High School Math Problems
2017
Week 17
Problem and Solution

Let a_1, a_2, $\ldots$, a_n, $\ldots$, be a sequence of real numbers, which form an arithmetic progression with common difference d. For every $n \in \mathbb{N}$, let $S_n = a_1 + a_2 + \cdots + a_n$. Further, for $x \in \mathbb{R}$ let $f(x) = x^2 + (2p-1)x + 1$, where $p \in \mathbb{R}$, and assume that

$$2S_n + 1 = f(n) \tag{1}$$

for all $n \in \mathbb{N}$. Suppose also that the roots of the equation $f(x) = 0$ are real. If, lastly,

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 - a_1 a_2 a_3 a_4 = \frac{15}{16}, \tag{2}$$

find a_1, d, and p.

Solution:

Since $S_1 = a_1$, from (1) with $n = 1$ we obtain that

$$\begin{aligned} 2a_1 + 1 &= 1 + 2p - 1 + 1 \\ a_1 &= p. \end{aligned} \tag{3}$$

From (1) with $n = 2$ we therefore further have that

$$\begin{aligned} 2(2p + d) + 1 &= 4 + 2(2p - 1) + 1 \\ d &= 1. \end{aligned}$$

Substituting, now, into (2), we obtain that

$$\begin{aligned} p^2 + (p+1)^2 + (p+2)^2 + (p+3)^2 - p(p+1)(p+2)(p+3) &= \frac{15}{16} \\ 4\left(p^2 + 3p\right) + 14 - \left(p^2 + 3p\right)\left(p^2 + 3p + 2\right) &= \frac{15}{16}. \end{aligned}$$

Letting $m = p^2 + 3p$, we obtain that the above equation becomes

$$4m - m(m+2) + 14 - \frac{15}{16} = 0$$
$$m^2 - 2m - \frac{209}{16} = 0$$
$$16m^2 - 32m - 209 = 0$$
$$m_{1,2} = \frac{16 \pm \sqrt{3600}}{16} = \frac{16 \pm 60}{16} = \begin{cases} \frac{19}{4} \\ -\frac{11}{4} \end{cases}.$$

We, therefore, have the following two equations for p:

$$p^2 + 3p = \frac{19}{4}$$
$$4p^2 + 12p - 19 = 0$$
$$p_{1,2} = \frac{-6 \pm \sqrt{112}}{4} = \frac{-6 \pm 4\sqrt{7}}{4} = \frac{-3 \pm 2\sqrt{7}}{2}$$

$$p^2 + 3p = -\frac{11}{4}$$
$$4p^2 + 12p + 11 = 0$$

This equation has the discriminant $D = 36 - 44 = -8 < 0$ and therefore has no real solutions.

(4)

Lastly, from the fact that the equation

$$f(x) = x^2 + (2p-1)x + 1 = 0$$

has real solutions, we obtain that its discriminant

$$D = (2p-1)^2 - 4 = 4p^2 - 4p - 3$$

must be non-negative.

Thus p must satisfy the inequality

$$4p^2 - 4p - 3 \geq 0,$$

whose solution is

$$p \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{3}{2}, \infty\right). \tag{5}$$

Combining (4) with (5) and (3), we obtain that

$$p = \frac{-3 - 2\sqrt{7}}{2} = a_1.$$