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2, L^2)$. Let now $\alpha_\pm \in C_0^\infty(T^*\mathbb{R}^n)$ supported near

$$N_\pm = C_\pm \circ \operatorname{supp}(\beta_\pm) \bigcap \big(\Pi_x \operatorname{supp} g_\pm \times \mathbb{R}^n\big),$$

and equal to 1 near this set. Then, the composition rules for h-FIOs implies

$$\begin{aligned} &\operatorname{Op}(\beta_+)\mathcal{S}(E, h)\operatorname{Op}(\beta_-) \\ &\qquad = c_1 \operatorname{Op}(\beta_+)\mathcal{M}_+^* \operatorname{Op}(\alpha_+)\mathcal{R}(E + i0)\operatorname{Op}(\alpha_-)\mathcal{M}_- \operatorname{Op}(\beta_-) + R, \end{aligned} \tag{4.19}$$

where $R = \mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2, L^2)$.

Note that N_+ is arbitrary close to

$$N_+^0 = \{\gamma_+(t, \theta^0, z_+^0, E_0);\ t \in \mathbb{R}\} \bigcap \big(\Pi_x \operatorname{supp} g_+ \times \mathbb{R}^n\big),$$

and N_- is arbitrary close to

$$N_-^0 = \{\gamma_-(t, \omega^0, z_-^0, E_0);\ t \in \mathbb{R}\} \bigcap \big(\Pi_x \operatorname{supp} g_- \times \mathbb{R}^n\big).$$

Every $(\rho_+, \rho_-) \in N_+^0 \times N_-^0$ is in $\widetilde{\Lambda_+ \times \Lambda_-}$ because $(\theta^0, z_+^0, \omega^0, z_-^0) \in \widetilde{\Lambda_+^\infty \times \Lambda_-^\infty}$. Up to a finite summation in (ρ_+, ρ_-) since $\Pi_x \operatorname{supp} g_\pm$ is compact, we can assume that $\alpha_\pm$ is localized in a small neighborhood of such a point $\rho_\pm$. To prove the theorem, we will compose the h-FIOs appearing in the formula (4.19).

The manifold $(\Lambda_+ \times \Lambda_-) \times C_-$ intersects $T^*\mathbb{R}^n \times \operatorname{diag}(T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \times T^*\mathbb{S}^{n-1}$ cleanly with excess 1 and

$$(\Lambda_+ \times \Lambda_-) \circ C_- \subset \Lambda_+ \times \Lambda_-^\infty.$$

Then the composition rules between the h-FIOs

$$\operatorname{Op}(\alpha_+) \mathcal{R}(E + i0) \operatorname{Op}(\alpha_-) \in \mathcal{I}_h^{1 - \frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \big(\mathbb{R}^n \times \mathbb{R}^n, \Lambda_+ \times \Lambda_-{}'\big),$$

with compactly supported symbol (see Theorem 2.2), and

$$\mathcal{M}_- \operatorname{Op}(\beta_-) \in \mathcal{I}_h^{-\frac{2n+3}{4}} \big(\mathbb{R}^n \times \mathbb{S}^{n-1}, C_-{}'\big),$$

with compactly supported symbol (see (4.18)), gives

$$\operatorname{Op}(\alpha_+) \mathcal{R}(E + i0) \operatorname{Op}(\alpha_-) \mathcal{M}_- \operatorname{Op}(\beta_-) \in \mathcal{I}_h^{\frac{3-2n}{4} - \frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \big(\mathbb{R}^n \times \mathbb{S}^{n-1}, \Lambda_+ \times \Lambda_-^\infty{}'\big), \tag{4.20}$$

with compactly supported symbol.

But now, taking the adjoint of (4.9), we obtain

$$\operatorname{Op}(\beta_+) \mathcal{M}_+^* \in \mathcal{I}_h^{-\frac{2n+3}{4}} \big(\mathbb{S}^{n-1} \times \mathbb{R}^n, C_+^{-1}{}'\big), \tag{4.21}$$

with compactly supported symbol. Here

$$C_+^{-1} = \big\{(\theta, z, x, \xi);\ (x, \xi, \theta, z) \in C_+\big\}.$$

The manifold $C_+^{-1} \times (\Lambda_+ \times \Lambda_-^\infty)$ intersects $T^*\mathbb{S}^{n-1} \times \operatorname{diag}(T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \times T^*\mathbb{S}^{n-1}$ cleanly with excess 1 and

$$C_+^{-1} \circ (\Lambda_+ \times \Lambda_-^\infty) \subset \Lambda_+^\infty \times \Lambda_-^\infty.$$

Then (4.19) and the composition rules between the h-FIOs given in (4.20) and (4.21) imply that

$$\operatorname{Op}(\beta_+)\mathcal{S}(E,h)\operatorname{Op}(\beta_-) \in \mathcal{I}_h^{\frac{1}{2}-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}\big(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}, \Lambda_+^\infty \times \Lambda_-^{\infty\prime}\big), \tag{4.22}$$

and this statement is Theorem 2.4.

4.3. Proof of Theorem 2.5. We explain briefly how to obtain from the preceding arguments the structure of the scattering matrix given in Theorem 2.5. It is clear that (4.19) holds also in the present case, and we have first to analyze the structure of the resolvent $\mathcal{R}(E+i0)$, or more precisely that of

$$\operatorname{Op}(\alpha_+)\mathcal{R}(E+i0)\operatorname{Op}(\alpha_-),$$

where $\alpha_\pm \in C_0^\infty(T^*\mathbb{R}^n)$ are now microlocally supported respectively near $\rho_- \in p^{-1}(E_0)$ and $\rho_+ = \exp(TH_p)(\rho_-)$ for some given T.

