High School Math Problems
2017
Week 6
Problem and Solution

Find the value of the real parameter a for which the equation

$$x\left(x^{12}-(2a+3)\,x^{6}+a^{2}\right)=0$$

has exactly five real solutions, which form an arithmetic progression.

Solution:

We observe that the equation

$$x\left(x^{12}-(2a+3)\,x^{6}+a^{2}\right)=0$$

has 0 as one of its root.

It is further easy to see that if r is a root of the equation, then $-r$ is also root of the equation.

Thus, if the given equation has the five real roots x_1, x_2, x_3, x_4, and x_5, which form an arithmetic progression with common difference $d \neq 0$, then these roots are $-2d$, $-d$, 0, d, and $2d$, respectively.

Substituting into the given equation, we now obtain that

$$d^{12}-(2a+3)\,d^{6}+a^{2}=0 \tag{1}$$

and

$$2^{12}d^{12}-(2a+3)\,2^{6}d^{6}+a^{2}=0. \tag{2}$$

Multiplying equation (1) by 2^6 and subtracting it from equation (2), we now obtain that

$$\left(2^{12}-2^{6}\right)d^{12}+\left(1-2^{6}\right)a^{2}=0.$$

From this we obtain that

$$d^{12}=\frac{2^{6}-1}{2^{12}-2^{6}}a^{2}=\frac{a^{2}}{2^{6}}.$$

Therefore

$$d^6 = \frac{|a|}{8}. \tag{3}$$

On the other hand, subtracting equation (1) from equation (2), we obtain that

$$\left(2^{12} - 1\right) d^{12} - (2a + 3) \left(2^6 - 1\right) d^6 = 0.$$

From this it follows that

$$d^6 = \frac{2a + 3}{2^6 + 1} = \frac{2a + 3}{65}. \tag{4}$$

From (3) and (4) it follows that

$$\frac{|a|}{8} = \frac{2a + 3}{65}. \tag{5}$$

We now have the following two cases:
Case 1: $a \geq 0$
Then equation (5) becomes

$$\frac{a}{8} = \frac{2a + 3}{65}$$
$$a = \frac{24}{49} \geq 0.$$

Case 2: $a < 0$
Then equation (5) becomes

$$-\frac{a}{8} = \frac{2a + 3}{65}$$
$$a = -\frac{8}{27} < 0.$$

From Cases 1 and 2 we have that the possible values for a are $\frac{24}{49}$ and $-\frac{8}{27}$.