High School Math Problems
2015
Week 31
Problem and Solution

Let a, b, and c be the lengths of the sides of a triangle and let α, β, and γ be the angles lying opposite to them, respectively.

Prove that the identity

$$a^4 = b^4 + c^4$$

is equivalent to

$$2\sin^2\alpha = \tan\beta\tan\gamma.$$

Proof:

I. Suppose first that $2\sin^2\alpha = \tan\beta\tan\gamma$. We shall prove that $a^4 = b^4 + c^4$.

Let R be the radius of the circumscribed circles of the triangle.

From the Law of Sines and the Law of Cosines we have that

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac} \text{ and } \sin\beta = \frac{b}{2R},$$

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab} \text{ and } \sin\gamma = \frac{c}{2R}.$$

Therefore

$$\begin{aligned}\tan\beta\tan\gamma &= \frac{\frac{b}{2R}}{\frac{a^2+c^2-b^2}{2ac}} \cdot \frac{\frac{c}{2R}}{\frac{a^2+b^2-c^2}{2ab}} \\ &= \frac{(abc)^2}{(a^2+b^2-c^2)(a^2+b^2-c^2)R^2}.\end{aligned}$$

Since from the Law of Sines

$$2\sin^2\alpha = \frac{a^2}{2R^2},$$

we, therefore, have that

$$\begin{aligned}
&\frac{(abc)^2}{(a^2+b^2-c^2)(a^2+c^2-b^2)R^2} = \frac{a^2}{2R^2} \\
&2b^2c^2 = a^4 - \left(b^2-c^2\right)^2 \\
&a^4 = b^4 + c^4.
\end{aligned}$$

II. Now suppose first that $a^4 = b^4 + c^4$. We shall prove that $2\sin^2\alpha = \tan\beta\tan\gamma$. The assertion follows by reversing the steps of the proof in part I above.