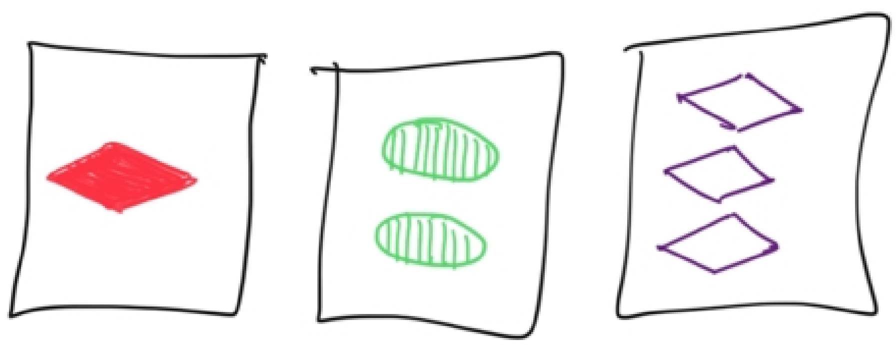
# SET and QUADS

Lauren Rose
Bard College

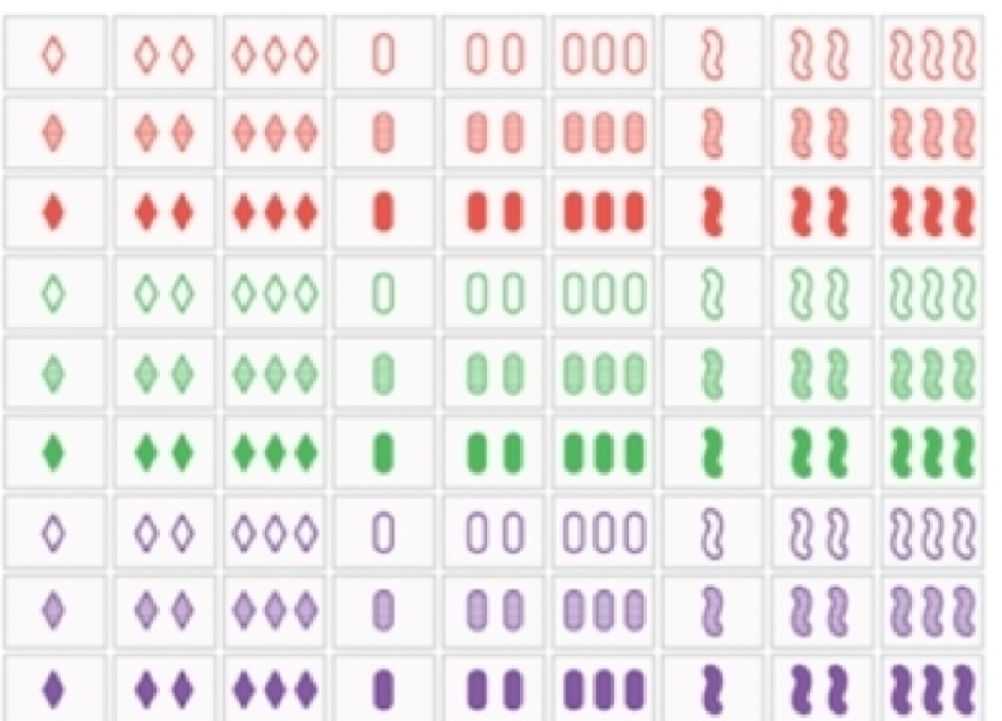


rose@bard.edu

① SET: 81 cards, 4 attributes, 3 states

color, #, shape, shading

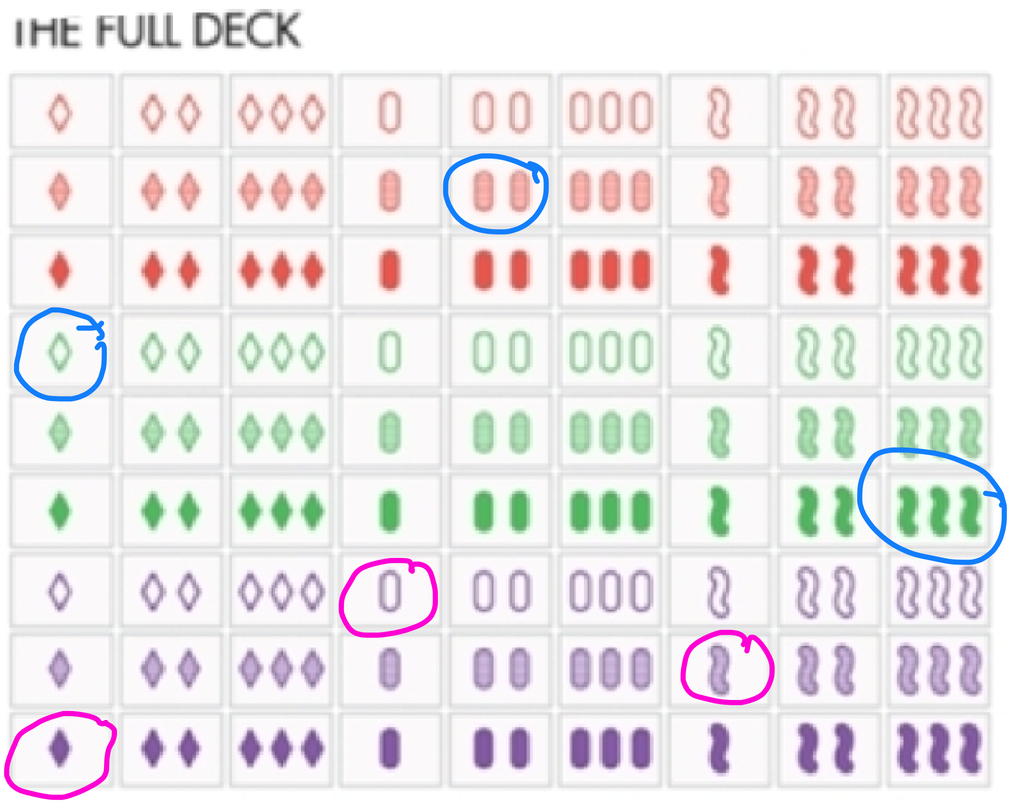

THE FULL DECK



A SET consists of 3 cards s.t. for each attribute, either all 3 states appear or all cards have the same state.

# ① SET: 81 cards, 4 attributes, 3 states

Color, #, shape, shading

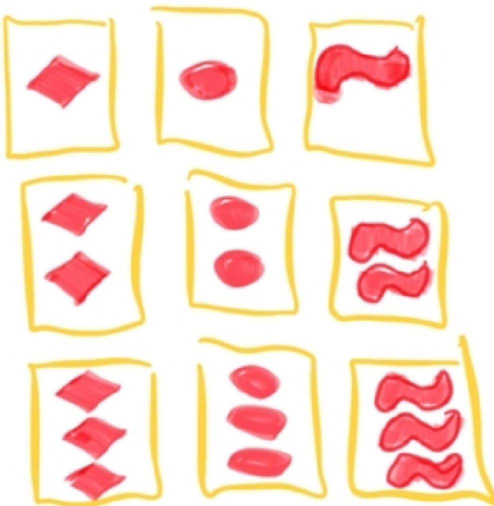

A SET consists of 3 cards s.t. for each attribute, either all 3 states appear or all cards have the same state.

EXAMPLES: The pink circles form a SET

The blue circles DO NOT.

