High School Math Problems
2017
Week 11
Problem and Solution

Let x, y, and m be real numbers such that

$$\left| \begin{array}{l} x + y = 2m - 1 \\ x^2 + y^2 = m^2 + 2m - 3 \end{array} \right. .$$

For what value(s) of m does $x \cdot y$ achieve its maximum and minimum values, if they exist?

Solution:

We have that

$$x \cdot y = \frac{(x+y)^2 - (x^2 + y^2)}{2} = \frac{(2m-1)^2 - (m^2 + 2m - 3)}{2} = \frac{3m^2 - 6m + 4}{2}.$$

Now, $\dfrac{3m^2 - 6m + 4}{2}$ is a quadratic polynomial in m with a positive coefficient in front of m^2. Therefore it achieves its minimum at $m_0 = 1$ and has no maximum value.

Thus the minimum value of $x \cdot y$ is $\dfrac{3m_0^2 - 6m_0 + 4}{2} = \dfrac{3 \cdot (1)^2 - 6 \cdot (1) + 4}{2} = \dfrac{1}{2}$ and $x \cdot y$ has no maximum value.