

1a. The set of players $\mathcal{I} = \{1, \dots, n\}$; the (pure) strategy set for player i is $S_i = [0, \infty)$; the payoff function for player i is given as:

$$u_i(b_i, b_{-i}) = \begin{cases} \frac{1}{m}(v_i - \max b_{-i}) & : \text{ if } b_i \geq \max b_{-i} \text{ where } m = \text{number of highest bidders} \\ 0 & : \text{ otherwise} \end{cases}$$

b. A strategy b_i^* is weakly dominant for player i if it is a best response against any strategy profile of other players, i.e. if for all b_i and all b_{-i} :

$$u_i(b_i^*, b_{-i}) \geq u_i(b_i, b_{-i}).$$

c. Show that $b_i = v_i$ is a weakly dominant strategy for player i .

(i) Suppose $\max b_{-i} > v_i$. Then $u_i(v_i, b_{-i}) = 0$. The only way play i can make a difference is to choose some $b_i \geq \max b_{-i} > v_i$, which entails negative payoff. (ii) Suppose $\max b_{-i} = v_i$. Then $u_i(v_i, b_{-i}) = 0$. If player i chooses $b_i < v_i$, he is sure not to win and if she chooses $b_i > v_i$ then his payoff is negative. (iii) Suppose $\max b_{-i} < v_i$. Then $u_i(v_i, b_{-i}) = v_i - \max b_{-i} > 0$. If player i chooses any $b_i > \max b_{-i}$ it wouldn't have any effect on his payoff; if $b_i = \max b_{-i}$ his (positive) payoff will be reduced to a fraction as he must share the object; if $b_i < \max b_{-i}$ his payoff will be zero. In summary, $b_i = v_i$ is always a best response to any combination of b_{-i} .

d. Find a Nash equilibrium of the second-price sealed-bid auction in which player $i = 1, \dots, n$ wins the object. Justify your answer.

There are many, e.g. $(b_i > v_n, b_{-i} = (0, \dots, 0))$. Note that a player other than i can win the object only if he bids no less than b_i , in which case his payoff will be negative.

Next consider the first-price auction format. Everything is the same as in the second-price auction above, except that *the winner pays his bid*. So the winner, say player i , receives a payoff $v_i - b_i$. All other players receive zero payoff.

e. Is there any weakly dominant strategy for each player? Justify your answer.

Answer: No. First the strategy $b_i = 0$ is not dominant. Given any strategy $b_i > 0$, if $\max b_{-i} < b_i$ then b_i is worse than all $b'_i \in (\max b_{-i}, b_i)$.

f. Can you find a pure-strategy Nash equilibrium? Explain your answer carefully.

Answer: There is no pure-strategy Nash. Suppose (b_1, \dots, b_n) is a Nash. (i) If there is one winner i : $b_i > \max b_{-i}$, then bidder i can increase his payoff by choosing some bid between $\max b_{-i}$ and b_i . Contradiction. (ii) Suppose there are more than one winner. Assume that the highest bid is \bar{b} and bidders i and j are among the winners. First, both i and j must receive nonnegative payoffs: $v_i - \bar{b} \geq 0$ and $v_j - \bar{b} \geq 0$, for they can always choose to bid less, say zero. Since $v_i \neq v_j$, this means at least one of them must receive positive payoff. Such a player can multiply his payoff by slightly increasing his bid. Impossible.

2a and g. Most text book theorems that give sufficient conditions for existence of competitive equilibrium include the hypothesis that [1] every consumer has convex preferences, i.e., for each x in the consumption set of a consumer, the preferred set at x (i.e., the set

of points at least as good as x for the consumer) is convex, and that [2] every firm has a convex production set. These conditions are stronger than necessary. For one thing, only the aggregate production set (the sum of the production sets of individual firms) matters, and it can be convex even though individual firms have nonconvex production sets. More generally, nonconvexities away from the equilibrium allocation may be compatible with existence of equilibrium, as shown in example 15.C.3(b) in the text by Mas-Colell et. al.

