High School Math Problems
2017
Week 25
Problem and Solution

For what values of the real parameter m do the equations

$$2x^2 - (3m+2)\,x + 3n = 0$$

and

$$4x^2 - (9m-2)\,x + 36 = 0$$

have a common root, if $n = \sqrt{9+4\sqrt{5}} - \sqrt{9-4\sqrt{5}}$?

Solution:

First we observe that

$$\begin{aligned} n &= \sqrt{9+4\sqrt{5}} - \sqrt{9-4\sqrt{5}} \\ &= \sqrt{2^2 + 2\cdot 2\cdot \sqrt{5} + \sqrt{5}^2} - \sqrt{2^2 - 2\cdot 2\cdot \sqrt{5} + \sqrt{5}^2} \\ &= \sqrt{\left(2+\sqrt{5}\right)^2} - \sqrt{\left(2-\sqrt{5}\right)^2} \\ &= 2 + \sqrt{5} - \left(\sqrt{5}-2\right) = 4. \end{aligned}$$

Therefore the first equation becomes

$$2x^2 - (3m+2)\,x + 12 = 0.$$

Let, now, x_0 be the common root of the two equations. Then we have that

$$\begin{aligned} 2x_0^2 - (3m+2)\,x_0 + 12 = 0 \\ 4x_0^2 - (9m-2)\,x_0 + 36 = 0. \end{aligned}$$

Multiplying the first equation by -2 and adding the result to the second equation, we now obtain

$$\begin{aligned} &(-3m+6)\,x_0 + 12 = 0 \\ &(m-2)\,x_0 = 4. \end{aligned}$$

Now, if $m = 2$, the last equation above has no solutions.
Working with $m \neq 2$, we, therefore, obtain that

$$x_0 = \frac{4}{m-2}.$$

Substituting into the first of the two given equations, we obtain

$$\begin{aligned}
&\frac{32}{(m-2)^2} - \frac{4(3m+2)}{m-2} + 12 = 0 \\
&32 - 4(3m+2)(m-2) + 12(m-2)^2 = 0 \\
&32 - 4\left(3m^2 - 4m - 4\right) + 12\left(m^2 - 4m + 4\right) = 0 \\
&32 - 12m^2 + 16m + 16 + 12m^2 - 48m + 48 = 0 \\
&-32m + 96 = 0 \\
&m = 3 \neq 2.
\end{aligned}$$

Therefore $m = 3$ is the solution.