As in Proposition 3.1, one can see that

$$\operatorname{Op}(\alpha_+)\mathcal{R}(E+i0)\operatorname{Op}(\alpha_-) = \operatorname{Op}(\alpha_+)\int \chi(t)e^{-it(P-E)/h}dt\operatorname{Op}(\alpha_-) + R,$$

with $\|R\|_{\mathcal{B}(L^2,L^2)} = \mathcal{O}(h^\infty)$, for some $\chi \in C_0^\infty(]0,2T[)$. From Lemma 3.3 (see also Remark 4.3), we then know that

$$\operatorname{Op}(\alpha_+)\mathcal{R}(E+i0)\operatorname{Op}(\alpha_-) \in \mathcal{I}_h^{1/2}(\mathbb{R}^n\times\mathbb{R}^n, \Lambda(E_0)'), \tag{4.23}$$

where

$$\begin{aligned}\Lambda(E_0) = \{(x,\xi,y,\eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n;\ p(x,\xi) = E_0,\\ \exists t \in \mathbb{R},\ (x,\xi) = \exp(tH_p)(y,\eta)\}.\end{aligned}$$

The scattering matrix is given by (4.19). Proceeding as in the previous section and using the fact that

$$C_+^{-1} \circ \Lambda(E_0) \circ C_- \subset \mathcal{SR}(E_0),$$

we obtain the theorem.

5. Microlocal representation of the scattering amplitude. Here we discuss the representation of the scattering amplitude as an oscillatory integral implied by Theorem 2.4. We also show that this leads to an integral kernel representation of the scattering amplitude.

For $\alpha \in \mathbb{S}^{n-1}$ we define the Lagrangian submanifolds $\Lambda_\alpha^\pm \subset T^*\mathbb{R}^n$ by

$$\Lambda_\alpha^\pm = \big\{\rho \in T^*\mathbb{R}^n;\ \lim_{t\to\pm\infty} \xi(t,\rho) = \sqrt{2E_0}\alpha\big\},$$

and the (modified) actions $S_\pm$ over the trajectories $\gamma_\pm = (x_\pm, \xi_\pm) \subset \Lambda_\pm$ as

$$S_\pm = \int_{-\infty}^{\infty} |\xi_\pm(t)|^2 - 2E_0 1_{\pm t>0} dt. \tag{5.1}$$

We now have the following

Lemma 5.1. *Let $\omega_0, \theta_0 \in \mathbb{S}^{n-1}$ be such that $\Lambda_{\omega_0}^-$ intersects Λ_- transversely in $p^{-1}(E_0)$ and $\Lambda_{\theta_0}^+$ intersects Λ_+ transversely in $p^{-1}(E_0)$. Then*

i) there exist open sets $O^\pm \subset \mathbb{S}^{n-1}$ with $\omega_0 \in O^-$ and $\theta_0 \in O^+$ such that for every $\omega \in O^-$ and every $\theta \in O^+$ the intersections of Λ_- with Λ_ω^- and of Λ_+ with Λ_θ^+ are transverse in $p^{-1}(E_0)$.

ii) there exist numbers $N_\pm \in \mathbb{N}$ such that for every $\omega \in O^-$ there are exactly N_- trajectories $\gamma_-^k(\omega)$ in Λ_- with initial direction ω and for every $\theta \in O^+$ there are exactly N_+ trajectories $\gamma_+^\ell(\theta)$ in Λ_+ with final direction θ.

iii) for $k \in \{1, \dots, N_-\}$ and $\ell \in \{1, \dots, N_+\}$, let $z_-^k(\omega)$ and $z_+^\ell(\theta)$ be the impact parameters of the curves $\gamma_-^k(\omega)$ and $\gamma_+^\ell(\theta)$ defined in (1.12). Then $\omega \mapsto z_-^k(\omega)$ and $\theta \mapsto z_+^\ell(\theta)$ are C^∞ functions in $O^\pm$.

Anticipating Lemma 5.2, we can now define the open sets

$$\Lambda_{-S_-^k} = \big\{\big(\omega, -\sqrt{2E_0} z_-^k(\omega)\big) \in T^*\mathbb{S}^{n-1};\ \omega \in O^-\big\} \subset \Lambda_-^\infty,$$
$$\Lambda_{S_+^\ell} = \big\{\big(\theta, -\sqrt{2E_0} z_+^\ell(\theta)\big) \in T^*\mathbb{S}^{n-1};\ \theta \in O^+\big\} \subset \Lambda_+^\infty.$$

Of course, the restrictions to $\Lambda_{-S_-^k}$ and to $\Lambda_{S_+^\ell}$ of the projection $\pi : T^*\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ are diffeomorphisms.

Proof. Let $\rho_0 \in \Lambda_+ \cap \Lambda_{\theta_0}^+$. Then, there exists a C^∞ function $f : p^{-1}(E_0) \to \mathbb{R}^{n-1}$ defined locally near ρ_0 such that, for ρ near ρ_0,

$$\rho \in \Lambda_+ \Longleftrightarrow f(\rho) = 0,$$

and the differential of f is of maximal rank. The same way, since Λ_θ^+ depend smoothly on θ, there exists a C^∞ functions $g : p^{-1}(E_0) \times \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ such that

$$\rho \in \Lambda_\theta^+ \Longleftrightarrow g(\rho, \theta) = 0.$$

and the differential, with respect to ρ, of g is of maximal rank.