On the other hand, if there is a competitive equilibrium, then the convex hull of every consumer's strictly preferred set at the equilibrium consumption lies outside the consumer's equilibrium budget set, and the equilibrium production vector of each firm maximizes the firm's profit over the convex hull of the firm's production set. This implies that the "convexified" economy in which strictly preferred sets and production sets are replaced by their convex hulls has an equilibrium that is also an equilibrium of the original economy. The original economy may have nonconvexities, but in its equilibrium behavior is just like the convexified economy.

Convexity of preferred sets production sets implies that there are nonincreasing marginal returns in consumption and production, in a way that can be made precise. This results in the optimal choice correspondence being continuous if the preferred sets and production sets are closed. Along with other sufficient conditions, the continuity ensures that supply and demand curves cross and equilibrium exists. This also shows why convexity is not necessary for equilibrium. Discontinuities in supply and demand curves away from equilibrium do not matter for existence of equilibrium.

No firm can have increasing returns to scale in a neighborhood of its competitive equilibrium production vector. Thus increasing returns over a wide range of output levels cannot be compatible with competitive behavior. On the other hand, when there is "no free lunch" (i.e., the aggregate production set Y satisfies $Y \cap \mathbb{R}_+^L = \{0\}$, where L is the number of goods) and all the firms' production sets are closed, the hypothesis that Y is convex implies that there is at least one price vector at which every firm has a profit maximizing production vector. Thus in that case there is a potential equilibrium price vector. (This answers part g.)

b. Convexity of preferences and technology are irrelevant for utility representation. Consider any utility function that is not quasiconcave, for example, $u(x_1, x_2) = x_1^2 + x_2^2$. It represents preferences that are not convex.

c. Convexity of the direct preferences is relevant for duality in consumer theory. If the preferred set for some x is not convex, then Hicksian compensated demand may not be continuous. It is necessarily discontinuous if some preferred set is not convex and the preferences are monotonic (i.e., if $z \geq y$, with y in the preferred set at x , then z is in the preferred set at x). In that case, there is some price vector at which there are two distinct cost-minimizing consumption vectors yielding a given utility level. The gradient of the expenditure function with respect to prices equals the Hicksian compensated demand vector, so if there is more than one such demand vector, the expenditure function is not differentiable with respect to prices.

Convexity of indirect preferences follows from optimizing behavior. The proof of Proposition 3.D.3(iii) in the text by Mas-Colell et. al. shows that a competitive consumer's indirect preference relation over prices and incomes is convex (the indirect utility function, if it exists, is quasiconvex) without any restrictions on the direct preferences for the goods.

d. Neither convexity of preferences nor of production sets is needed for the first welfare

theorem. On the other hand, the first welfare theorem only applies if a competitive equilibrium exists. The theorem is vacuous unless the economy has some limited form of local convexity, so that equilibrium can exist, as discussed in part a above. Convexity of preferences and production sets is typically assumed in statements of the second welfare theorem. They are not necessary for the conclusion of the theorem to hold, as can be seen in example 15.C.3(b) in Mas-Colell et. al. But, again, some form of local convexity is needed. The conclusion of the second welfare theorem fails to hold under the same kind of nonconvexity that prevents existence of equilibrium.

e. The main implications in revealed preference theory are consequences of optimizing behavior with transitive, locally nonsatiated preferences, without any need for convexity of preferences. For example, if x is chosen at price vector p and wealth w and \bar{x} is chosen at (\bar{p}, \bar{w}) , with $p\bar{x} \leq w$ then x must be at least as good as \bar{x} , so $\bar{p}x \geq \bar{w}$. If not, then with local nonsatiation and transitivity, there is a point x' preferred to x and hence to \bar{x} with $\bar{p}x' \leq \bar{w}$, which contradicts the optimality of \bar{x} in that budget set. This conclusion is called the weak axiom of revealed preference.

