High School Math Problems
2016
Week 4
Problem and Solution

Solve the equation

$$\frac{1}{2n+nx}-\frac{1}{2x-x^2}=\frac{2\left(n+3\right)}{x^3-4x},$$

where n is a real parameter.

Solution:

The domain of the parameter is $D_n=\{n:n\neq 0\}$ and the domain of the variable is $D_x=\{x:x\neq 0,\pm 2\}$.

$$\begin{aligned}
&\frac{1}{2n+nx}-\frac{1}{2x-x^2}=\frac{2\left(n+3\right)}{x^3-4x}\\
&\frac{1}{\left(2+x\right)n}+\frac{1}{\left(x-2\right)x}-\frac{2\left(n+3\right)}{x\left(x-2\right)\left(x+2\right)}=0\\
&x\left(x-2\right)+\left(2+x\right)n-2n\left(n+3\right)=0\\
&x^2+\left(n-2\right)x-2n^2-4n=0
\end{aligned}$$

The discriminant of this equation is

$$D=\left(n-2\right)^2-4\left(-2n^2-4n\right)=n^2-4n+4+8n^2+16n=9n^2+12n+4=\left(3n+2\right)^2.$$

Therefore the roots of the equation are

$$x_{1,2}=\frac{2-n\pm\sqrt{\left(3n+2\right)^2}}{2}=\frac{2-n\pm\left(3n+2\right)}{2}=\begin{cases}n+2\\-2n\end{cases}.$$

Now, $n+2\in D_x$ if and only if $n+2\neq 0$, $n+2\neq 2$, and $n+2\neq -2$, which is equivalent to $n\neq -4,-2,0$.

Similarly, $-2n\in D_x$ if and only if $-2n\neq 0$, $-2n\neq 2$, and $-2n\neq -2$, which is equivalent to $n\neq -1,0,1$.

Thus we have that

- if $n = 0$, the equation is not defined
- if $n = -4$ or $n = -, 2$ the solution to the equation is $x = -2n$
- if $n = -1$ or $n = 1$, the solution to the equation is $x = n + 2$
- if $n \neq -4, -2, -1, 1$, then the solutions to the equation are $x_1 = n + 2$ and $x_2 = -2n$.