High School Math Problems
2017
Week 27
Problem and Solution

For $a > b > 0$, prove that

$$b\sqrt{2} \cdot \frac{2a + \sqrt{a^2 - b^2}}{\sqrt{a + \sqrt{a^2 - b^2}}} = \sqrt{(a+b)^3} - \sqrt{(a-b)^3}.$$

Solution:

We proceed as follows

$$\begin{aligned}
&\left(\sqrt{(a+b)^3} - \sqrt{(a-b)^3}\right)\sqrt{a + \sqrt{a^2 - b^2}} \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{(a+b)^3} - \sqrt{(a-b)^3}\right)\sqrt{\left(\sqrt{a+b} + \sqrt{a-b}\right)^2} \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{a+b} - \sqrt{a-b}\right)\left(a + b + \sqrt{a^2 - b^2} + a - b\right)\left(\sqrt{a+b} + \sqrt{a-b}\right) \\
&= \frac{1}{\sqrt{2}}(a + b - a + b)\left(2a + \sqrt{a^2 - b^2}\right) \\
&= b\sqrt{2} \cdot \left(2a + \sqrt{a^2 - b^2}\right).
\end{aligned}$$

Therefore

$$b\sqrt{2} \cdot \frac{2a + \sqrt{a^2 - b^2}}{\sqrt{a + \sqrt{a^2 - b^2}}} = \sqrt{(a+b)^3} - \sqrt{(a-b)^3}.$$