Let $a$, $b$, and $c$ be the lengths of the sides of a triangle and let $\alpha$, $\beta$, and $\gamma$ be the angles lying opposite to them, respectively.

Prove that the identity

$$a^4 = b^4 + c^4$$

is equivalent to

$$2\sin^2 \alpha = \tan \beta \tan \gamma.$$

Proof:

I. Suppose first that $2\sin^2 \alpha = \tan \beta \tan \gamma$. We shall prove that $a^4 = b^4 + c^4$.

Let $R$ be the radius of the circumscribed circles of the triangle. From the Law of Sines and the Law of Cosines we have that

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac} \quad \text{and} \quad \sin \beta = \frac{b}{2R},$$

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab} \quad \text{and} \quad \sin \gamma = \frac{c}{2R}.$$

Therefore

$$\tan \beta \tan \gamma = \frac{b}{a^2 + c^2 - b^2} \cdot \frac{c}{a^2 + b^2 - c^2} \cdot \frac{b}{2R} \cdot \frac{b}{2R} = \frac{(abc)^2}{(a^2 + b^2 - c^2)(a^2 + b^2 - c^2)R^2}.$$

Since from the Law of Sines

$$2\sin^2 \alpha = \frac{a^2}{2R^2},$$
we, therefore, have that

\[
\frac{(abc)^2}{(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)R^2} = \frac{a^2}{2R^2}
\]

\[
2b^2c^2 = a^4 - (b^2 - c^2)^2
\]

\[
a^4 = b^4 + c^4.
\]

II. Now suppose first that \(a^4 = b^4 + c^4\). We shall prove that \(2\sin^2 \alpha = \tan \beta \tan \gamma\). The assertion follows by reversing the steps of the proof in part I above.