② **Math**: In order to explore this, we restrict to 2 attributes

SET CARDS $n=2$



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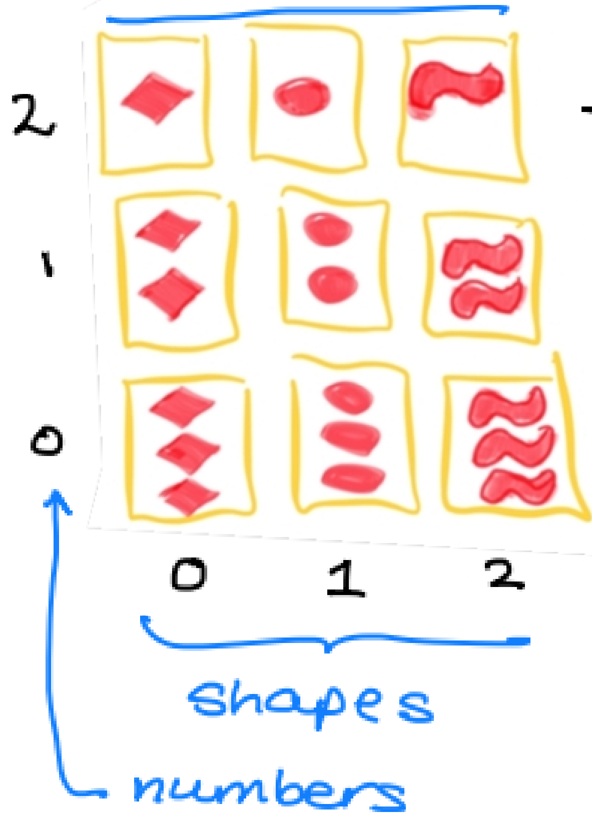
**SET CARDS**




**VECTORS mod 3**

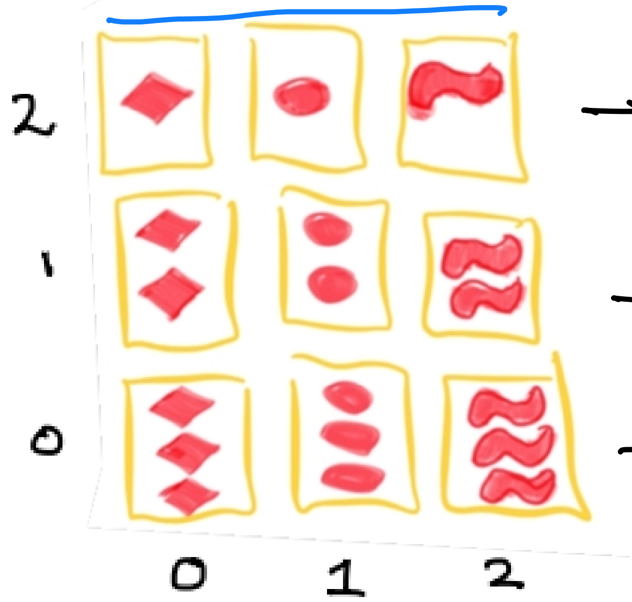
→ (0,2) (1,2) (2,2)

→ (0,1) (1,1) (1,2)

→ (0,0) (1,0) (2,0)

## ② Math: In order to explore this, we restrict to 2 attributes

SET CARDS



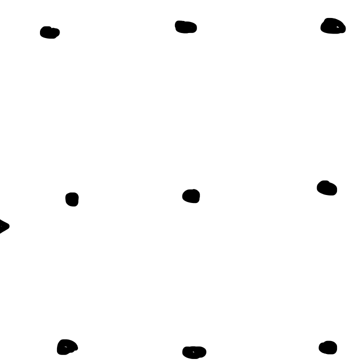

VECTORS mod 3

→ (0,2) (1,2) (2,2)

→ (0,1) (1,1) (1,2) →

→ (0,0) (1,0) (2,0)

