

High School Math Problems
2017
Week 26
Problem and Solution

Can the numbers $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ be elements of the same arithmetic progression, not necessarily consecutive?

Solution:

If $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ are elements of an arithmetic progression, then there exist $a \in \mathbb{R}$, $d \in \mathbb{R}_+$, $k, l, m \in \mathbb{N}$ such that

$$\sqrt{3} = a + kd, \quad \sqrt{5} = a + ld, \quad \sqrt{7} = a + md.$$

We observe that, since $\sqrt{7} \neq \sqrt{5} \neq \sqrt{3} \neq \sqrt{7}$, it follows that $k \neq l \neq m \neq k$.

Then

$$\frac{\sqrt{7} - \sqrt{5}}{\sqrt{7} - \sqrt{3}} = \frac{m - l}{m - k}. \quad (1)$$

However,

$$\frac{\sqrt{7} - \sqrt{5}}{\sqrt{7} - \sqrt{3}} \in \mathbb{I}$$

and

$$\frac{m - l}{m - k} \in \mathbb{Q}$$

and therefore (1) is impossible.

Thus numbers $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ cannot be elements of the same arithmetic progression.