High School Math Problems
2015
Week 22
Problem and Solution

Solve the equation

$$\sqrt[3]{\frac{a+x}{a-x}}-\sqrt[3]{\frac{a-x}{a+x}}=\sqrt[3]{\frac{b+x}{b-x}}-\sqrt[3]{\frac{b-x}{b+x}}, \tag{1}$$

where a and b are real parameters.

Solution:
The domain of the equation is $D_x=\{x : x\neq \pm a, \pm b\}$.
We raise both sides of (1) to power 3 and obtain

$$\frac{a+x}{a-x}-3\left(\sqrt[3]{\frac{a+x}{a-x}}-\sqrt[3]{\frac{a-x}{a+x}}\right)-\frac{a-x}{a+x}=\frac{b+x}{b-x}-3\left(\sqrt[3]{\frac{b+x}{b-x}}-\sqrt[3]{\frac{b-x}{b+x}}\right)-\frac{b-x}{b+x}. \tag{2}$$

Multiplying, now, (1) by 3 and adding it to (2), we obtain

$$\begin{aligned}
&\frac{a+x}{a-x}-\frac{a-x}{a+x}=\frac{b+x}{b-x}-\frac{b-x}{b+x}\\
&\frac{4ax}{a^2-x^2}=\frac{4bx}{b^2-x^2}\\
&ax\left(b^2-x^2\right)=bx\left(a^2-x^2\right)\\
&x\left(ab^2-ax^2-a^2b+bx^2\right)=0\\
&(b-a)\,x\left(x^2+ab\right)=0.
\end{aligned}$$

Therefore we have the following cases:
Case 1: $a=b$
Then any $x\in D_x$ is a solution.
Case 2: $a\neq b$ and $ab>0$
Then $x=0$ is the only solution.
Case 3: $a\neq b$ and $ab<0$

Then $x = 0$ and $x = \pm\sqrt{-ab}$ are the solutions.
Therefore, for the solutions of the equation we have that

$$\begin{aligned}
&\text{if } a = b, && \text{then } x \neq a, b \\
&\text{if } a \neq b \text{ and } ab > 0, && \text{then } x = 0 \\
&\text{if } a \neq b \text{ and } ab < 0, && \text{then } x = 0 \text{ and } x = \pm\sqrt{-ab}.
\end{aligned}$$