Compression of Spherical Whittaker Functions in Type A

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Abstract

We exhibit a Haglund, Haiman, Loehr (HHL) type formula for the spherical Whittaker function, i.e. as a sum over *fillings* of a Young diagram. This new formula is purely combinatorial and is derived via compression from an *alcove walk* formula. We then compress our HHL-type formula to a semistandard Young tableaux (SSYT) formula known as Tokuyama's formula using a binary generation tree.

Spherical Whittaker Functions

Whittaker functions are a basic tool in the theory of automorphic forms, e.g., in the construction of L-functions. They can be viewed as functions from weights λ to the field of fractions $\mathcal{K}$ of Laurent polynomials on the weight lattice, which also depend on a parameter t. They have numerous applications to:

- Weyl Group multiple Dirichlet series,
- combinatorial representation theory,
- Schubert calculus on flag varieties.

Compression

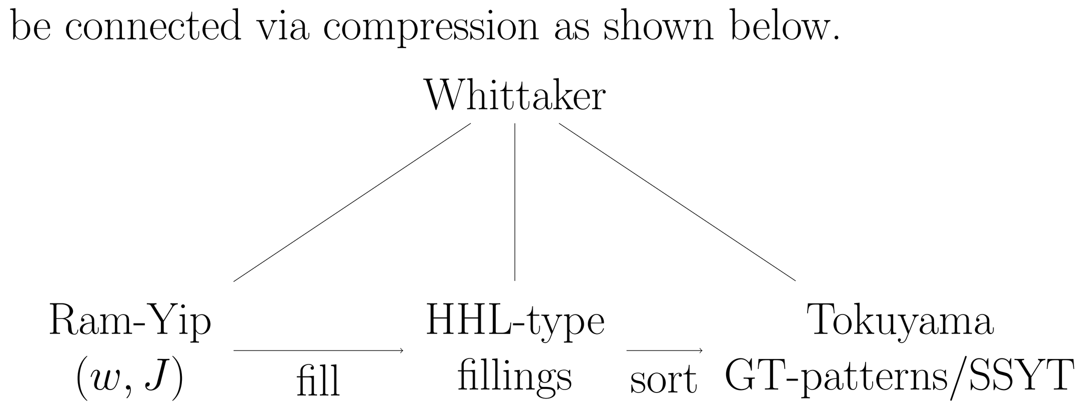
Let F be a function. Let A and B be sets such that,

$$F(x) = \sum_{a \in A} x(a) \quad (1)$$
$$= \sum_{b \in B} x(b) \quad (2)$$

we say that (2) is a *compressed form* of (1) if $\exists$ a surjection $f : A \longrightarrow B$ such that

$$x(b) = \sum_{a \in f^{-1}(b)} x(a)$$

The following descriptions of spherical Whittaker functions can be connected via compression as shown below.



Alcove Walk Formula in Type A

Let λ be a dominant weight/partition. We denote the transposition of values i and j by (i, j). Consider a chain of roots denoted $\Gamma(k)$ given by:

$$\Gamma(k) = \begin{matrix} ((1,k+1), & (1,k+2), & \ldots, & (1,n), \\ (2,k+1), & (2,k+2), & \ldots, & (2,n) \\ & & \ldots & \\ (k,k+1), & (k,k+2), & \ldots, & (k,n)) \end{matrix}$$

Now define a chain Γ as a concatenation $\Gamma := \Gamma_1 \ldots \Gamma_{\lambda_1}$ where $\Gamma_j = \Gamma(\lambda'_j)$ with a small exception when j is the first column of length λ'_j read from left to right.

$$\mathcal{A}(\Gamma) = \{(w, K) \in W \times 2^{[m]} : \; K = \{k_1, \ldots, k_s\},$$
$$u < ur_{k_1} < ur_{k_1}r_{k_2} < \ldots < ur_{k_1} \ldots r_{k_s} = w\}$$

Alcove Walk Formula in Type A cont.

