

1. Consider a population of workers consisting of equal numbers of two types, $t = l, h$. The marginal productivity of a worker with education level $e \in [0, 16]$ is e for $t = l$ and $2e$ for $t = h$. Workers know their types but the firms cannot identify them directly. The firms are risk neutral. They base their wage offers on e , which is observable, and compete in a Bertrand manner: they make wage offers to a worker that is equal to their expected marginal productivity of the worker. Utility functions $u^t(w, e)$ of workers $t = h, l$ are strictly concave and satisfy $u_w^t(w, e) > 0, u_e^t(w, e) < 0$ (subscripts indicate partial derivatives), and the *single crossing property*: $-u_e^l(w, e)/u_w^l(w, e) > -u_e^h(w, e)/u_w^h(w, e)$ at each (w, e) . The wage expectation of a worker with education e is denoted by $w(e)$. A worker of type t chooses $e = e_t$ that maximizes $u^t(w(e), e)$. For simplicity, we assume that all maximizers are unique in this problem. Let $\mu(e)$ be the belief of firms that a worker is type h given their education level of e .

a. Show why the single crossing assumption implies that an indifference curve of type h worker meets an indifference curve of type l worker at most once.

Answer At each (w, e) , $u_w^t(w, e)dw + u_e^t(w, e)de = 0$. So the slope of indifference curve at (w, e) for type t worker is $dw/de = -u_e^t(w, e)/u_w^t(w, e)$. If the indifference curves cross more than once, the condition will be violated.

b. Draw the game tree of the above signaling game.

Answer: a game tree is given in MWG.

c. What is a (pure) strategy of a worker? What is a (pure) strategy of a firm?

Answer: a pure strategy of a worker is a function from $\{h, l\}$ to $[0, 16]$. A pure strategy of a firm is a function mapping $e \in [0, 16]$ to an wage offer $w(e)$.

d. What would be an ‘equilibrium’ if employers could identify ability? Draw a graph to illustrate your answer.

Answer. Figure 17.3 of Kreps.

e. It can be shown that in a sequential equilibrium, firms’ beliefs on the type of workers are the same and thus they offer the same $w(e)$ after observing e . Show that in a sequential equilibrium, competing firms offer a wage $w(e)$ between e and $2e$ to a worker with education level e .

Answer: $w(e) = \mu(e)2e + (1 - \mu(e))e$, where $\mu(e) \in [0, 1]$.

f. Under what assumption on $\mu(e)$, would the wage offer function $w(e)$ be increasing in e in a sequential equilibrium?

Answer: Since $w(e) = \mu(e)2e + (1 - \mu(e))e$, $w(e)$ is increasing in e if and only if $\mu(e) + 1 + \mu'(e)e > 0$ almost everywhere.

g. Let $\hat{e} = \arg \max_e u^l(e, e)$. We assume that $\hat{e} < 16$. Show that type l obtains at least the utility level $u^l(\hat{e}, \hat{e})$ at any (separating or pooling) sequential equilibrium.

Answer: The worst wage that a type t worker can get after obtaining education level e is e . Thus, a type l worker can assure himself at least $u^l(\hat{e}, \hat{e})$.

h. Consider an education level e satisfying $u^l(2e, e) < u^l(\hat{e}, \hat{e})$. Explain why a type l would not choose such an education level.

Answer: type l can guarantee itself a utility level $u^l(\widehat{e}, \widehat{e})$ by choosing \widehat{e} whereas by choosing e , the best he can possibly get is $u^l(2e, e)$. The firms know this and faced with such e (if it is not eliminated by type h as well), consider the worker as high quality and pay $2e$.

i. Let \widehat{f}_1 be the level of education satisfying $u^l(2\widehat{f}_1, \widehat{f}_1) = u^l(\widehat{e}, \widehat{e})$ and let $\widehat{f}_2 = \arg \max_{\widehat{f}_1 \leq e \leq 16} u^h(2e, e)$. That is, $e_h = \widehat{f}_2$ maximizes $u^h(2e, e)$ for any education level e at least as large as \widehat{f}_1 . Assume in the rest of the problem 1. $\widehat{f}_1 < 16$, 2. $\widehat{f}_2 > \widehat{f}_1$ and 3. type h indifference curve passing through $(32, 16)$ does not meet (e, e) . Explain why type h worker can guarantee himself the utility level of $u^h(2\widehat{f}_2, \widehat{f}_2)$ in any sequential equilibrium.

