High School Math Problems
2017
Week 7
Problem and Solution

For $x > 0$ let

$$f\left(x\right)=\left(x\sqrt[n]{1+\left(a^{n}x^{-n}\right)^{\frac{1}{n+1}}}+a\sqrt[n]{1+\left(x^{n}a^{-n}\right)^{\frac{1}{n+1}}}\right)\cdot\left(\frac{b}{5}\right)^{-1},$$

where $b > a > 0$ and $n \in \mathbb{N}$.

Let $p = f\left(x_0\right)$, where $x_0 = \left(b^{\frac{n}{n+1}} - a^{\frac{n}{n+1}}\right)^{\frac{n+1}{n}}$.

Solve the equation

$$\sqrt{1+\frac{20}{p}y}+\sqrt{\frac{20}{p}-y^{2}}+\sqrt{y^{2}+z^{2}-8z-3}=\sqrt[4]{y^{4}-16}+p-\left(z^{6}\right)^{\frac{1}{6}}. \tag{1}$$

Solution:

We first compute

$$\left(a^{n}x_0^{-n}\right)^{\frac{1}{n+1}}=\left(\frac{a^{n}}{x_0^{n}}\right)^{\frac{1}{n+1}}=\left(\frac{a^{n}}{\left(b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}\right)^{n+1}}\right)^{\frac{1}{n+1}}=\frac{a^{\frac{n}{n+1}}}{b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}}.$$

Therefore

$$\begin{aligned}
p=f\left(x_0\right)&=\left(\left(b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}\right)^{\frac{n+1}{n}}\cdot\sqrt[n]{1+\frac{a^{\frac{n}{n+1}}}{b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}}}+a\cdot\sqrt[n]{1+\frac{b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}}{a^{\frac{n}{n+1}}}}\right)\cdot\frac{5}{b}\\
&=\left(\left(b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}\right)^{\frac{n+1}{n}}\cdot\sqrt[n]{\frac{b^{\frac{n}{n+1}}}{b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}}}+a\cdot\sqrt[n]{\frac{b^{\frac{n}{n+1}}}{a^{\frac{n}{n+1}}}}\right)\cdot\frac{5}{b}\\
&=\left(b^{\frac{1}{n+1}}\left(b^{\frac{n}{n+1}}-a^{\frac{n}{n+1}}\right)+a^{\frac{n}{n+1}}b^{\frac{1}{n+1}}\right)\cdot\frac{5}{b}\\
&=\left(b-a^{\frac{n}{n+1}}b^{\frac{1}{n+1}}+a^{\frac{n}{n+1}}b^{\frac{1}{n+1}}\right)\cdot\frac{5}{b}=5.
\end{aligned}$$

Therefore equation (1) becomes

$$\begin{aligned}
&\sqrt{1+4y}+\sqrt{4-y^2}+\sqrt{y^2+z^2-8z-3}=\sqrt[4]{y^4-16}+4-|z| \\
&\sqrt{1+4y}+\sqrt{4-y^2}+\sqrt{y^2+z^2-8z-3}=\sqrt[4]{\left(y^2-4\right)\left(y^2+16\right)}+4-|z|\,.
\end{aligned}$$

Therefore the domain of this equation consists of the pairs (y,z) such that

$$\left|\begin{array}{l} 1+4y\geq 0 \\ 4-y^2\geq 0 \\ y^2-4\geq 0 \\ y^2+z^2-8z-3\geq 0 \end{array}\right. \qquad \left|\begin{array}{l} y\geq -\dfrac{1}{4} \\ y\in[-2,2] \\ y\in(-\infty,-2]\cup[2,\infty) \\ y^2+z^2-8z-3\geq 0 \end{array}\right. . \tag{2}$$

The first three inequalities in (2), now, imply that $y=2$.
Therefore the last inequality in (2) becomes

$$z^2-8z+1\geq 0,$$

whose solution is

$$z\in\left(-\infty,4-\sqrt{15}\right]\cup\left[4+\sqrt{15},\infty\right).$$

Thus the domain of equation (1) is

$$D=\left\{(y,z):y=2,z\in\left(-\infty,4-\sqrt{15}\right]\cup\left[4+\sqrt{15},\infty\right)\right\}.$$

Therefore equation (1) becomes

$$\begin{aligned}
&\sqrt{9}+\sqrt{z^2-8z+1}=4-|z| \\
&\sqrt{z^2-8z+1}=1-|z| \\
&z^2-8z+1=1-2|z|+z^2 \\
&4z=|z| \\
&z=0.
\end{aligned}$$

Since $(2,0)\in D$, it follows that $(y,z)=(2,0)$ is the solution to the equation.