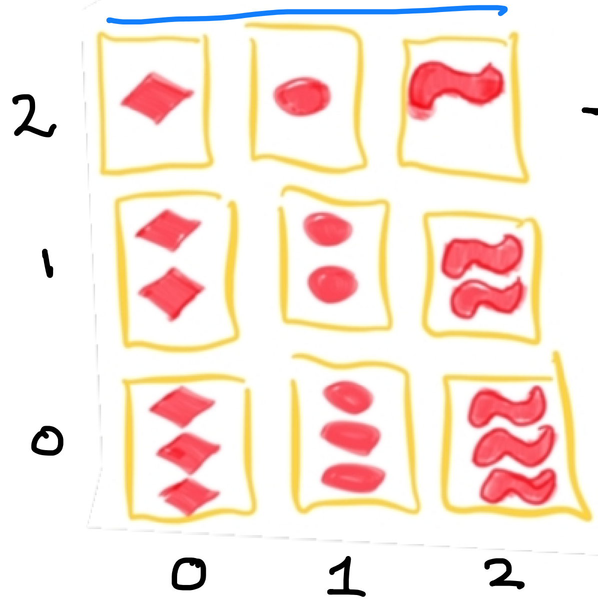
Geometry



$\underbrace{\qquad\qquad}_{\mathbb{Z}_3 \times \mathbb{Z}_3}$

## ② Math: In order to explore this, we restrict to 2 attributes

**SET CARDS**



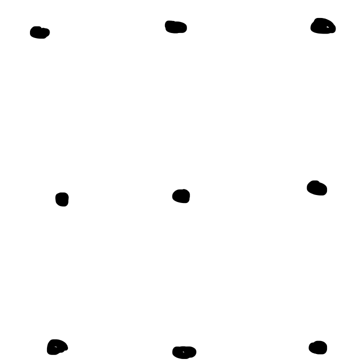

**VECTORS mod 3**

$(0,2)\quad(1,2)\quad(2,2)$

$(0,1)\quad(1,1)\quad(1,2)$

$(0,0)\quad(1,0)\quad(2,0)$

**Geometry**



$\mathbb{Z}_3 \times \mathbb{Z}_3$

SETS are rows, columns, diagonals, off diagonals

## ② Math: In order to explore this, we restrict to 2 attributes

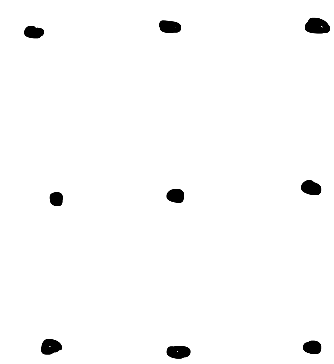
| SET CARDS | VECTORS mod 3 | Geometry |
|---|---|---|
| | | |

2 → (0,2) (1,2) (2,2)

1 → (0,1) (1,1) (1,2) →

0 → (0,0) (1,0) (2,0)

0 1 2



$\mathbb{Z}_3 \times \mathbb{Z}_3$

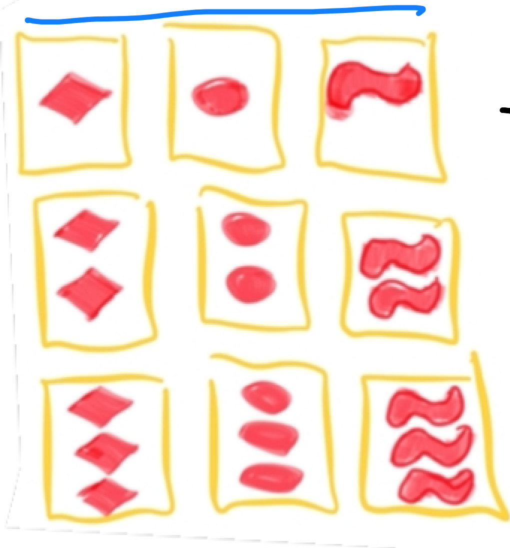
SETS are rows, columns, diagonals, off diagonals

SETS have form $\left\{ \begin{array}{l} A+B+C \equiv \vec{0} \\ \text{mod } 3 \end{array} \right.$

# ② Math: In order to explore this, we restrict to 2 attributes

**SET CARDS**

2
1
0



0 1 2

SETS are rows, columns, diagonals, off diagonals

**VECTORS mod 3**

→ (0,2) (1,2) (2,2)