Answer: For $\widehat{f}_2 > \widehat{f}_1$, $u^l(2\widehat{f}_2, \widehat{f}_2) < u^l(\widehat{e}, \widehat{e})$. Thus, type l worker will not choose \widehat{f}_2 . The indifference curve of type h passing through $(32, 16)$ does not meet (e, e) line. Thus, there is no e such that $u^h(e, e) > u^h(2\widehat{f}_2, \widehat{f}_2)$. This means that the education level \widehat{f}_2 is not eliminated by type h by such a consideration as in $(h.)$. Thus, if the education level $\widehat{f}_2 > \widehat{f}_1$ is observed, a firm believes that the worker type is h . and offers $2\widehat{f}_2$.

j. Explain why at a separating equilibrium, $e_h = \widehat{f}_2$. Illustrate such a separating equilibrium with a graph.

Answer: At a separating equilibrium, e_h cannot be smaller than \widehat{f}_1 and get $2e_h$; Otherwise, type l would prefer these over $(\widehat{e}, \widehat{e})$. For $e_h = \widehat{f}_2$, type h gets an wage offer of $2\widehat{f}_2$ since $\mu(\widehat{f}_2) = 1$. Likewise, for any $e_h > \widehat{f}_1$, $w(\widehat{f}_1) = 2\widehat{f}_1$. One can draw $w(\cdot)$ function supporting $\{e_l = \widehat{e}, e_h = \widehat{f}_2\}$ and respecting above properties.

k. Under the assumption that $\widehat{f}_2 > \widehat{f}_1$, there cannot be a pooling equilibrium. Explain.

Answer: A pooling equilibrium needs to be on the $(1.5e, 1.5e)$ line. But in this case, no point on the $(1.5e, 1.5e)$ line can give type h at least $u_h(2\widehat{f}_2, \widehat{f}_2)$, that type h can guarantee himself.

2. A society is composed of three agents, A , B and C with von Neumann and Morgernstern (Bernoulli) utilities for money $u_A(m) = \sqrt{m}$, $u_B(m) = m$ and $u_C(m) = m^2$, respectively. In the absence of redistribution, the agents' gross incomes would be $m_A = 46$, $m_B = 92$ and $m_C = 138$, respectively.

Now, imagine that the agents were behind a (thin) *veil of ignorance* in which they know their preferences (as above), but not which of the three income levels they will be assigned. That is, with equal probability, one of the agents will receive gross income 46, one will receive 92 and one will receive 138, but it is not yet known which.

a. Devise a social welfare system, or a redistribution scheme, to be implemented after incomes are realized, so as to maximize the "outcome" of the "worst off" individual (i.e., the lowest ranked agent according to that outcome measure) when the outcome is measured by:

i. income

Answer: equate income at \$92

ii. expected utility (tax conditional solely on income)

Answer: equate income

iii. ex post utility (tax can be conditional on identity)

Answer: equate utility at $\mathbf{m} = (256, 16, 4)$

b. Such a *Rawlsian* social objective is often taken to rationalize the provision of social insurance among risk averse agents. For example, if agent *A* were to vote over redistributive tax policies, its preferences would accord with those of a Rawlsian social planner. Is this social objective function appropriate here? And why?

Answer: neither 1 nor 3 is risk averse

c. Contrast your answers to part (a) with the case of a *Utilitarian* social planner who seeks to maximize the aggregate “outcome”. Again consider each of the three outcome measures.

Answers:

(i) maximize $\sum m_i$ – all allocations equivalent

(ii) give all to any one agent

(iii) give all to 3

Next, consider a *thick* veil of ignorance behind which the agents do not even know their preferences (although it is known they are expected utility maximizers.) Now assume there is an equal chance of each agent being assigned to each utility function and receiving any of the three income levels.

d. How does this affect your answers to part (a)?

Answers:

(i) equate income

(ii) equate income

(iii) equate utility at $\mathbf{m} = (256, 16, 4)$

3. Several trucking firms buy diesel oil from refineries at a port and deliver it to a large city. The oil is used for home heating and as fuel for trucks. Let q_j be the amount of diesel oil delivered to the city by trucking firm j . The inverse demand function for diesel oil in the city is $p = a - bq$, where p is the price of diesel oil sold in the city and q is the total quantity delivered by the trucking firms ($a > 0$, $b > 0$). The refineries charge r per gallon for diesel oil. Each trucking firm fuels its own trucks using $\gamma < 1$ gallons of diesel oil per gallon it delivers to the city. It also has other variable costs of w per gallon delivered and fixed costs.

a. Consider the game played by the trucking firms, in which they independently choose the quantities of diesel oil they deliver, and their payoffs are their profits from the sale of oil in the city. Find a pure strategy Nash equilibrium (NE) of the game in terms of the exogenous variables of the problem. Is there only one pure strategy Nash equilibrium?

