High School Math Problems
2017
Week 9
Problem and Solution

For $n \in \mathbb{N}$ let $A_n = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n \text{ radicals}}$.

Prove that for every $n \in \mathbb{N}$

$$\frac{2-A_n}{2-A_{n-1}} > \frac{1}{4}.$$

Solution:

We have that

$$\frac{2-A_n}{2-A_{n-1}} = \frac{\left(2-\sqrt{2+A_{n-1}}\right)\left(2+\sqrt{2+A_{n-1}}\right)}{(2-A_{n-1})\left(2+\sqrt{2+A_{n-1}}\right)} = \frac{2-A_{n-1}}{(2-A_{n-1})\left(2+\sqrt{2+A_{n-1}}\right)} = \frac{1}{2+A_n}.$$

Now,

$$A_n = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n \text{ radicals}} < \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{4}}}}_{n \text{ radicals}} = 2$$

and therefore

$$\frac{2-A_n}{2-A_{n-1}} = \frac{1}{2+A_n} > \frac{1}{4}.$$

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