High School Math Problems
2016
Week 49
Problem and Solution

Let $k \in \mathbb{N}$, $k > 3$. Let, further, $(a_l)_{l=1}^k \subset \mathbb{R}$ be an arithmetic progression such that $\left(\sqrt{1-a_l^2}\right)_{l=1}^k$ is a geometric progression.

Prove that $a_1 = a_2 = \cdots = a_k$.

Solution:

Fist we observe that since $\sqrt{1-a_l^2}$ is defined for all $l = 1, 2, \ldots, k$, it follows that

$$a_l \in [-1, 1] \text{ for all } l = 1, 2, \ldots, k. \tag{1}$$

Now, since $(a_l)_{l=1}^k$ is an arithmetic progression, it follows that

$$a_l = \frac{a_{l-1} + a_{l+1}}{2} \text{ for all } l = 2, 3, ..., k-1, \tag{2}$$

and since $\left(\sqrt{1-a_l^2}\right)_{l=1}^k$ is a geometric progression, we have that

$$1 - a_l^2 = \sqrt{\left(1 - a_{l-1}^2\right)\left(1 - a_{l+1}^2\right)} \text{ for all } l = 2, 3, ..., k-1. \tag{3}$$

Substituting, now, (2) into (3) and simplifying, we obtain that

$$
\begin{aligned}
&1-\left(\frac{a_{l-1}+a_{l+1}}{2}\right)^2=\sqrt{\left(1-a_{l-1}^2\right)\left(1-a_{l+1}^2\right)}\\
&1-\frac{a_{l-1}^2+2a_{l-1}a_{l+1}+a_{l+1}^2}{4}=\sqrt{\left(1-a_{l-1}^2\right)\left(1-a_{l+1}^2\right)}\\
&4-a_{l-1}^2-2a_{l-1}a_{l+1}-a_{l+1}^2=4\sqrt{1-a_{l-1}^2-a_{l+1}^2+a_{l-1}^2a_{l+1}^2}\\
&\left(4-a_{l-1}^2-2a_{l-1}a_{l+1}-a_{l+1}^2\right)^2=16\left(1-a_{l-1}^2-a_{l+1}^2+a_{l-1}^2a_{l+1}^2\right)\\
&16+a_{l-1}^4+4a_{l-1}^2a_{l+1}^2+a_{l+1}^4-8a_{l-1}^2-16a_{l-1}a_{l+1}-8a_{l+1}^2+4a_{l-1}^3a_{l+1}+2a_{l-1}^2a_{l+1}^2+4a_{l-1}a_{l+1}^3\\
&\quad=16-16a_{l-1}^2-16a_{l+1}^2+16a_{l-1}^2a_{l+1}^2\\
&a_{l-1}^4+a_{l+1}^4-10a_{l-1}^2a_{l+1}^2+4a_{l-1}^3a_{l+1}+4a_{l-1}a_{l+1}^3+8a_{l-1}^2+8a_{l+1}^2-16a_{l-1}a_{l+1}=0\\
&\left(a_{l-1}^2-a_{l+1}^2\right)^2-8a_{l-1}^2a_{l+1}^2+4a_{l-1}^3a_{l+1}+4a_{l-1}a_{l+1}^3+8\left(a_{l-1}^2+a_{l+1}^2-2a_{l-1}a_{l+1}\right)=0\\
&\left(a_{l-1}^2-a_{l+1}^2\right)^2+4a_{l-1}a_{l+1}\left(-2a_{l-1}a_{l+1}+a_{l-1}^2+a_{l+1}^2\right)+8\left(a_{l-1}-a_{l+1}\right)^2=0\\
&\left(a_{l-1}^2-a_{l+1}^2\right)^2+4a_{l-1}a_{l+1}\left(a_{l-1}-a_{l+1}\right)^2+8\left(a_{l-1}-a_{l+1}\right)^2=0\\
&\left(a_{l-1}-a_{l+1}\right)^2\left(a_{l-1}+a_{l+1}\right)^2+4a_{l-1}a_{l+1}\left(a_{l-1}-a_{l+1}\right)^2+8\left(a_{l-1}-a_{l+1}\right)^2=0\\
&\left(a_{l-1}-a_{l+1}\right)^2\left(\left(a_{l-1}+a_{l+1}\right)^2+4a_{l-1}a_{l+1}+8\right)=0.
\end{aligned}
\tag{4}
$$

Since from (1) we have that $a_{l-1}a_{l+1}\in[-1,1]$ for every $l=2,3,\ldots,k-1$, it follws that $\left(a_{l-1}+a_{l+1}\right)^2+4a_{l-1}a_{l+1}+8>0$ for all $l=2,3,\ldots,k-1$.

From the last line of (4) we, therefore, obtain that

$$\left(a_{l-1}-a_{l+1}\right)^2=0 \text{ for } l=2,3,\ldots,k-1,$$

or, equivalently, that

$$a_{l-1}=a_{l+1} \text{ for } l=2,3,\ldots,k-1.$$

Now, from (2) we obtain that

$$a_l=\frac{a_{l-1}+a_{l+1}}{2}=a_{l-1}=a_{l+1}$$

for $l=2,3,\ldots,k-1$, which completes the proof of the assertion.