High School Math Problems
2017
Week 23
Problem and Solution

Let m, n, p, and q be non-zero real numbers such that, if x_1 and x_2 are the roots of the equation $x^2+px+q=0$ and y_1 and y_2 are the roots of the equation $y^2+my+n=0$, then they satisfy the relation $x_1y_2=x_2y_1$.

Prove that

$$\frac{q}{n}=\frac{p^2}{m^2}.$$

Solution:

We observe that, since n and q are not zero, the roots $x_{1,2}$ and $y_{1,2}$ are also non-zero.

Therefore, from $x_1y_2=x_2y_1$ we have that

$$x_1=\frac{x_2y_1}{y_2}.$$

Now, from Vieta's Formulas we have that

$$p=-\left(x_1+x_2\right), \quad q=x_1x_2, \quad m=-\left(y_1+y_2\right), \quad n=y_1y_2.$$

Therefore

$$\frac{q}{n}=\frac{x_1x_2}{y_1y_2}=\frac{\dfrac{x_2y_1}{y_2}\cdot x_2}{y_1y_2}=\frac{x_2^2}{y_2^2}.$$

On the other hand

$$\frac{p^2}{m^2}=\frac{\left(x_1+x_2\right)^2}{\left(y_1+y_2\right)^2}=\frac{\left(\dfrac{x_2y_1}{y_2}+x_2\right)^2}{\left(y_1+y_2\right)^2}=\frac{x_2^2}{y_2^2}.$$

Therefore

$$\frac{q}{n}=\frac{p^2}{m^2}.$$