Find the value of the real parameter $a$ for which the equation

$$x \left( x^{12} - (2a + 3) x^6 + a^2 \right) = 0$$

has exactly five real solutions, which form an arithmetic progression.

Solution:

We observe that the equation

$$x \left( x^{12} - (2a + 3) x^6 + a^2 \right) = 0$$

has 0 as one of its root.

It is further easy to see that if $r$ is a root of the equation, then $-r$ is also root of the equation.

Thus, if the given equation has the five real roots $x_1, x_2, x_3, x_4,$ and $x_5$, which form an arithmetic progression with common difference $d \neq 0$, then these roots are $-2d, -d, 0, d,$ and $2d$, respectively.

Substituting into the given equation, we now obtain that

$$d^{12} - (2a + 3) d^6 + a^2 = 0 \quad (1)$$

and

$$2^{12} d^{12} - (2a + 3) 2^6 d^6 + a^2 = 0. \quad (2)$$

Multiplying equation (1) by $2^6$ and subtracting it from equation (2), we now obtain that

$$(2^{12} - 2^6) d^{12} + (1 - 2^6) a^2 = 0.$$  

From this we obtain that

$$d^{12} = \frac{2^6 - 1}{2^{12} - 2^6} a^2 = \frac{a^2}{2^6}. $$
Therefore

\[ d^6 = \frac{|a|}{8}. \]  

(3)

On the other hand, subtracting equation (1) from equation (2), we obtain that

\[(2^{12} - 1) d^{12} - (2a + 3) (2^6 - 1) d^6 = 0.\]

From this it follows that

\[ d^6 = \frac{2a + 3}{2^6 + 1} = \frac{2a + 3}{65}. \]  

(4)

From (3) and (4) it follows that

\[ \frac{|a|}{8} = \frac{2a + 3}{65}. \]  

(5)

We now have the following two cases:

Case 1: \( a \geq 0 \)

Then equation (5) becomes

\[ a = \frac{2a + 3}{65} \]

\[ a = \frac{24}{49} \geq 0. \]

Case 2: \( a < 0 \)

Then equation (5) becomes

\[ -a = \frac{2a + 3}{65} \]

\[ a = -\frac{8}{27} < 0. \]

From Cases 1 and 2 we have that the possible values for \( a \) are \( \frac{24}{49} \) and \( -\frac{8}{27} \).