If the consumer has locally nonsatiated preferences that are strictly convex (i.e., convex preferences such that the indifference surfaces contain no segments), then the consumer's demand satisfies the strong axiom, hence also the weak axiom of revealed preference. In that case, in the notation above, $p\bar{x} \leq w \Rightarrow \bar{p}x > \bar{w}$ when $x \neq \bar{x}$. (To see this, note that under the hypothesis, $\bar{p}x \geq \bar{w}$ follows from the argument above. If this last condition holds with equality, then $\bar{p}x = \bar{p}\bar{x}$ and the midpoint of the segment joining x and \bar{x} must be strictly preferred to \bar{x} , since x is at least as good as \bar{x} . This implies that \bar{x} is not optimal when it is chosen—a contradiction.)

f. In expected utility theory, with state independent utility, the utility of a random prospect paying x_s in state of nature s is the expected value of $u(x_s)$ with the expected value taken over all states. The preference relation over all prospects is convex if u is concave (and this condition is almost necessary for convexity of the preferences over prospects). Concavity of u implies weak risk aversion. Replacing any prospect by one that gives the expected value of the original prospect in every state makes the consumer at least as well off.

3. Consider a two person household with collective preferences represented by $u(z_1, z_2) = z_1^\beta z_2^{1-\beta}$, $\beta \in (0, 1)$, where z_i denotes the home goods produced by person i . These are produced according to the technologies

$$\begin{aligned} z_1 &= x_1^\alpha t_1^{1-\alpha}, \alpha \in (0, 1), \\ z_2 &= x_2^\gamma t_2^{1-\gamma}, \gamma \in (0, 1), \end{aligned}$$

where t_i is the amount of time spent by i on home production, and x_i is purchased in the market at price p . Person i produces good z_i and sells time in the marketplace at price s_i . The household also has exogenous income w . Each agent has total amount of time T , and both pool their resources and seek to maximize $u(z_1, z_2)$.

a. Set up the household's optimization problem.

Answer:

$$\begin{aligned} \max & (x_1^\alpha t_1^{1-\alpha})^\beta (x_2^\gamma t_2^{1-\gamma})^{1-\beta} \\ \text{subject to} & p(x_1 + x_2) \leq w + s_1(T - t_1) + s_2(T - t_2) \end{aligned} \tag{0.1}$$

b. Find the optimal z_i , x_i and t_i .

Answer: Let $I = w + T(s_1 + s_2)$.

$$\begin{aligned}
 x_1 &= \alpha \beta \frac{I}{p} \\
 x_2 &= \gamma(1 - \beta) \frac{I}{p} \\
 t_1 &= (1 - \alpha) \beta \frac{I}{s_1} \\
 t_2 &= (1 - \gamma)(1 - \beta) \frac{I}{s_2} \\
 z_1 &= \left(\frac{\alpha}{p}\right)^\alpha \left(\frac{1 - \alpha}{s_1}\right)^{1 - \alpha} \beta I \\
 z_2 &= \left(\frac{\gamma}{p}\right)^\gamma \left(\frac{1 - \gamma}{s_2}\right)^{1 - \gamma} (1 - \beta) I
 \end{aligned} \tag{0.2}$$

c. Determine the effects of changes in s_i , p and w on z_i , x_i and t_i .

Answer: For $i, j = 1, 2$, $j \neq i$, $\partial t_i / \partial s_i < 0$, $\partial t_i / \partial s_j > 0$, and $\partial z_i / \partial s_j > 0$, since $\partial I / \partial s_j > 0$. Letting $k = (\alpha/p)^\alpha (1 - \alpha)^{1 - \alpha} \beta$, we have $\partial z_1 / \partial s_1 = k[(\alpha - 1)I s_1^{\alpha - 2} + T s_1^{\alpha - 1}]$, which is positive when $s_1 > (1 - \alpha)I/T$, hence when $s_1 > (1 - \alpha)(w + T s_2)/(\alpha T)$; and when this last inequality is reversed, $\partial z_1 / \partial s_1 \leq 0$. An increase in s_1 raises z_1 if s_1 is sufficiently big. The effect of s_2 on z_2 can be derived in the same way, with α replaced by γ and β replaced by $1 - \beta$.

d. Show that the z_i and t_i obtained in part b are the same as those that would have been obtained by solving

$$\begin{aligned}
 &\max u(z_1, z_2) \\
 &\text{subject to } \sum_i \bar{s}_i t_i = T(s_1 + s_2) + w \\
 &\quad z_1 = \alpha_1 t_1 \\
 &\quad z_2 = \alpha_2 t_2
 \end{aligned}$$

where $\alpha_1 = \left(\frac{s_1}{p} \frac{\alpha}{1 - \alpha}\right)^\alpha$, $\alpha_2 = \left(\frac{s_2}{p} \frac{\gamma}{1 - \gamma}\right)^\gamma$, $\bar{s}_1 = \frac{s_1}{1 - \alpha}$ and $\bar{s}_2 = \frac{s_2}{1 - \gamma}$. Explain why.

