High School Math Problems
2017
Week 13
Problem and Solution

Solve the inequality

$$6\sqrt{y} > \frac{y^2 + 12y + 4}{y + 2}.$$

Solution:

The domain of the inequality is $D = \{y : y \geq 0\}$.

We now consider the following two cases:

Case 1: $y = 0$

Then the inequality becomes $0 > 2$, which is incorrect.

Therefore $y = 0$ is not a solution.

Case 2: $y > 0$

Then $y + 2 > 0$ and from the inequality we obtain

$$\begin{aligned}
6\sqrt{6}\,(y+2) &> y^2 + 12y + 4 \\
\sqrt{6}\,(y+2) &> (y+2)^2 + 8y \\
\frac{6\,(y+2)}{\sqrt{y}} &> \left(\frac{y+2}{\sqrt{y}}\right)^2 + 8.
\end{aligned}$$

Letting, now, $z = \dfrac{y+2}{\sqrt{y}}$, we have that

$$z = \sqrt{y} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y}} \geq 2 + \frac{1}{\sqrt{y}} > 2 \tag{1}$$

and we obtain that the above inequality becomes

$$6z > z^2 + 8. \tag{2}$$

The last inequality is equivalent to

$$z^2 - 6z + 8 > 0,$$

from which we obtain that

$$z \in (-\infty, 2) \cup (4, \infty)\,. \tag{3}$$

From (1) and (3) we obtain that

$$\begin{aligned} z = \frac{y+2}{\sqrt{y}} > 4 \\ y - 4\sqrt{y} + 2 > 0. \end{aligned}$$

Setting, now, $t = \sqrt{y} > 0$, we obtain

$$\begin{aligned} & t^2 - 4t + 2 > 0 \\ & t \in \left(-\infty, 2 - \sqrt{2}\right) \cup \left(2 + \sqrt{2}, \infty\right). \end{aligned}$$

Since $t > 0$, we therefore must have that

$$t = \sqrt{y} \in \left(0, 2 - \sqrt{2}\right) \cup \left(2 + \sqrt{2}, \infty\right).$$

Therefore

$$y \in \left(0, \sqrt{2 - \sqrt{2}}\right) \cup \left(\sqrt{2 + \sqrt{2}}, \infty\right).$$