Now we define

$$F : \begin{cases} p^{-1}(E_0) \times \mathbb{S}^{n-1} \longrightarrow & \mathbb{R}^{2n-2} \\ (\rho, \theta) & (f(\rho), g(\rho, \theta)) \end{cases}$$

and we note that

$$\rho \in \Lambda_+ \cap \Lambda_\theta^+ \Longleftrightarrow F(\rho, \theta) = 0.$$

Since the intersection $\Lambda_+ \cap \Lambda_{\theta_0}^+$ is transverse, the differential of F, with respect to ρ, is of maximal rank for $\theta = \theta_0$. By continuity, this property remains true for θ near θ_0 and *i)* follows.

In particular, up to a reordering of the coordinates, we can assume that $d_{\rho'} F(\rho, \theta)$ is invertible for (ρ, θ) in a neighborhood of (ρ_0, θ_0). Here ρ' denotes the $2n-2$ variables $(\rho_2, \ldots, \rho_{2n-1})$. Then, by the implicit function theorem, there exist a C^∞ function $G : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{2n-2}$ such that

$$\rho \in \Lambda_+ \cap \Lambda_\theta^+ \Longleftrightarrow \rho' = G(\rho_1, \theta).$$

Thus, for θ fixed, $\Lambda_+ \cap \Lambda_\theta^+$ is locally a one dimensional manifold. Since $\Lambda_+ \cap \Lambda_\theta^+$ is necessarily stable by the Hamiltonian flow, $\Lambda_+ \cap \Lambda_\theta^+$ is locally a unique Hamiltonian curve and

$$\rho \in \Lambda_+ \cap \Lambda_\theta^+ \Longleftrightarrow \exists t \in \mathbb{R}, \quad \rho = \exp(tH_p)(\rho_{0,1}, G(\rho_{0,1}, \theta)),$$

locally near ρ_0 (here, $\rho_{0,1}$ can be replaced by any real number close to this value). Then *ii)* follows from a compactness argument on $\Lambda_+ \cap \{|x| = \varepsilon\}$.

Let now $z_+^\ell(\theta)$ be the impact parameter of the trajectory $t \mapsto \exp(tH_p)$ $(\rho_{1,0}, G(\rho_{1,0}, \theta))$ defined in (1.12). From (1.10) and the fact that G is smooth, $z_+^\ell(\theta)$ is a C^∞ function in O^+ if O^+ is a small enough neighborhood of θ_0. $\square$

For $m \in \{1, \ldots, N_+\}$ or $m \in \{1, \ldots, N_-\}$ and $\theta \in O^+$ or $\omega \in O^-$ we shall use the superscript m to denote objects related to the unique trajectory $\gamma_\pm^m$ with final direction θ or initial direction ω. In particular, we let $S_+^m(\theta)$, $\theta \in O^+$, denote the (modified) action, given by (5.1), over the m-th trajectory with final direction θ. With $S_-^m(\omega)$ for $\omega \in O^-$ defined mutatis mutandis, we now have the following lemma which is analogous to [5, Lemma 5] and Equation (3.13).

Lemma 5.2. *For $m \in \{1, \dots, N_\pm\}$, we have* $\Lambda_{\pm S_\pm^m} = \big\{(\alpha, \pm\partial_\alpha S_\pm^m(\alpha));$ $\alpha \in O^\pm\big\}$.

Proof. We will only calculate $\Lambda_{S_+^\ell}$. The case of the manifold $\Lambda_{-S_-^k}$ can be treated the same way. Here, we will use the notation

$$\begin{cases} x_+(t,\theta) = x_+(t,\theta,z_+^\ell(\theta)) \\ \xi_+(t,\theta) = \xi_+(t,\theta,z_+^\ell(\theta)). \end{cases}$$

We recall from [6, Equation (7.11)] that

$$\begin{aligned} \Psi_+\big(x_+(t,\theta),\theta\big) =& 2E_0 t 1_{t>0} - \int_t^{+\infty} |\xi_+(s,\theta)|^2 - 2E_0 1_{s>0}\, ds \\ =& - S_+^\ell(\theta) + \int_{-\infty}^t |\xi_+(s,\theta)|^2 ds. \end{aligned} \tag{5.2}$$

From Lemma 4.2 and Lemma 5.1,

$$\Lambda_{S_+^\ell} = \big\{\big(\theta, (\partial_\theta\Psi_+)(x_+(t,\theta),\theta)\big);\ \theta \in O^+\big\}, \tag{5.3}$$

for any $t \in \mathbb{R}$. Combining (4.14) and (5.2), we obtain

$$\begin{aligned} (\partial_\theta\Psi_+)\big(x_+(t,\theta),\theta\big) &= \partial_\theta\big(\Psi_+(x_+(t,\theta),\theta)\big) - (\partial_x\Psi_+)(x_+(t,\theta),\theta)\partial_\theta(x_+(t,\theta)) \\ &= - \partial_\theta S_+^\ell(\theta) + \int_{-\infty}^t \partial_\theta\big(|\xi_+(s,\theta)|^2\big) ds - \xi_+(t,\theta)\partial_\theta(x_+(t,\theta)). \end{aligned} \tag{5.4}$$

Since the energy is constant on the Hamiltonian curves, we have, as in (3.14),

$$\begin{aligned} \partial_t\big(\xi_+(t,\theta)\partial_\theta(x_+(t,\theta))\big) &= \xi_+(t,\theta)\partial_\theta(\xi_+(t,\theta)) - (\partial_x V)(x_+(t,\theta))\partial_\theta(x_+(t,\theta)) \\ &= \frac{1}{2}\partial_\theta\big(|\xi_+(t,\theta)|^2\big) - \partial_\theta\big(V(x_+(t,\theta))\big) \\ &= \frac{1}{2}\partial_\theta\big(|\xi_+(t,\theta)|^2\big) - \partial_\theta\Big(E_0 - \frac{1}{2}|\xi_+(t,\theta)|^2\Big) \\ &= \partial_\theta\big(|\xi_+(t,\theta)|^2\big). \end{aligned} \tag{5.5}$$