Answer: Maximizing with respect to t_1 and t_2 , we obtain from first order conditions

$$\frac{t_2}{t_1} = \frac{1 - \beta}{\beta} \frac{1 - \gamma}{1 - \alpha} \frac{s_1}{s_2}$$

It is straightforward to verify that the solutions above satisfy this and the budget equation.

Why? In light of the “nested” Cobb-Douglas structure, we can think of this as either a single-stage maximization in x_1, x_2, t_1, t_2 as in part (a) or as a two-stage maximization: first, determining the optimal ratio of $\frac{x_i}{t_i}$ and then allocating total wealth $T(s_1 + s_2) + w$ between t_1 and t_2 at prices \bar{s}_1 and \bar{s}_2 .

e. Examine the optimal $\frac{t_1}{t_2}$ and explain the nature of its dependence on α , β , γ , and $\frac{s_1}{s_2}$.

Answer: $\partial(t_1/t_2)/\partial\alpha < 0$, $\partial(t_1/t_2)/\partial\beta > 0$, $\partial(t_1/t_2)/\partial\gamma > 0$, $\partial(t_1/t_2)/\partial(s_1/s_2) < 0$.

f. Show that the elasticities of the optimal z_i , x_i and t_i with respect to w are all less than unity, while the same elasticities with respect to “full income” equal unity.

Answer: z_i , x_i and t_i are linear in full income I , so their elasticities with respect to I equal 1. The observed income elasticities (with respect to w) equal $w/I < 1$. To see why, note that the elasticity of a composite function $g(w) = f(I(w))$ is $wg'(w)/g(w) = wf'(I(w))I'(w)/g(w) = w/I(w)$ if $f(I)$ is linear and $I = w + T(s_1 + s_2)$.

4 a and e. In part e, if each firm is charged t per unit output and an additional F if the output is positive, then firm j maximizes $(a - c - t - q)q_j - K - F$ with respect to q_j , where q is total industry output. If the firm produces, it produces q_j with $a - c - t - q - q_j = 0$, so all firms that produce have the same output level $q_j = a - c - t - q$. If n firms produce, then $q = nq_j$ and $q_j = (a - c - t)/(n + 1)$. The profit of a producing firm j in Nash equilibrium (NE) is $\pi_j = (a - c - t - q)q_j - K - F = q_j^2 - K - F = (a - c - t)^2(n + 1)^{-2} - K - F$. Let the largest $n \leq J$ for which this expression is nonnegative be \bar{n} (or if no such n exists, let $\bar{n} = 0$). Then there is a NE in which \bar{n} firms produce. There may also be Nash equilibria in which fewer than \bar{n} firms produce (if no firm can enter and make positive profit with the other firms producing at their NE levels).

In part a, $t = F = 0$, so there is a NE in which the number of producing firms is the largest $n \leq J$ with $(a - c)^2/(n + 1)^2 \geq K + F$ (or is 0 if no such n exists). Each producing firm produces $q_j = (a - c)/(n + 1)$. For all J firms to produce, we need $\pi_j \geq 0$ when $n = J$, hence $(a - c)^2/K \geq (J + 1)^2$.

