High School Math Problems
2016
Week 30
Problem and Solution

Let α, β, and γ be the angles of a triangle and let a, b, and c be the lengths of the corresponding sides of the triangle.

Prove that

$$\sin\frac{\alpha}{2} \leq \frac{a}{2\sqrt{bc}}$$
$$\sin\frac{\beta}{2} \leq \frac{b}{2\sqrt{ac}}$$
$$\sin\frac{\gamma}{2} \leq \frac{c}{2\sqrt{ab}}.$$

Proof. Since $\frac{\alpha}{2} \in \left(0, \frac{\pi}{2}\right]$, it follows that $\sin\frac{\alpha}{2} > 0$ and therefore we have that

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}.$$

From the Law of Cosines we, therefore, have that

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\frac{b^2+c^2-a^2}{2bc}}{2}} = \sqrt{\frac{a^2-(b-c)^2}{4bc}}$$
$$= \frac{\sqrt{a^2-(b-c)^2}}{2\sqrt{bc}} \leq \frac{a}{2\sqrt{bc}},$$

from which we also obtain that equality is achieved when $b = c$. $\square$