Moreover, as $t \to -\infty$, we have $\xi_+(t,\theta) \to 0$ and

$$\partial_\theta(x_+(t,\theta)) = d\Pi_x \circ d\exp(tH_p)\big(\partial_\theta x_+(0,\theta), \partial_\theta\xi_+(0,\theta)\big) \longrightarrow 0,$$

since $(x_+(0,\theta),\xi_+(0,\theta)) \in \Lambda_+$ for all θ and 0 is a unstable node of H_p restricted to Λ_+. Then, (5.5) yields

$$\xi_+(t,\theta)\partial_\theta(x_+(t,\theta)) = \int_{-\infty}^{t} \partial_\theta\big(|\xi_+(s,\theta)|^2\big)ds.$$

Using this equality, the lemma follows from (5.3) and (5.4). □

We now have the following

Theorem 5.3. *Let $E = E_0 + hE_1$, with $E_1 \in]-C_0, C_0[$ for some $C_0 > 0$, and $\omega^0, \theta^0 \in \mathbb{S}^{n-1}$ satisfy $\omega^0 \neq \theta^0$. Then*

i) for every $(\theta^0, -\sqrt{2E_0}z_+^0, \omega^0, -\sqrt{2E_0}z_-^0) \in \widetilde{\Lambda_+^\infty \times \Lambda_-^\infty}$ there exist $m \in \mathbb{N}$, a symbol $a \in S_{2n+m-2}^{\frac{1}{2}-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}+\frac{m}{2}}(1)$, and a non-degenerate phase function $\varphi \in C^\infty(\mathbb{R}^{2n+m-2})$ such that, microlocally near $(\theta^0, -\sqrt{2E_0}z_+^0, \omega^0, -\sqrt{2E_0}z_-^0)$,

$$\mathcal{A}(E,h)(\theta,\omega) = \int_{\mathbb{R}^m} e^{i\varphi(\theta,\omega,\tau)/h} a(\theta,\omega,\tau;E,h)\,d\tau.$$

ii) Assume that Λ_ω^- intersects Λ_- transversely and Λ_θ^+ intersects Λ_+ transversely. For every $(\theta^0, -\sqrt{2E_0}z_+^0, \omega^0, -\sqrt{2E_0}z_-^0) \in \widetilde{\Lambda_+^\infty \times \Lambda_-^\infty}$, there exists a symbol $a \in S_{2n-2}^{\frac{1}{2}-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}(1)$ such that, microlocally near $(\theta^0, -\sqrt{2E_0}z_+^0, \omega^0, -\sqrt{2E_0}z_-^0)$,

$$\mathcal{A}(E,h)(\theta,\omega) = e^{i(S_+(\theta)+S_-(\omega))/h} a(\theta,\omega;E,h),$$

where $S_+(\theta)$ and $S_-(\omega)$ are the actions defined before Lemma 5.2 associated to the paths in $\Lambda_+ \cap \Lambda_\theta^+$ and $\Lambda_- \cap \Lambda_\omega^-$ close to $\gamma_+(t,\theta^0,z_+^0,E_0)$ and $\gamma_-(t,\omega^0,z_-^0,E_0)$.

iii) Assume $O^- \cap O^+ = \emptyset$ and $\langle g_+(\rho_+), g_-(\rho_-)\rangle \neq 0$ for all $(\rho_+,\rho_-) \in \Lambda_+ \times \Lambda_-$ such that $\pm \lim_{t\to\pm\infty} \xi(t,\rho_\pm) \in \sqrt{2E_0}O^\pm$. Let N_∞ be the number of (ω,θ)-trajectories. For $j \in \{1,\dots,N_\infty\}$, $k \in \{1,\dots,N_-\}$ and $\ell \in \{1,\dots,N_+\}$, there exist $m_j, m_{k,\ell} \in \mathbb{N}$, non-degenerate phase functions

$$\varphi_j \in C^\infty\left(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^{m_j}\right) \quad \textit{and} \quad \varphi_{k,\ell} \in C^\infty\left(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}^{m_{k,\ell}}\right),$$

and symbols

$$a_j \in S^{\frac{m_j}{2}}_{2n-2+m_j}(1) \quad \text{and} \quad a_{k,\ell} \in S^{\frac{1}{2}-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}+\frac{m_{k,\ell}}{2}}_{2n-2+m_{k,\ell}}(1),$$

such that, in $C^\infty(O^+ \times O^-)$,

$$\mathcal{A}(E,h)(\theta,\omega) = \sum_{j=1}^{N_\infty} \int_{\mathbb{R}^{m_j}} e^{i\varphi_j(\theta,\omega,\tau)/h} a_j(\theta,\omega,\tau;E,h)\, d\tau + \sum_{k=1}^{N_-}\sum_{\ell=1}^{N_+} \int_{\mathbb{R}^{m_{k,\ell}}} e^{i\varphi_{k,\ell}(\theta,\omega,\tau)/h} a_{k,\ell}(\theta,\omega,\tau;E,h)\, d\tau + \mathcal{O}(h^\infty).$$

Proof. *i)* The first part is a direct consequence of Theorem 2.4 and [2, Theorem 1].

