Compression of Spherical Whittaker Functions in Type A

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Discrete Math Day

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Spherical Whittaker Functions

Whittaker functions are a basic tool in the theory of automorphic forms, e.g., in the construction of L-functions. They can be viewed as functions from weights λ to the field of fractions $\mathcal{K}$ of Laurent polynomials on the weight lattice, which also depend on a parameter t. They have numerous applications to:

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- Weyl Group multiple Dirichlet series,
- combinatorial representation theory,
- Schubert calculus on flag varieties.

Compression

Let F be a function. Let A and B be sets such that,

$$F(x) = \sum_{a \in A} x(a) \tag{1}$$

$$= \sum_{b \in B} x(b) \tag{2}$$

we say that (2) is a *compressed form* of (1) if $\exists$ a surjection $f : A \longrightarrow B$ such that

$$x(b) = \sum_{a \in f^{-1}(b)} x(a)$$

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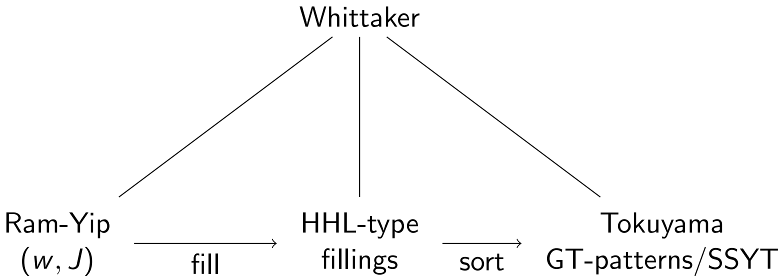

Ram-Yip Formula

Let λ be a dominant weight/partition. We denote the transposition of values i and j by (i, j). Consider a chain of roots denoted $\Gamma(k)$ given by:

$$\Gamma(k) = \begin{matrix} ((1, k+1), & (1, k+2), & \ldots, & (1, n), \\ (2, k+1), & (2, k+2), & \ldots, & (2, n) \\ & & \ldots & \\ (k, k+1), & (k, k+2), & \ldots, & (k, n)) \end{matrix}$$

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Now define a chain Γ as a concatenation $\Gamma := \Gamma_1 \ldots \Gamma_{\lambda_1}$ where $\Gamma_j = \Gamma(\lambda'_j)$ with a small exception when j is the first column of length λ'_j read from left to right.

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$$\mathcal{A}(\Gamma) = \{(w,K) \in W \times 2^{[m]} : \; K = \{k_1, \ldots, k_s\},$$
$$\text{condition on chain}(w,K)\}$$

Ram-Yip Formula

We have,

Theorem (Lenart,Orr, Shimozono)

$$\widetilde{\mathcal{W}}_\lambda = \sum_{(w,K)\in\mathcal{A}(\Gamma)} (-1)^{\ell(wK)} t^{\frac{1}{2}(\ell(w)+\ell(wK)-|K|)} (t-1)^{|J|} x^{-uwt(J)-\rho}$$

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Where ℓ is the length function, $\rho = (n-1, n-2, \ldots, 1)$ and wt is the weight function.

Fill Map

The *filling map* is the map f which maps a pair (w, T) to a filling σ of shape λ. Let $n = 4$ and $\lambda = (2, 1, 0, 0)$. We have

$$\Gamma = \Gamma_1\Gamma_2 = ((1,4),(2,4)|(1,3),(1,4))$$

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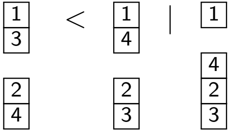

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$$f(w,T) = \begin{array}{|c|c|}\hline 1 & 3\\\hline 4 & \\\cline{1-1}\end{array}$$

HHL fillings

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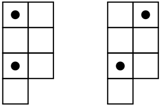
Pairs of entries in the following positions,



are not equal.

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So by summing over alcove walks in the preimage of a filling with respect to *fill*. We get the following:

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Theorem (Lenart, S.)

$$\widetilde{\mathcal{W}}_\lambda = \sum_{\sigma \in F(\lambda+\rho,n)} (-1)^{n-a_0+\ell(C)} t^{n-a_0+inv(\sigma)} (1-t)^{des(\sigma)} x^{ct(\sigma)},$$

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where $F(\lambda+\rho, n)$ are all HHL fillings of shape $\lambda+\rho$ with entries in $[n]$.

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The *inversion statistic* on a filling σ, denoted $\text{inv}(\sigma)$ is the number of configurations:

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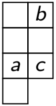
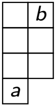


and



where the three entries satisfy $a < b < c$.

Tokuyama's Formula

For any weakly decreasing partition λ of length n, we have

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$$\widetilde{\mathcal{W}}_\lambda = \sum_{T \in SGT(\lambda+\rho)} (1-t)^{z(T)}(-t)^{l(T)}x^{m(T)}$$

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$SGT(\lambda+\rho)$ are particular types of SSYT. While $z(T)$ and $l(T)$ are statistics on these SSYT.

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We assign the corresponding t-coefficients and pair entries which have the same parent node.

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We then sum up the tree, assigning the sums of t-coefficients to each parent node.

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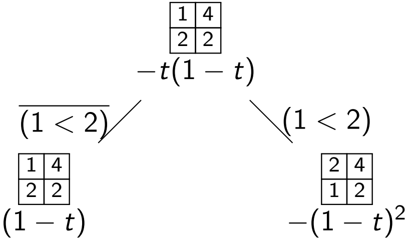

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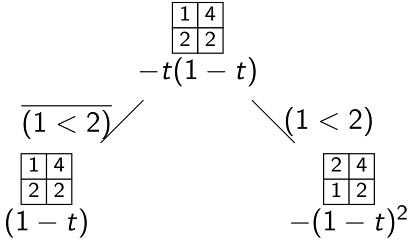

So we have compression of the form:

$$(1-t)-(1-t)^2=(1-t)(1-(1-t))=-t(1-t)$$

Main Result

Theorem (Lenart, S.)

Fix a SSYT σ. We have,

$$\sum_{T : T \in sort^{-1}(\sigma)} (-1)^{\ell(T[1])} t^{inv(T)} (1-t)^{des(T)} =$$

$$(-t)^{\sum_{i=2}^{(\lambda+\rho)_1} (\check{N}(i) + \hat{N}(i))} (1-t)^{\sum_{i=2}^{(\lambda+\rho)_1} p(i)}$$

if the Non-overlapping Condition is satisfied for all columns in the root $\widehat{T}$. Otherwise the sum is 0.

Thank You!