Let $\alpha, \beta,$ and $\gamma$ be the angles of a triangle and let $a, b,$ and $c$ be the lengths of the corresponding sides of the triangle.

Prove that

$$\sin \frac{\alpha}{2} \leq \frac{a}{2\sqrt{bc}}$$

$$\sin \frac{\beta}{2} \leq \frac{b}{2\sqrt{ac}}$$

$$\sin \frac{\gamma}{2} \leq \frac{c}{2\sqrt{ab}}.$$

Proof. Since $\frac{\alpha}{2} \in \left(0, \frac{\pi}{2}\right)$, it follows that $\sin \frac{\alpha}{2} > 0$ and therefore we have that

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}}.$$

From the Law of Cosines we, therefore, have that

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 - \frac{b^2 + c^2 - a^2}{2bc}}{2}} = \sqrt{\frac{a^2 - (b - c)^2}{4bc}}$$

$$= \sqrt{\frac{a^2 - (b - c)^2}{2\sqrt{bc}}} \leq \frac{a}{2\sqrt{bc}},$$

from which we also obtain that equality is achieved when $b = c.$

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