For what values of the parameter $a$ do the equations $x^2 - ax + 5 = 0$ and $x^2 + 5x - a = 0$ have a common root?

Solution:

Let $x_0$ be the common root of the two equations. Then we have that

$$x_0^2 - ax_0 + 5 = x_0^2 + 5x_0 - a$$

$$(5 + a) x_0 = 5 + a. \tag{1}$$

We now consider the following two cases:

Case 1: $a = -5$

Then equation (1) becomes

$$0x_0 = 0$$

and has as a solution every $x_0 \in \mathbb{R}$.

The two given quadratic equations further both become

$$x^2 + 5x + 5 = 0$$

and thus have identical roots.

Case 2: $a \neq -5$

Then from equation (1) we have that

$$x_0 = 1$$

is the common root of the two quadratic equations.

Substituting into the first equation, we then have that

$$1^2 - a + 5 = 0$$

$$a = 6.$$

From Cases 1 and 2 we have that the two equations have a common root when $a = -5$ or $a = 6$. 