→ (0,1) (1,1) (1,2) →

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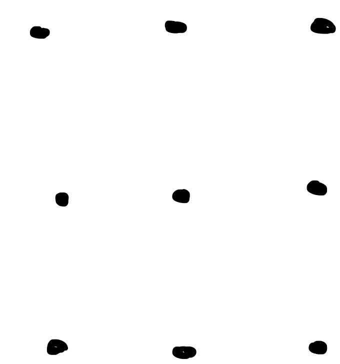
SETS have the form $\begin{cases} A+B+C \equiv 0 \\ \text{mod } 3 \end{cases}$

**Geometry**



$\mathbb{Z}_3 \times \mathbb{Z}_3$

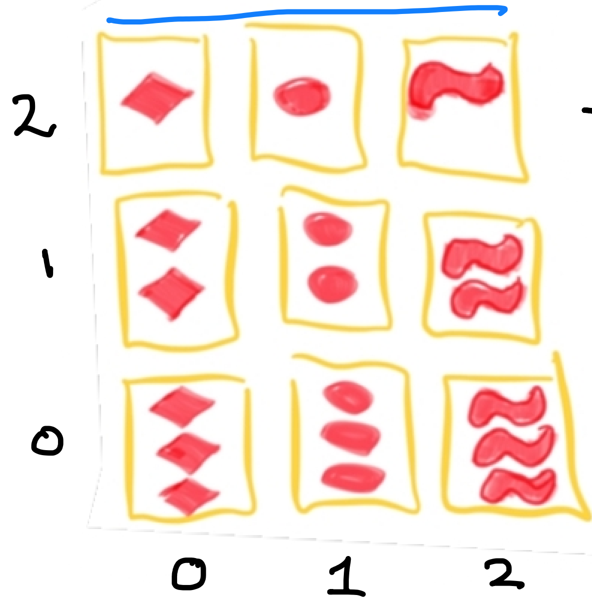
SETS are lines in the affine Geometry $A(2,3)$

dimension

field cardinality

## ② math : In order to explore this, we restrict to 2 attributes

**SET CARDS**




SETS are
rows, columns,
diagonals, off diagonals

**VECTORS mod 3**

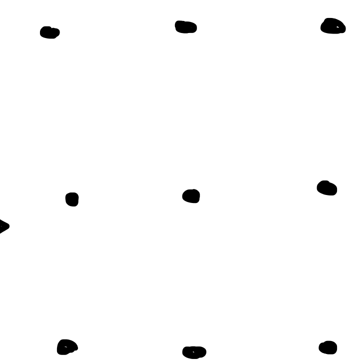
$\rightarrow$ (0,2) (1,2) (2,2)

$\rightarrow$ (0,1) (1,1) (1,2) $\rightarrow$

$\rightarrow$ (0,0) (1,0) (2,0)

SETS have the form $\left\{ \begin{array}{l} A+B+C \equiv \vec{0} \\ \text{mod } 3 \end{array} \right.$

**Geometry**



$\mathbb{Z}_3 \times \mathbb{Z}_3$

SETS are lines in the affine Geometry A(2, 3)

dimension

field cardinality

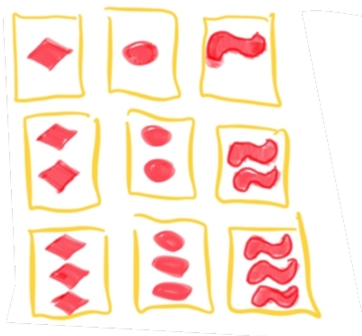
**COMBINATORICS**

A SET deck is a Steiner Triple System, since each pair of cards lies in exactly one SET, i.e. the 3rd card of a SET is uniquely determined.

③ The maximal cap problem (in Finite Geometry)

Q: What is the size of the largest Anti-SET?

SET CARDS n=2



n = 3 attributes

n = 4 attributes

## ③ The maximal cap problem (in Finite Geometry)

Q: What is the size of the largest Anti-SET?