ii) The second part follows from Theorem 2.4, Lemma 5.2, and [2, Theorem 1].

iii) To establish the last part of the theorem, it suffices to prove that $WF_h^i(\chi\mathcal{A}(E,h)) = \emptyset$ for every $\chi \in C_0^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1} \setminus \operatorname{diag}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}))$. Recall the representation (4.3) of the scattering amplitude. From [28, page 166], we have, in the sense of oscillatory integrals,

$$K_{T_{\pm 1}} = \int e^{i\psi_\pm(\theta,\omega,x)/h} k_{\pm b}(x,\sqrt{2E}\omega;h)\overline{a_+}(x,\sqrt{2E}\theta;h)\, dx,$$

with $k_{\pm b} = e^{-i\Phi_\pm/h}\big(-\frac{h^2}{2}\Delta + V - \frac{1}{2}\xi^2\big)e^{i\Phi_\pm/h}b_\pm \in A_{-1}$ and $\psi_\pm(\theta,\omega,x) = \Phi_\pm(x,\sqrt{2E}\omega) - \Phi_+(x,\sqrt{2E}\theta)$. Since $O^+\cap O^- = \emptyset$, there exists $C>0$ such that $|\partial_x\psi_\pm| > C$ for $(\theta,\omega)\in O^+\times O^-$. Then, integrating by parts with respect to x, we see that the distribution $K_{T_{\pm 1}}$ is a C^∞ function on $O^+\times O^-$. Moreover this function and all its derivatives are bounded by $\mathcal{O}(h^\infty)$. Therefore,

$$WF_h^i\big(K_{T_{\pm 1}}|_{O^+\times O^-}\big) = \emptyset. \tag{5.6}$$

From (4.7), it is clear that $(\theta,\omega)\mapsto G(\theta,\omega)$ is C^∞ with respect to (θ,ω). In some coordinate chart and for any $f_+(\theta)$, $f_-(\omega)$ in $C_0^\infty(\mathbb{R}^{n-1})$ supported in this chart, we have

$$\big|\big(\mathcal{F}_h(f_+f_-G)\big)(\xi,\eta)\big| = \left|\iiint e^{-i(\xi\theta+\eta\omega)/h}e^{-i\Phi_+(x,\sqrt{2E_0}\theta)/h} f_+\overline{g_+}\mathcal{R}\big(e^{i\Phi_-(y,\sqrt{2E_0}\omega)/h}f_-g_-\big)\, dx\, d\theta\, d\omega\right|,$$

for $\xi, \eta \in \mathbb{R}^{n-1}$. For $|\xi|$ large enough, we have

$$|\partial_\theta(\xi\theta + \Phi_+(x, \sqrt{2E_0}\theta))| \gtrsim \langle \xi \rangle,$$

on the support of $\overline{g_+}$. The same way, for $|\eta|$ large enough, we have

$$|\partial_\omega(\eta\omega - \Phi_-(y, \sqrt{2E_0}\omega))| \gtrsim \langle \eta \rangle,$$

on the support of g_-. Then, performing integration by parts with respect to θ or ω, we obtain $|(\mathcal{F}_h(f_+f_-G))(\xi, \eta)| = \mathcal{O}(h^\infty \langle \xi, \eta \rangle^{-\infty})$ for $\langle \xi, \eta \rangle$ large enough, and therefore

$$WF_h^i(G) = \emptyset. \tag{5.7}$$

To treat, now, the terms in (4.4) containing the operators whose norms are estimated in Lemma 4.1, we use the following result, the proof of which we present later.

Lemma 5.4. *Let $T \in \mathcal{B}(L^2_{-\gamma}(\mathbb{R}^n), L^2_\gamma(\mathbb{R}^n))$ satisfy $\|T\|_{\mathcal{B}(L^2_{-\gamma}, L^2_\gamma)} = \mathcal{O}(h^\infty)$ for all $\gamma \gg 1$ and let $E > 0$. Then*

$$WF_h^i\big(K_{F_0(E,h)TF_0^*(E,h)}\big) = \emptyset.$$

From (4.3), (5.6), (4.4), (5.7), Lemma 4.1, and Lemma 5.4 we now conclude that

$$WF_h^i(\chi\mathcal{A}(E, h)) = \emptyset. \tag{5.8}$$

For $j \in \{1, \ldots, N_\infty\}$ we now let $\mathcal{SR}_j(E_0)$ denote the scattering relation near the j-th (ω, θ)-trajectory, defined in (1.17) and indicated in Figure 1. From Theorem 2.5,

$$\mathcal{S}(E, h) \in \mathcal{I}_h^0\left(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \mathcal{SR}_j(E_0)'\right),$$

microlocally near the limit points of the j-th (ω, θ)-trajectory. Moreover $\mathcal{S}(E, h) = 0$ microlocally near the other points given in Theorem 2.6. From (5.8), it is enough to know the scattering amplitude microlocally in a compact set. Then, the conclusion of the theorem follows from these observations, (1.7), Theorem 2.4 and [2, Theorem 1]. □

