High School Math Problems
2017
Week 4
Problem and Solution

Solve the equation

$$\sqrt[3]{p+y}+\sqrt{12-y}=6, \tag{1}$$

where p is the product of the integer values of b for which the equation

$$\frac{4b}{x+1}=3b-5 \tag{2}$$

has only negative values.

Solution:

The domain of equation (2) is $D_2=\{x : x\neq -1\}$.

To solve equation (2), we consider the following two cases:

Case 1: $b=\dfrac{5}{3}$

Then equation (2) becomes

$$\frac{20}{3\,(x+1)}=0,$$

which has no solution.

Case 2: $b\neq\dfrac{5}{3}$

Then from equation (2) we obtain that

$$x=\frac{4b}{3b-5}-1=\frac{4b-3b+5}{3b-5}=\frac{b+5}{3b-5}.$$

Now, $x=\dfrac{b+5}{3b-5}\in D_2$ if and only if

$$\begin{aligned}
&\frac{b+5}{3b-5}\neq -1\\
&b+5\neq -3b+5\\
&4b\neq 0\\
&b\neq 0.
\end{aligned} \tag{3}$$

Further, the solution of (2) is negative if and only if

$$\frac{b+5}{3b-5} < 0$$
$$b \in \left(-5, \frac{5}{3}\right),$$

which, combined with (3), gives that the integer values of b for which equation (2) has only negative values are $-4, -3, -2$, and -1.

Therefore $p = (-4) \cdot (-3) \cdot (-2) \cdot (-1) = 24$.

Thus equation (1) becomes

$$\sqrt[3]{24+y} + \sqrt{12-y} = 6.$$

The domain of equation (1) is $D_1 = \{y : y \leq 12\}$.

Let

$$z = \sqrt[3]{24+y}.$$

Then

$$y = z^3 - 24 \quad (4)$$

and equation (1) becomes

$$z + \sqrt{36 - z^3} = 6$$
$$\sqrt{36 - z^3} = 6 - z,$$

which is defined for $z \leq 6$.

Squaring both sides of the last equation, we obtain

$$36 - z^3 = 36 - 12z + z^2$$
$$z^3 + z^2 - 12z = 0$$
$$z\left(z^2 + z - 12\right) = 0$$
$$z(z+4)(z-3) = 0$$
$$z_1 = -4, z_2 = 0, z_3 = 3.$$

Substituting into (4), we obtain that the corresponding values of y are

$$y_1 = z_1^3 - 24 = -88 \in D_1,$$
$$y_2 = z_2^3 - 24 = -24 \in D_1,$$
$$y_3 = z_3^3 - 24 = 3 \in D_1.$$

Thus the solutions to (1) are $y_1 = -88$, $y_2 = -24$, and $y_3 = 3$.