High School Math Problems
2017
Week 12
Problem and Solution

Solve the inequality

$$\sqrt{x-1}+\sqrt{x^2-1}<\sqrt{x^3}.$$

Solution:

The domain of the inequality is

$$D=\{x : x\in[1,\infty)\}\,. \tag{1}$$

Since both sides of the inequality are non-negative we can square both sides and obtain

$$\begin{aligned} &x-1+x^2-1+2\sqrt{(x-1)\left(x^2-1\right)}<x^3\\ &2\sqrt{x^3-x^2-x+1}<x^3-x^2-x+2. \end{aligned}$$

Let $y=x^3-x^2-x+1\geq 0$.

Then the above inequality becomes

$$\begin{aligned} &2y<y^2+1\\ &y^2-2y+1>0\\ &(y-1)^2>0\\ &y\in[0,1)\cup(1,\infty)\,. \end{aligned}$$

This is equivalent to

$$\begin{aligned} &x\in D\\ &\sqrt{x^3-x^2-x+1}\neq 1. \end{aligned} \tag{2}$$

To find the values of x, which satisfy the latter condition, we now consider

$$\begin{aligned} &x^3-x^2-x+1=1\\ &x^3-x^2-x=0\\ &x\left(x^2-x-1\right)=0\\ &x_1=0, x_{2,3}=\frac{1\pm\sqrt{5}}{2}. \end{aligned} \tag{3}$$

From (2), (3), and (1) we now obtain that

$$x \in \left[1, \frac{1+\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right).$$