Proof of Lemma 5.4. In some coordinate chart and for any $f_+(\theta)$, $f_-(\omega)$ in $C_0^\infty(\mathbb{R}^{n-1})$ supported in this chart, we have

$$\begin{aligned} K(\xi,\eta) &= \big(\mathcal{F}_h K_{f_+ F_0(E,h) T F_0^*(E,h) f_-}\big)(\xi,\eta) \\ &= c_2 \iiint e^{-i(\theta\xi+\sqrt{2E}x\theta)/h} f_+(\theta) T\big(e^{i(\sqrt{2E}y\omega-\omega\eta)/h} f_-(\omega)\big)\, dx\, d\theta\, d\omega, \end{aligned}$$

with $c_2 = (2\pi h)^{-n}(2E)^{\frac{n-2}{2}}$. In particular, for $\alpha,\beta \in \mathbb{N}^{n-1}$,

$$\xi^\alpha \eta^\beta K(\xi,\eta) = c_2 \iiint e^{-i\theta\xi/h}(-ih\partial_\theta)^\alpha \big(e^{-i\sqrt{2E}x\theta/h} f_+(\theta)\big)$$
$$T\big(e^{-i\omega\eta/h}(-ih\partial_\omega)^\beta\big(e^{i\sqrt{2E}y\omega/h} f_-(\omega)\big)\big)\, dx\, d\theta\, d\omega,$$

We remark that

$$\begin{aligned} &e^{-i\theta\xi/h}(-ih\partial_\theta)^\alpha\big(e^{-i\sqrt{2E}x\theta/h} f_+(\theta)\big) \in L^2_{-n/2-1-|\alpha|}(\mathbb{R}^n_x), \\ &e^{-i\omega\eta/h}(-ih\partial_\omega)^\beta\big(e^{i\sqrt{2E}y\omega/h} f_-(\omega)\big) \in L^2_{-n/2-1-|\beta|}(\mathbb{R}^n_y), \end{aligned}$$

uniformly with respect to h, ξ, η, θ, ω. Combining $\|T\|_{\mathcal{B}(L^2_{-n/2-1-|\beta|}, L^2_{n/2+1+|\alpha|})} = \mathcal{O}(h^\infty)$ with these estimates and the compactness of $\mathbb{S}^{n-1}$, we get

$$\xi^\alpha \eta^\beta K(\xi,\eta) = \mathcal{O}(h^\infty),$$

uniformly in ξ,η and the lemma follows. $\square$

Remark 5.5. It is clear that all estimates in the above proof can be made uniform in the energy if that is allowed to vary in a bounded set.

Appendix A. Elements of semiclassical analysis.

A.1. Semiclassical distributions. Here we recall some of the elements of semiclassical analysis which we use throughout the paper. A family $(u_h)_{h\in]0,h_0]}$ of distributions in $\mathcal{D}'(\mathbb{R}^n)$ is called a semiclassical distribution when

$$\forall \chi \in C_0^\infty(\mathbb{R}^n), \quad \exists N \in \mathbb{N}, \quad \mathcal{F}_h(\chi u)(\xi) \lesssim h^{-N}\langle\xi\rangle^N,$$

where $\mathcal{F}_h$ is the h-Fourier transform

$$\mathcal{F}_h(\chi u)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi/h}\chi u(x)\, dx.$$

The space of semiclassical distributions is denoted by $\mathcal{D}'_h(\mathbb{R}^n)$. We define the semiclassical wavefront set of $u = (u_h) \in \mathcal{D}'_h(\mathbb{R}^n)$ as follows.

Definition A.1. *Let $u \in \mathcal{D}'_h(\mathbb{R}^n)$ and let $(x_0,\xi_0) \in T^*\mathbb{R}^n \sqcup T^*\mathbb{S}^{n-1}$. We shall say that (x_0,ξ_0) does not belong to the semiclassical wavefront set of u if:*

- *If* $(x_0, \xi_0) \in T^*\mathbb{R}^n$*: there exist* $\chi \in C_0^\infty(\mathbb{R}^n)$ *with* $\chi(x_0) \neq 0$ *and an open neighborhood* U *of* ξ_0*, such that* $\forall N \in \mathbb{N}$, $\forall \xi \in U$,

$$|\mathcal{F}_h(\chi u)(\xi)| \leq C_{N,U} h^N.$$

We shall denote the complement of the set of all such points by $WF_h^f(u)$.

- *If* $(x_0, \xi_0) \in T^*\mathbb{S}^{n-1}$*: there exist* $\chi \in C_0^\infty(\mathbb{R}^n)$ *with* $\chi(x_0) \neq 0$ *and a conic neighborhood* U *of* ξ_0*, such that* $\forall N \in \mathbb{N}$, $\forall \xi \in U \cap \left\{|\xi| \geq \frac{1}{K}\right\}$ *for some* $K > 0$,

$$|\mathcal{F}_h(\chi u)(\xi)| \leq C_{N,U,K} h^N \langle \xi \rangle^{-N}.$$

We shall denote the complement of the set of all such points by $WF_h^i(u)$.

We shall further use $WF_h(u) = WF_h^f(u) \sqcup WF_h^i(u)$ *to denote the semiclassical wavefront set of* u.

A family $u = (u_h)$ of temperate distributions in $\mathcal{S}'(\mathbb{R}^n)$ is called a semiclassical temperate distribution when, for some $N \in \mathbb{R}$,

$$\langle (x, hD) \rangle^{-N} u = \mathcal{O}(h^{-N}),$$

in $L^2(\mathbb{R}^n)$. The space of semiclassical temperate distributions is denoted by $\mathcal{S}_h'(\mathbb{R}^n)$.