We have,

$$\widetilde{\mathcal{W}}_\lambda = \sum_{(w,K) \in \mathcal{A}(\Gamma)} (-1)^{\ell(wK)} t^{\frac{1}{2}(\ell(w)+\ell(wK)-|K|)} (t-1)^{|J|} x^{-\mathrm{uwt}(J)-\rho} \quad (3)$$

Where ℓ is the length function, $\rho = (n-1, n-2, \ldots, 1)$ and wt is the weight function.

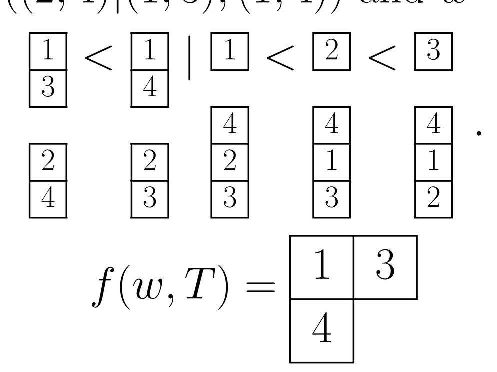
Fill Map

The *filling map* is the map f which maps a pair (w, T) to a filling σ of shape λ.

Ex: Let $n = 4$ and $\lambda = (2, 1, 0, 0)$. We have

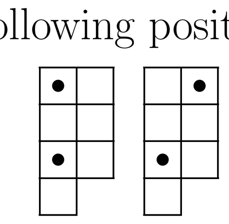
$$\Gamma = \Gamma_1\Gamma_2 = ((1,4),(2,4)|(1,3),(1,4))$$

Let $T = T_1T_2 = ((2,4)|(1,3),(1,4))$ and $w = 1324$.



HHL-type Formula for Spherical Whittaker Functions

A *Haglund-Haiman-Loehr (HHL) filling* is a filling which is weakly increasing across each row and satisfies the condition: Pairs of entries in the following positions,



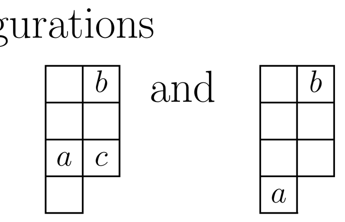
are not equal.

Let $F(\lambda + \rho, n)$ be the set of HHL fillings. By summing over all HHL-type fillings we have a new formula for the spherical Whittaker function of dominant weight λ,

$$\widetilde{\mathcal{W}}_\lambda = \sum_{\sigma \in F(\lambda+\rho,n)} (-1)^{n-a_0+\ell(C)} t^{n-a_0+\mathrm{inv}(\sigma)} (1-t)^{\mathrm{des}(\sigma)} x^{ct(\sigma)},$$

where *content of* σ is denoted $ct(\sigma)$.

The *inversion statistic* on a filling σ, denoted $\mathrm{inv}(\sigma)$ is the number of configurations



where the three entries satisfy $a < b < c$.

The *descent statistic* on a filling σ, denoted $\mathrm{des}(\sigma)$ is the number of cells such that we have a strict increase in a row when read from left to right.

Finally, a_0 is the unique entry absent from the first column, C, of σ.

Tokuyama's Formula

A Gelfand-Tsetlin (GT) pattern is a triangular array of non-negative integers of the form

$$\begin{matrix} a_{1,1} & & a_{1,2} & & a_{1,3} & & \ldots & & a_{1,n} \\ & a_{2,2} & & a_{2,3} & & \ldots & & a_{2,n} & \\ & & \ldots & & \ldots & & \ldots & & \\ & & & a_{n-1,n-1} & & a_{n-1,n} & & & \\ & & & & a_{n,n} & & & & \end{matrix}$$

Furthermore, a *strict* GT pattern is one in which each row is strictly decreasing. Denote the set of all strict GT patterns with top row α as $SGT(\alpha)$.

