High School Math Problems
2017
Week 21
Problem and Solution

Find the value of the real parameter a, if the equation $x^4 - 10x^2 + a = 0$ has four real roots, which form an arithmetic progression.

Solution:

We first observe that if $r \in \mathbb{R}$ is a solution of the equation, then $-r$ is also a solution. Since the roots of the equation also form an arithmetic progression, it, now, follows that they have the form $q, q + d$, $q + 2d$, and $q + 3d$, where $q < 0$, $d > 0$, and

$$\begin{aligned} q &= -(q + 3d) \\ q + d &= -(q + 2d). \end{aligned}$$

From the last two equations we obtain that

$$q = -\frac{3}{2}d.$$

Therefore the roots of the equation are

$$-\frac{3}{2}d, -\frac{1}{2}d, \frac{1}{2}d, \frac{3}{2}d.$$

From this obtain that

$$\begin{aligned} x^4 - 10x^2 + a &= \left(x + \frac{3}{2}d\right)\left(x + \frac{1}{2}d\right)\left(x - \frac{1}{2}d\right)\left(x - \frac{3}{2}d\right) \\ &= \left(x^2 - \frac{1}{4}d^2\right)\left(x^2 - \frac{9}{4}d^2\right) \\ &= x^4 - \frac{10}{4}d^2x^2 + \frac{9}{16}d^4. \end{aligned}$$

Therefore

$$\left| \begin{array}{l} 1 = \dfrac{d^2}{4} \\ a = \dfrac{9}{16} 4d^4 \end{array} \right. ,$$

from which we obtain that $d = 2$ and $a = 9$.