A.2. Pseudodifferential operators. We now define briefly the semiclassical pseudodifferential operators (see the book of Dimassi and Sjöstrand [9]). A positive function $m : \mathbb{R}^p \to]0, +\infty[$ is called an *order function* if there exists $C > 0$ such that

$$m(X) \leq C \langle X - Y \rangle^C m(Y),$$

for all $X, Y \in \mathbb{R}^p$. We denote by $S^q(m) = S_p^q(m)$ the set of (families of) functions $a(X; h) \in C^\infty(\mathbb{R}^p)$ such that, for all $\alpha \in \mathbb{N}^p$,

$$\partial_X^\alpha a(X; h) = \mathcal{O}(h^{-q} m(X)).$$

If $a(x, \xi; h)$ is a symbol of class $S_{2n}^q(m)$, we define the h-pseudodifferential operator, in Weyl quantization, $\mathrm{Op}(a)$ with symbol a by

(A.1) $\forall u \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h} a\Big(\frac{x+y}{2}, \xi; h\Big) u(y)\, dy\, d\xi,$$

extending the definition to $\mathcal{S}'(\mathbb{R}^n)$ by duality. We also denote by $\Psi^q(m)$ the space of operators $\mathrm{Op}(S^q_{2n}(m))$.

We extend these notions to compact manifolds through the following definition of semiclassical pseudodifferential operators on compact manifolds. Let M be a smooth compact manifold and $\kappa_j : M_j \to X_j$, $j = 1, \dots, N$, be a set of local charts. A linear continuous operator $A : C^\infty(M) \to \mathcal{D}'_h(M)$ belongs to $\Psi^q(1, M)$ if for all $j \in \{1, \dots, N\}$ and $u \in C_0^\infty(M_j)$ we have $Au \circ \kappa_j^{-1} = A_j\big(u \circ \kappa_j^{-1}\big)$ with $A_j \in \Psi^q(1)$, and $\chi_1 A \chi_2 : \mathcal{D}'_h(M) \to h^\infty C^\infty(M)$ if $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$ (see [11, Section E.2] for more details).

A.3. Microlocal Properties. We can now define what we mean by "microlocally". We will only work on $\mathbb{R}^n$. Using the previous paragraph, this definition can be extended to the case of compact manifolds.

Let $u, v \in \mathcal{S}'_h(\mathbb{R}^n)$. We say that $u = v$ *microlocally* near a set $U \subset T^*\mathbb{R}^n$, if there exists $a \in S^0(1)$, $a = 1$ in a neighborhood of U, such that

$$\mathrm{Op}(a)(u - v) = \mathcal{O}(h^\infty),$$

in $L^2(\mathbb{R}^n)$. We also say that $u \in \mathcal{S}'_h(\mathbb{R}^n)$ satisfies a property $\mathcal{P}$ *microlocally* near a set $U \subset T^*\mathbb{R}^n$ if there exist $v \in \mathcal{S}'_h(\mathbb{R}^n)$ such that $u = v$ microlocally near U and v satisfies property $\mathcal{P}$.

Definition A.2. *Let $A, B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ be linear operators bounded by $\mathcal{O}(h^{-N})$, $N > 0$ and $(\rho, \widetilde{\rho}) \in T^*\mathbb{R}^m \times T^*\mathbb{R}^n$. We say that*

$$A = B \textit{ microlocally near } (\rho, \widetilde{\rho}),$$

if there exists $\alpha \in C_0^\infty(T^*\mathbb{R}^m)$ (resp. $\beta \in C_0^\infty(T^*\mathbb{R}^n)$) equal to 1 near ρ (resp. $\widetilde{\rho}$) such that

$$\mathrm{Op}(\alpha)(B - A)\,\mathrm{Op}(\beta) = \mathcal{O}(h^\infty),$$

in $\mathcal{B}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^m))$.

A.4. Semiclassical Fourier integral operators. We now define global semiclassical Fourier integral operators. For the general theory of the FIOs in the classical setting, we refer to Hörmander [16, Section 25.2]. The theory of the semiclassical FIOs can be found in the books of Ivrii [19, Section 1.2], Robert [26], in the PhD thesis of Dozias [10] or in the article of the first author [2]. We will develop this theory in $\mathbb{R}^n$. Using local charts, the following definitions and theorem can easily be extended to the case of compact manifolds.

Let $\varphi(x, y, \theta) \in C^{\infty}(\Omega)$ where Ω is an open set of $\mathbb{R}^{m+n+d}$. We say that φ is a *non-degenerate phase function* if, for all $(x, y, \theta) \in C_{\varphi}$ with

$$C_{\varphi} = \{(x, y, \theta) \in \Omega;\ \partial_{\theta}\varphi = 0\},$$

the d differentials $d\partial_{\theta_1}\varphi, \dots, d\partial_{\theta_d}\varphi$ are linearly independant.

If φ is a non-degenerate phase function, C_{φ} is a $(m+n)$-dimensional manifold and

$$j_{\varphi} : \begin{cases} C_{\varphi} \longrightarrow & T^*\mathbb{R}^m \times T^*\mathbb{R}^n \\ (x, y, \theta) & (x, \partial_x\varphi, y, \partial_y\varphi) \end{cases}$$

is locally a diffeomorphism whose image is a Lagrangian manifold for the symplectic form $d\xi \wedge dx + d\eta \wedge dy$ ((x, ξ) and (y, η) are the standard coordinates on $T^*\mathbb{R}^m$ and $T^*\mathbb{R}^n$). We set $\Lambda_{\varphi} = j_{\varphi}(C_{\varphi})$.

Definition A.3. *A submanifold $\Lambda \subset T^*\mathbb{R}^m \times T^*\mathbb{R}^n$ is a canonical relation from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^m$ if Λ is a Lagrangian manifold for the symplectic form $d\xi \wedge dx - d\eta \wedge dy$.*

A canonical relation Λ is given by a canonical transformation if there exists a symplectic diffeomorphism $\kappa : T^*\mathbb{R}^n \to T^*\mathbb{R}^m$ such that $\Lambda = \operatorname{graph}(\kappa)$.