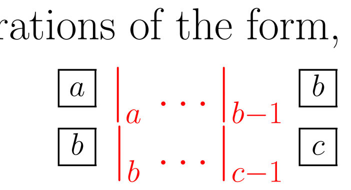
For any weakly decreasing partition λ of length n, we have

$$\widetilde{\mathcal{W}}_\lambda = \sum_{T \in SGT(\lambda+\rho)} (1-t)^{z(T)} (-t)^{l(T)} x^{m(T)}$$

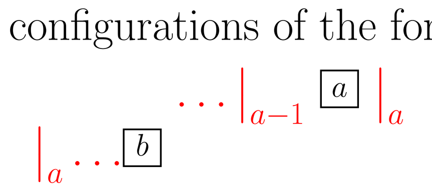
We can translate GT patterns into SSYT via a classical bijection. The corresponding statistics on GT patterns in Tokuyama's formula are then defined on *separating walls* on SSYT.

Let T be a SSYT. A *separating wall of index* k, denoted $|_k$ is a decoration on T, placed after the last box of T with entry less than or equal to k and before the first box with entry greater than k with respect to the reading order; walls may be placed at the beginning or end of a row.

The *boxed, not circled statistic* on a SSYT, T, denoted $l(T)$ counts configurations of the form,



The *not boxed, not circled statistic* on a SSYT, T, denoted $z(T)$ counts configurations of the form,



Generation Algorithm

In order to compress our HHL-type formula to Tokuyama's formula we need to uniquely generate all HHL fillings which sort to the same SSYT. We construct a binary generation tree.

The root of our tree is known as $\widehat{C}$ and is constructed by first matching common entries, then placing distinct entries in the relative order determined by C'.

Ex: $C' = \begin{smallmatrix}4\\7\\3\\6\end{smallmatrix}$ and $S = \{1,2,3,4\}$ gives $\widehat{C} = \begin{smallmatrix}4\\2\\3\\1\end{smallmatrix}$.

We construct the generation tree by defining a sequence of transpositions which act on the root configuration. We then iterate this procedure from right to left, to produce the entire preimage of a SSYT under the map *sort*.

Generation Algorithm cont.