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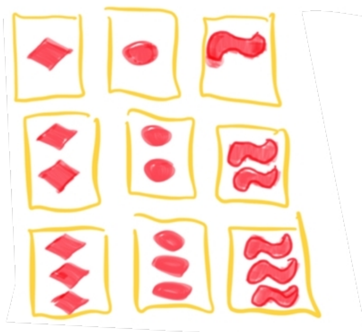


→ max capsize = 4

n = 3 attributes

→ max capsize = 9

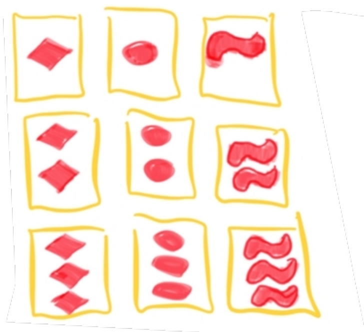
n = 4 attributes

→ max capsize = 20

## 3 The maximal cap problem (in Finite Geometry)

Q: What is the size of the largest Anti-SET?

SET CARDS $n=2$



→ max capsize = 4

$n = 3$ attributes

→ max capsize = 9

$n = 4$ attributes

→ max capsize = 20

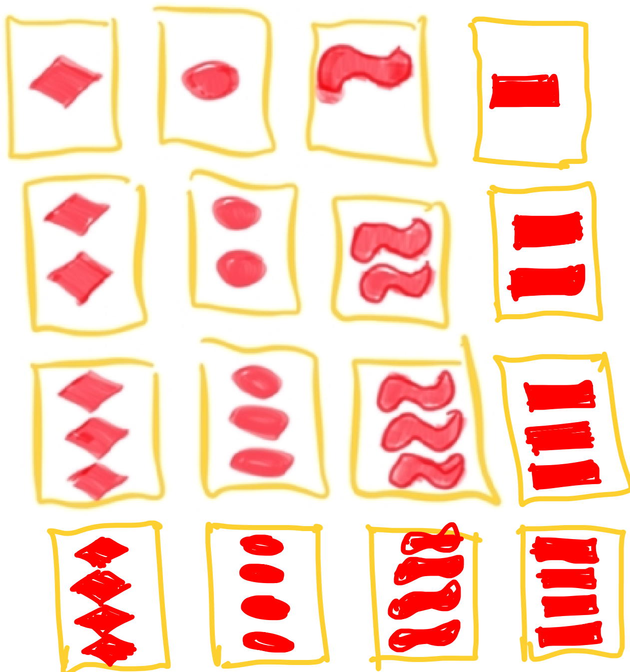
$n = 5$: max = 45

$n = 6$: max = 112

$n \geq 7$: unknown

# 4 QUADS

QUADS
~~SET~~ CARDS $n=2$



**Idea:**
Add another state to each attribute.

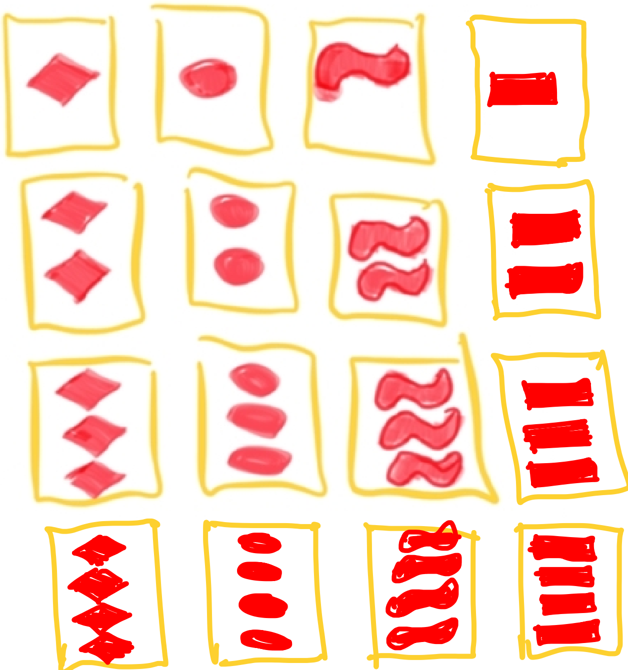
Numbers: 1, 2, 3, 4

Shapes: ◇, 0, S, ▯

Def: A QUAD is 4 cards ....
ALL the same or all different

# ④ QUADS

QUADS
~~SET~~ CARDS $n=2$



Idea:
Add another state to each attribute.