As usual, if $\Lambda \subset T^*\mathbb{R}^m \times T^*\mathbb{R}^n$, we set

$$\Lambda' = \{(x, \xi, y, -\eta);\ (x, \xi, y, \eta) \in \Lambda\},$$

which is also a subset of $T^*\mathbb{R}^m \times T^*\mathbb{R}^n$. In particular, for a non-degenerate phase function φ, the manifold Λ'_{φ} is a canonical relation (if φ is restricted to a small set).

Definition A.4. *Let $r \in \mathbb{R}$, Λ be a canonical relation from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^m$ and $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ be a linear operator bounded by $\mathcal{O}(h^{-N})$, $N > 0$. Then A is called a h-Fourier integral operator (h-FIOs) of order r associated to Λ and we denote by*

$$A \in \mathcal{I}_h^r(\mathbb{R}^m \times \mathbb{R}^n, \Lambda'),$$

if, for all $(\rho, \widetilde{\rho}) \in T^*\mathbb{R}^m \times T^*\mathbb{R}^n$, A is equal to

$$h^{-r-\frac{n+m}{4}-\frac{d}{2}} \int_{\theta \in \mathbb{R}^d} e^{i\varphi(x,y,\theta)/h} a(x, y, \theta; h)\, d\theta. \tag{A.2}$$

microlocally near $(\rho, \widetilde{\rho})$. Here, the symbol $a \in S^0(1)$ has compact support in the variables x, y, θ (uniformly with respect to h). The function φ is a non-degenerate phase function defined near the support of a with ${\Lambda_{\varphi}}' \subset \Lambda$.

A h-FIO A will be called a h-Fourier integral operator with compactly supported symbol if, modulo an operator $\mathcal{O}(h^\infty)$ in $\mathcal{B}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^m))$, A is a finite sum of operators of the form (A.2).

Lastly, we give the composition law for h-Fourier integral operators (see *e.g.* [10] for the proof). The following theorem is a semiclassical version of Theorem 25.2.3 of Hörmander [16]. Since all the h-FIOs which appear in the present paper (except the one in Lemma 3.2) have compactly supported symbol, we give the composition law only in that case.

Let $A_1 \in \mathcal{I}_h^{r_1}(\mathbb{R}^m \times \mathbb{R}^n, {\Lambda_1}')$ and $A_2 \in \mathcal{I}_h^{r_2}(\mathbb{R}^n \times \mathbb{R}^p, {\Lambda_2}')$ be two h-FIOs with compactly supported symbols, associated with $\Lambda_1 \subset T^*\mathbb{R}^m \times T^*\mathbb{R}^n$ and $\Lambda_2 \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^p$ respectively. We set

$$\begin{aligned} X &= T^*\mathbb{R}^m \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^p \\ Y &= \Lambda_1 \times \Lambda_2 \subset X \\ Z &= T^*\mathbb{R}^m \times \operatorname{diag}(T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \times T^*\mathbb{R}^p \subset X. \end{aligned}$$

Definition A.6. *We say that Y and Z intersect cleanly if $Y \cap Z$ is a manifold and $T_\rho(Y \cap Z) = T_\rho Y \cap T_\rho Z$ at each $\rho \in Y \cap Z$. The excess of the intersection is*

$$e = \dim X + \dim Y \cap Z - \dim Y - \dim Z.$$

Let

$$\pi : Y \cap Z \longrightarrow T^*\mathbb{R}^m \times T^*\mathbb{R}^p, \tag{A.3}$$

be the natural projection. The image of π is

$$\Lambda_2 \circ \Lambda_1 = \{(\rho_3, \rho_1) \in T^*\mathbb{R}^m \times T^*\mathbb{R}^p;\ \exists \rho_2 \in T^*\mathbb{R}^n,\ (\rho_3, \rho_2) \in \Lambda_2 \text{ and } (\rho_2, \rho_1) \in \Lambda_1\}.$$

Definition A.7. *We say that Y and Z intersect connectedly if, for all $\gamma \in T^*\mathbb{R}^m \times T^*\mathbb{R}^p$, the set $\pi^{-1}(\gamma)$ is connected.*

When Y and Z intersect cleanly and connectedly, the set $\Lambda_2 \circ \Lambda_1$ is a Lagrangian submanifold of $T^*\mathbb{R}^m \times T^*\mathbb{R}^p$. In general, the intersection $Y \cap Z$ is also assumed to be proper. This means that π, defined in (A.3), is proper. But since A_1 and A_2 have compactly supported symbol, we don't have to make such a hypothesis.

Theorem A.7. *Let* $A_1 \in \mathcal{I}_h^{r_1}(\mathbb{R}^m \times \mathbb{R}^n, {\Lambda_1}')$ *and* $A_2 \in \mathcal{I}_h^{r_2}(\mathbb{R}^n \times \mathbb{R}^p, {\Lambda_2}')$ *be two h-FIOs with compactly supported symbols. If* Y *and* Z *intersect connectedly and cleanly with excess* e*, then*

$$A_2 \circ A_1 \in \mathcal{I}_h^{r_1+r_2+e/2}(\mathbb{R}^m \times \mathbb{R}^p, \Lambda_2 \circ {\Lambda_1}'),$$

is a h-FIO with compactly supported symbol.

As stated in [16, Page 18], the hypothesis "Y and Z intersect connectedly" is made to avoid self-intersections of $\Lambda_2 \circ \Lambda_1$. In particular, this assumption can be replaced by "$\Lambda_2 \circ \Lambda_1$ is a manifold". Note that all the compositions in this paper satisfy this last statement.

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