Firm j maximizes $\pi_j = (a - bq)q_j - rq_j - r\gamma q_j - wq_j - F_j$, where F_j is its fixed cost and where $q = \sum q_i$. The first order condition is $a - r(1 + \gamma) - w - bq - bq_j \leq 0$, with equality if $q_j > 0$. If J firms deliver oil, they all deliver the same amount q_j with $bq = bJq_j = J[a - r(1 + \gamma) - w - bq]$, hence $q_j = q/J = [a - r(1 + \gamma) - w]/[(J + 1)b]$ and, from the first order condition, $\pi_j = [a - bq - r(1 + \gamma) - w]q_j - F_j = bq_j^2 - F_j = b^{-1}(J + 1)^{-2}[a - r(1 + \gamma) - w]^2 - F_j$. To see how many firms deliver, it is easiest to renumber the firms so that F_j is nondecreasing in j . Let J^* be the largest J such that

$$b^{-1}(J + 1)^{-2}[a - r(1 + \gamma) - w]^2 - F_J > 0. \quad (*)$$

There is a Nash equilibrium with J^* firms delivering since each of those J^* firms maximizes its profit, given the quantities the others deliver, and no other firm can make positive profit by delivering oil. If the left side of (*) equals 0 when $J = J^* + 1$, then there is also NE with $J^* + 1$ firms delivering, assuming there are at least that many firms. The only way there can be just one pure strategy NE is if there are no more than J^* firms. Otherwise there are different NE with different sets of J^* firms delivering.

b. If the price r were slightly higher what would be the effect on the equilibrium profit of a typical trucking firm and on the price p of diesel oil in the city in the Nash equilibrium you found in part a? Is the change in equilibrium p as large as the change in r ? Interpret your answer.

Slightly higher r reduces the equilibrium π_j without changing the number of firms that deliver: $\partial\pi_j/\partial r = -2(1 + \gamma)[a - r(1 + \gamma) - w]/[b(J^* + 1)^2] < 0$. The equilibrium values of q_j and q also fall, so p rises: $\partial p/\partial r = -\partial(bq)/\partial r = J^*(1 + \gamma)/(J^* + 1)$. This derivative is bigger or smaller than 1 = $\partial r/\partial r$, depending on whether $J^*\gamma$ is bigger or smaller than 1. If the initial refinery price r and per unit variable cost w are small relative to a (the lowest price at which there is 0 demand for oil in the city), and if the fixed costs are small, then many firms deliver in equilibrium, and a small increase in the refinery price increases the price in the city by more. The refinery price rise is magnified because it raises the transportation cost along with the cost of the oil being delivered. If instead the fixed costs are sufficiently high, given the other parameters, then few firms deliver in equilibrium. A rise in the refinery price raises the price of oil in the city by less. The imperfectly competitive trucking industry partially insulates the city market from the input price rise.

c. How would the equilibrium profit of trucking firm 1 differ if γ were slightly smaller? Be as specific as possible.

A small decrease in γ has the same effect as some decrease in r . It raises the equilibrium profit of firm 1: $-\partial\pi_1/\partial\gamma = 2r[a - r(1 + \gamma) - w]/[b(J^* + 1)^2] > 0$.

d. Carefully formulate a system of equations characterizing Nash equilibrium if the trucks of firm 1 used their fuel more efficiently, while the fuel efficiency of all the other trucks (along with all other exogenous variables) remained as before. You do not need to find the new equilibrium. Compare the effect of this improvement in efficiency to the effect of a small reduction in γ (the effect found in part c). Explain the comparison. Can the envelope theorem be applied to compute the effect of a slight improvement in the fuel efficiency of the trucks of firm 1 on that firm's profit? Explain why or why not.