Numbers: 1, 2, 3, 4

Shapes: ◇, O, S, ▯

Def: A QUAD is 4 cards .... ALL the same or all different

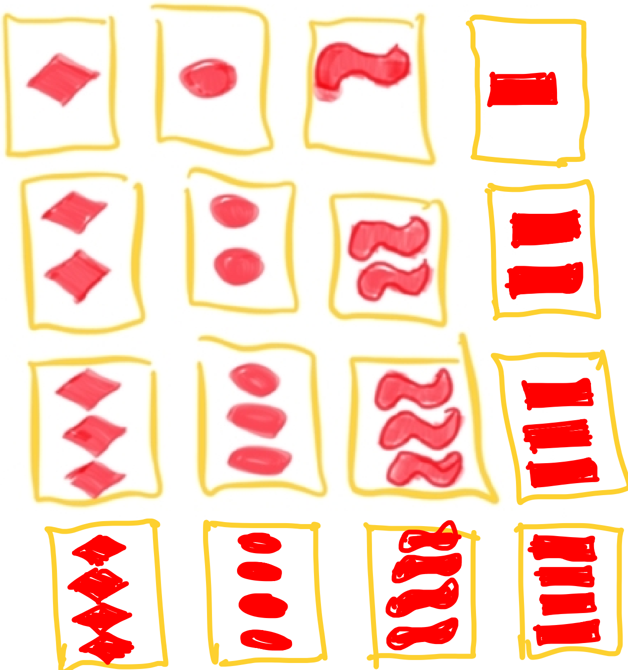
We have a problem

EXAMPLE:  make a QUAD with .

But  don't make a QUAD with any card.

# ④ QUADS

QUADS
~~SET~~ CARDS $n=2$



Idea:
Add another state to each attribute

Numbers: 1, 2, 3, 4

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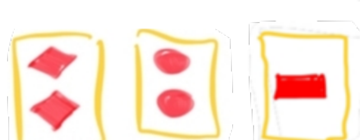
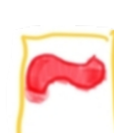
Def: A QUAD is 4 cards ....
ALL the same or all different

We have a problem

EXAMPLE:  make a QUAD with .

But  don't make a QUAD with any card.

We solve this problem by introducing ½ and ½ sets of states

 ⟶ 

shape: ◇ O ▭ → ∾ ⟵ all different shapes
number: $\underbrace{2 \quad 2}$ $\underbrace{1 \to 1}$ ⟵ Two 1's, Two 2's

# 5 Mathematics of QUADS


→ vectors in $\mathbb{Z}_4 \times \mathbb{Z}_4$? → Geometry in $\mathbb{Z}_4 \times \mathbb{Z}_4$ ?

→ Steiner Quadruple Systems?

# 5 Mathematics of QUADS


→ NO! ~~vectors in $\mathbb{Z}_4 \times \mathbb{Z}_4$~~?

$A+B+C+D \neq \vec{0}$ mod 4

→ No! Geometry in $\mathbb{Z}_4 \times \mathbb{Z}_4$?

Not an Affine Geometry

→ Steiner Quadruple Systems?

YES! This works.

# ⑤ Mathematics of QUADS


→ NO! ~~vectors in $\mathbb{Z}_4 \times \mathbb{Z}_4$~~ ? $A+B+C+D \neq \vec{0}$ mod 4 → NO! Geometry in $\mathbb{Z}_4 \times \mathbb{Z}_4$ ? Not an Affine Geometry

→ Steiner Quadruple Systems ?
YES! This works.