b. There are infinitely many Pareto efficient allocations which differ in the distribution of the output among consumers and firm owners. In every efficient allocation, the goods go to the consumers who value them most, and total surplus (the sum of consumer surplus and total profit) is maximized. When n firms produce and q is total output, consumer surplus is $\int_0^q (a - Q)dQ - (a - q)q = q^2/2$, and total profit is $(a - q)q - cq - tq - nK - nF$, so total surplus is $(a - c - t)q - (1/2)q^2 - nK - nF$. Letting $t = F = 0$, total surplus is maximized at $q = 0$ or at $q > 0$ with $a - c - q = 0$, hence $q = a - c$ (corresponding to the price $p = a - q = c$ equal to marginal cost). The efficient number of firms is the smallest possible n (which is $n = 1$ if efficient output is positive, and is $n = 0$ otherwise). If $n = 0$, then total surplus is 0, so for $n = 1$ to be efficient, we must have $0 \leq (a - c)q - (1/2)q^2 - K = (1/2)(a - c)^2 - K$. If the fixed cost is sufficiently low ($K < (a - c)^2/2$), then in an efficient allocation only one firm produces and it produces $q = a - c$. Such allocations are also efficient if $K = (a - c)^2/2$, but in that case it is also efficient to have no firm produce. If $K > (a - c)^2/2$, then only allocations in which no firm produces are efficient.

c. From the answer to part b, if at least one firm produces in every Pareto efficient allocation, then $K < (a - c)^2/2$.

d. We are assuming that the condition in part c holds. Therefore in a Pareto efficient allocation, one firm produces. A NE allocation without taxation cannot be Pareto efficient. The answer to part a implies that in NE with one firm producing, total output is $(a - c)/2$, less than the Pareto efficient level.

f. Consider a NE in part a with $n \geq 1$ firms producing. Assume first that an additional entrant would make a loss. We consider small values of t and F that do not change the number of firms producing in NE. From part a, total output is $q(t) = n(a - c - t)/(n + 1)$ and $a - c - t = (n + 1)q(t)/n$. Net tax revenue is $tq(t) + nF$. Assuming it is 0, F

can be treated as a function of t with $nF(t) = -tq(t)$. From part b, total surplus is $s(t) = (a - c - t)q(t) - (1/2)[q(t)]^2 - nK - nF(t) = \{[(n+1)/n] - (1/2)\}[q(t)]^2 - nK + tq(t)$. Since $q'(t) = -n/(n+1)$, we have $s'(0) = -2n\{[(n+1)/n] - (1/2)\}q(0)/(n+1) + q(0) = \{[n/(n+1)] - 1\}q(0) < 0$. Thus a small subsidy to output ($t < 0$), financed by a fee $F(t) > 0$ increases total surplus. In fact, both consumers and producers gain since the price falls and the derivative of total profit with respect to t is nonnegative (strictly positive if $n > 1$). The subsidy induces the firms that produce to increase their output.

If instead, starting from NE with $t = F = 0$, an additional entrant would make 0 profit, then the tax subsidy scheme described above would induce an entrant to enter. This also can be shown to lead to Pareto improvement. Alternatively, the tax scheme in the paragraph above can be modified so that F is raised slightly more than enough to balance the government budget. Then no entrant enters, but the Pareto improvement remains as above.

If there is no NE in which a firm produces, then the only way to achieve a Pareto improvement is to have a firm produce. If a single firm produces, its output is $q = (a - c - t)/2$, hence $t = a - c - 2q$. Government budget balance requires $F = -tq = -(a - c)q + 2q^2$, and the firm's profit is $q^2 - K - F = -q^2 + (a - c)q - K$. The profit is maximized at $q = (a - c)/2$, with $t = 0$, and at this q (hence at every $q \geq 0$) the profit is negative (otherwise there is a NE in which a firm produces). Thus if there is no NE with positive output in part a, then no Pareto improvement is possible under the tax scheme of part e. Pareto improvement requires additional taxes on the consumers.

g. To obtain a Pareto efficient NE, the government must ensure that only one firm produces. From part a, the firm produces $q = (a - c - t)/2$, so for Pareto efficiency, $t = -(a - c)$. Government budget balance requires $F = -tq = (a - c)^2 = q^2$. But since $a - c - t = 2q$, the profit of a single producing firm is $(a - q)q - cq - tq - K - F = 2q^2 - q^2 - K - F = -K < 0$, so no firm chooses to produce. This shows that it is not possible to attain a Pareto efficient NE using the tax scheme in part e.