High School Math Problems
2017
Week 28
Problem and Solution

For what values of the real parameter m does the inequality

$$-3 < \frac{x^2 + mx - 2}{x^2 - x + 1} < 2$$

hold for all $x \in \mathbb{R}$.

Solution:

We first observe that the polynomial $d(x) = x^2 - x + 1$ has the discriminant $D = 1 - 4 = -3 < 0$ and thus

$$d(x) = x^2 - x + 1 > 0 \tag{1}$$

for all $x \in \mathbb{R}$ since also the coefficient in front of x^2 in d is $1 > 0$.

Thus the given inequality is equivalent to the system of inequalities

$$\left| \begin{array}{l} \dfrac{-x^2 + (m+2)x - 4}{x^2 - x + 1} < 0 \\ \dfrac{4x^2 + (m-3)x + 1}{x^2 - x + 1} > 0 \end{array} \right. .$$

Using again (1), we find that the system above is equivalent to the system

$$\left| \begin{array}{l} -x^2 + (m+2)x - 4 < 0 \\ 4x^2 + (m-3)x + 1 > 0 \end{array} \right. \text{ for all } x \in \mathbb{R},$$

or

$$\left| \begin{array}{l} x^2 - (m+2)x + 4 > 0 \\ 4x^2 + (m-3)x + 1 > 0 \end{array} \right. \text{ for all } x \in \mathbb{R}.$$

The above inequalities will hold if and only if the discriminants of each quadratic polynomial is negative. Thus we must have that

$$\left| \begin{array}{l} (m+2)^2 - 16 < 0 \\ (m-3)^2 - 16 < 0 \end{array} \right.$$

or, equivalently, that

$$\left| \begin{array}{l} (m+2)^2 < 16 \\ (m-3)^2 < 16 \end{array} \right.$$

$$\left| \begin{array}{l} |m+2| < 4 \\ |m-3| < 4 \end{array} \right.$$

$$\left| \begin{array}{l} -6 < m < 2 \\ -1 < m < 7 \end{array} \right. .$$

Therefore we must have that

$$-1 < m < 2.$$