Let γ_1 be the efficiency parameter of firm 1. The first order conditions characterizing NE remain the same for every firm $j > 1$. For firm 1, the first order condition becomes $a - r(1 + \gamma_1) - w - bq - bq_1 \leq 0$, with equality if $q_1 > 0$. In part c, every equilibrium q_j is higher when γ is smaller. When only γ_1 changes, the equilibrium q_1 is higher, and as a result every other q_j is lower. This makes the equilibrium q_1 and the profit of firm 1 higher in part d than in part c.

In an optimization problem, the value function is the optimized value of the objective function treated as a function of the parameters. The envelope theorem says that the derivative of the value function with respect to a parameter equals the partial derivative of the objective function with respect to that parameter when the control variables are fixed at their values that are optimal for the initial values of the parameters. When the parameter γ_1 changes, the profit of firm 1 changes, but the derivative of the profit is not equal to the partial derivative with the control variable held fixed unless we take account of the effect of a change in γ_1 on the equilibrium levels of the other firms' outputs q_j . The envelope theorem does not apply here since the other q_j variables are not control variables in the optimization problem of firm 1, but their equilibrium values do change in response to a change in γ_1 .

e. Starting from the original Nash equilibrium in part a, suppose that without bearing any additional costs firm 1 could save a small fixed amount of oil by improving the fuel efficiency of its trucks. What would be the value of this efficiency improvement to firm 1 per unit of oil saved? Is it the price r that the firm pays for oil at the refinery, or the price p at which it sells oil in the city, or something in between? Explain.

The easiest way to approach the problem is to let q_j continue to represent oil delivered. For any amount delivered, the firm now can buy x units less from the refinery than in the original problem, saving itself rx . The formula for the profit of firm 1 becomes $\pi_1 = (a - bq)q_1 - rq_1 - wq_1 - r\gamma q_1 - F_1 + rx$. Since x is fixed and small, the change is like a change in fixed cost and has no effect on the equilibrium values of q_j , so the value to the firm per unit of oil saved is r .

4. Consider the following strategic-form game.

	L	C	R
T	-1, 2	2, -1	0, 0
M	2, -1	-1, 2	0, 0
D	0, 0	0, 0	-1, -1

- (a) Find a Nash equilibrium.
 (b) Show that the equilibrium is unique.

Next consider the game with all payoffs multiplied by -1 :

	L	C	R
T	1, -2	-2, 1	0, 0
M	-2, 1	1, -2	0, 0
D	0, 0	0, 0	1, 1

- (c) Show that there is a unique Nash equilibrium for this game.

Answer:

(a) The Nash is $(\frac{1}{2}T + \frac{1}{2}M, \frac{1}{2}L + \frac{1}{2}C)$.

(b) One can show that D and R are strictly dominated. For example, if the column player plays L, C, R with probabilities ℓ, c, r , then row player's expected payoffs from her three choices are:

$$T: -\ell + 2c$$

$$M: 2\ell - c$$

$$D: -r$$

Some combination of T and M, say $\frac{1}{2}T + \frac{1}{2}M$, dominates D , as

$$\frac{1}{2}(\ell + c) > -r.$$

Then uniqueness can be established by showing that in the 2×2 game

	L	C
T	-1, 2	2, -1
M	2, -1	-1, 2

there is no pure strategy Nash and the above equilibrium is the only mixed strategy Nash. Alternatively, one could plot best response functions to demonstrate there is only one intersection point.

(c) One should see that there is a Nash (D, R). To prove uniqueness, we can follow the following steps.

Let $((t, m, d), (\ell, c, r))$ be a Nash.

First show that $t = 0$, i.e. row player never plays T . Suppose otherwise.

Note that row player's expected payoffs from three choices are

$$U(T) = \ell - 2c$$

$$U(M) = -2\ell + c$$

$$U(D) = r$$

Therefore $t > 0$ implies $\ell - 2c \geq r$, which implies $\ell > 0$. Now column player's payoffs from three choices are:

$$U(L) = m - 2t$$

$$U(C) = t - 2m$$

$$U(R) = d$$

Therefore $\ell > 0$ implies $m - 2t \geq d$, which implies

$$m > 0.$$

But $t > 0$ and $m > 0$ imply

$$\ell - 2c = -2\ell + c \text{ (row player must be indifferent between } T \text{ and } M)$$

It follows $\ell = c > 0$, which implies $U(T) = U(M) < 0 \leq U(D)$! A contradiction.

We thus established $t = 0$. Similar reasoning will show that $m = \ell = c = 0$. Hence (D, R) is the only Nash.