High School Math Problems
2017
Week 5
Problem and solution

Evaluate the expression

$$\left(\frac{x\sqrt{x}}{\sqrt{1-x^3}}+\frac{\sqrt{1-x^3}}{x\sqrt{x}}\right)^{-1},$$

if

$$x=\sqrt[3]{\frac{a-\sqrt{a^2-b^2}}{2a}},$$

where $a,b\in\mathbb{R}$ are such that $0<|a|\geq|b|>0$.

Solution:

$$\begin{aligned}\left(\frac{x\sqrt{x}}{\sqrt{1-x^3}}+\frac{\sqrt{1-x^3}}{x\sqrt{x}}\right)^{-1}&=\left(\frac{x^3+1-x^3}{x\sqrt{x}\sqrt{1-x^3}}\right)^{-1}\\&=\left(\frac{1}{\sqrt{x^3\left(1-x^3\right)}}\right)^{-1}=\sqrt{x^3\left(1-x^3\right)}\\&=\sqrt{\frac{a-\sqrt{a^2-b^2}}{2a}\left(1-\frac{a-\sqrt{a^2-b^2}}{2a}\right)}\\&=\sqrt{\frac{a-\sqrt{a^2-b^2}}{2a}\cdot\frac{a+\sqrt{a^2-b^2}}{2a}}\\&=\sqrt{\frac{a^2-a^2+b^2}{4a^2}}\\&=\frac{|b|}{2\,|a|}.\end{aligned}$$