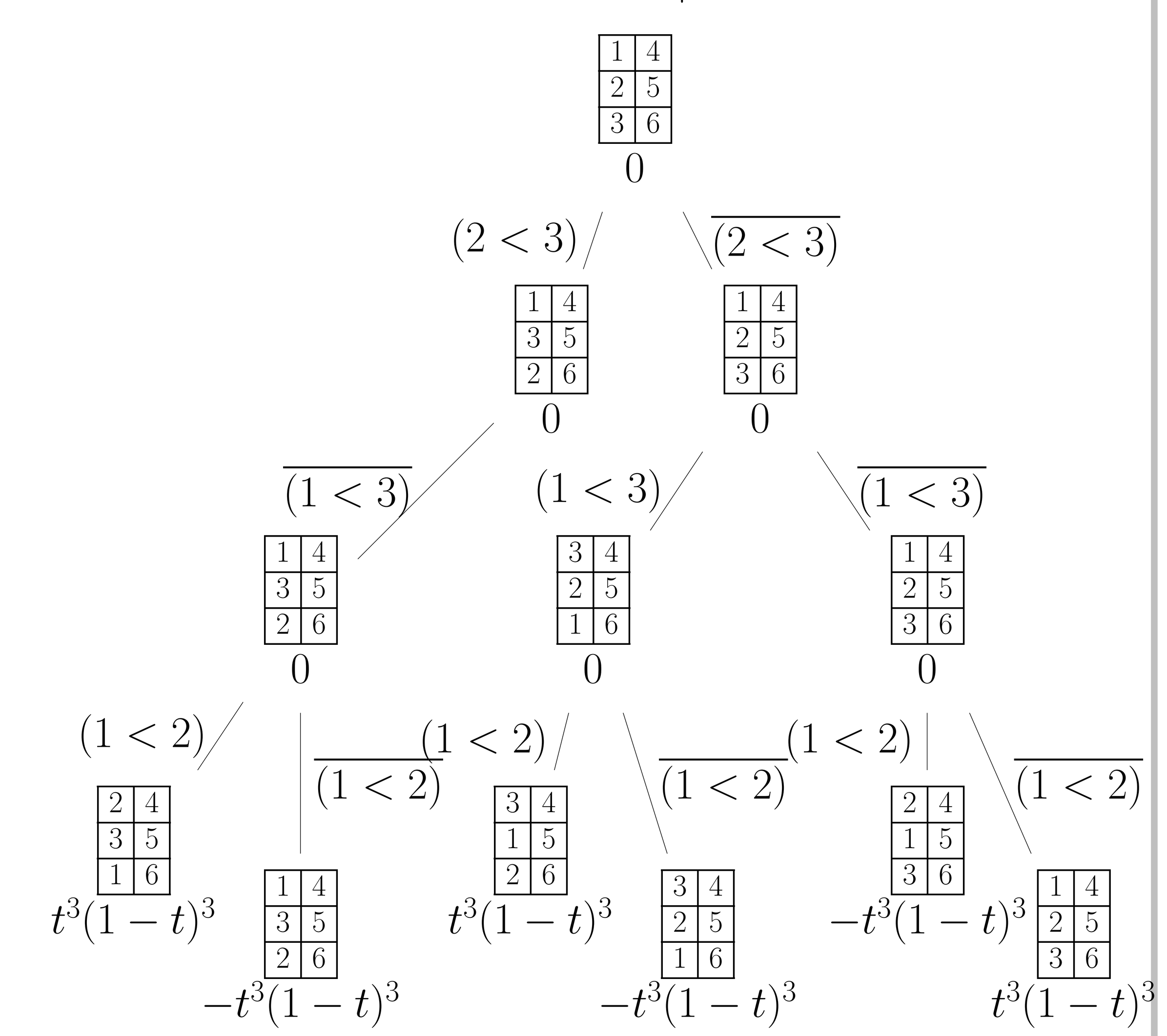
Call row i a *pivot* if $\widehat{C}(i) < C'(i)$. If i is a pivot row, call $\widehat{C}(i)$ and $C'(i)$ *pivot entries.*

Let $a_1 > a_2 > \ldots a_m$ be the set of entries in the left column. Let $b_1 > b_2 > \ldots > b_p$ be the pivot entries in the left column. The pivot entries strictly less that a_i are denoted $b_{j_i} > \ldots > b_p$. We have the following sequence of transpositions:

$$\beta = \begin{matrix} (b_{j_1} < a_1) \; (b_{j_1+1} < a_1) \; (b_{j_1+2} < a_1) \ldots (b_p < a_1) \\ (b_{j_2} < a_2) \; (b_{j_2+1} < a_2) \; (b_{j_2+2} < a_2) \ldots (b_p < a_2) \\ \ldots \\ (b_{j_c} < a_c) \; (b_{j_c+1} < a_c) \; (b_{j_c+2} < a_c) \ldots (b_p < a_c) \end{matrix}$$

Ex: Let $\widehat{C}C' = \begin{smallmatrix}1&4\\2&5\\3&6\end{smallmatrix}$

$$\beta = (2<3)\;(1<3) \;|\; (1<2)$$



Future Work

1. Apply compression to Non-symmetric Macdonald polynomials specialized at $q = 0$ to derive HHL formulas in this case.
2. The Alcove Walk formula generalizes to any root system, so we can use these techniques to derive Tokuyama-type formulas in type B and C.

References

References

[1] C. Lenart. Working Alcove Formulas for Iwahori-Whittaker Functions.

[2] A. Puskas. Gelfand-Tsetlin coefficients on Young tableaux.