Solution:

- Use $\mathbb{Z}_2 \times \mathbb{Z}_2$ instead $\mathbb{Z}_4$

When $n=2$, cards are in $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$

For example:  $\longrightarrow ((0,1),(1,0)) \to (01,10)$

Now: $A+B+C+D$ is a QUAD $\Longleftrightarrow \equiv \vec{0} \bmod (\mathbb{Z}_2 \times \mathbb{Z}_2)$

# ⑤ Mathematics of QUADS


$\rightarrow$

vectors in $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$

(00, 00)
(00, 01)
⋮
(11, 10)
(11, 11)

Finite Geometry ?

- Yes,
  cards = A(2,4)
  dim 2
  field = $\mathbb{F}_4$

- But NO, QUADS are not necessarily LINES in A(2,4)

There is a solution!

# ⑤ Mathematics of QUADS


vectors in $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$

(00, 00)
(00, 01)
⋮
(11, 10)
(11, 11)

$$\underbrace{\left.\begin{array}{c}(0,0,0,0)\\(0,0,0,1)\\ \vdots \\(1,1,1,0)\\(1,1,1,1)\end{array}\right\}}_{\text{vectors in } (\mathbb{Z}_2)^4} = A(4,2)$$

dim 4 over $\mathbb{Z}_2$

Finite Geometry ?

# ⑤ Mathematics of QUADS


vectors in $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$

Finite Geometry?

$$\begin{matrix} (00,00) \\ (00,01) \\ \vdots \\ (11,10) \\ (11,11) \end{matrix} \rightarrow \underbrace{\left.\begin{matrix} (0,0,0,0) \\ (0,0,0,1) \\ \vdots \\ (1,1,1,0) \\ (1,1,1,1) \end{matrix}\right\}}_{\text{vectors in } (\mathbb{Z}_2)^4} = A(4,2)$$

dim 4 over $\mathbb{Z}_2$

- QUADS are: rows, columns, diags, off diags, squares, trapezoids, parallelograms and swap diagonals
- QUADS: sum to $\vec{0}$ mod $\mathbb{Z}_2 \times \mathbb{Z}_2$
- QUADS: sum to $\vec{0}$ mod 2
- QUADS: Are planes in $A(4,2)$!

## ⑥ Quad Questions:

- what is the maximal size of a NON-QUAD? (we have partial results)
- What properties of SET still hold for QUADS? (with adjustments)
- what properties are different?
- What properties of SET and QUADS hold for other Steiner systems?
- What if we add more states to each attribute? Can we still get a "SET-like" game?

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- What properties of SET and QUADS hold for other Steiner systems?
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## ⑦ SET References (among many...)


- "The Joy of SET" by McMahon, Gordon, Gordon, and Gordon
- "The Card Game SET" by Davis and Maclagan.
- "On Large Subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression" by Ellenberg and Gijswijt.

## ⑧ FUN FACT:

→ QUADS* is the official card deck for the Association for Women in Mathematics 50th Anniversary in 2021.

→ There will be profiles of female mathematicians on one side of each card, and a QUAD deck on the other side.

Thanks to the organizers, speakers, and attendees

* It will have different symbols, and possibly a different name.

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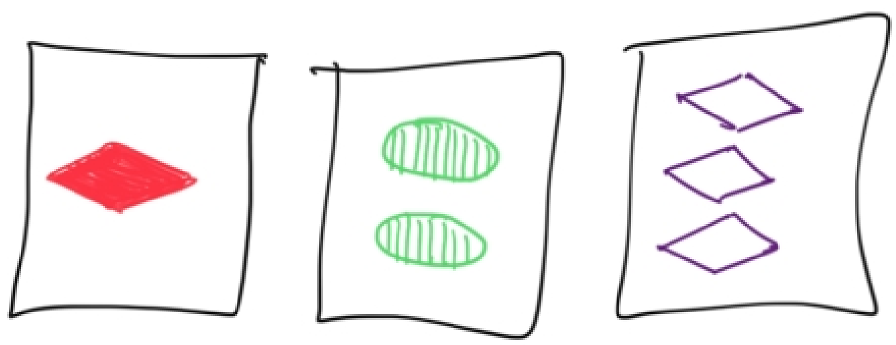

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