Tight Surface Graphs

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Introduction

In this project, we study inductive constructions for graphs that are embedded in surfaces without edge crossings. In particular, for $(2,2)$-tight graphs on a torus we exhibit a complete inductive construction for such graphs. We also give a geometric application of this result to representations of graphs as contact graphs of configurations of circular arcs. This work forms part of a joint project with James Cruikshank, Derek Kitson and Stephen Power.

Surface Graphs

- Given a surface Σ, a Σ-graph is an embedding of an abstract graph $\Gamma = (V, E)$ in Σ without edge crossings.
- A face of a Σ-graph G is a connected component of the image of $\Sigma - \Gamma$. We let f_i to be the number of faces with i edges in the boundary.

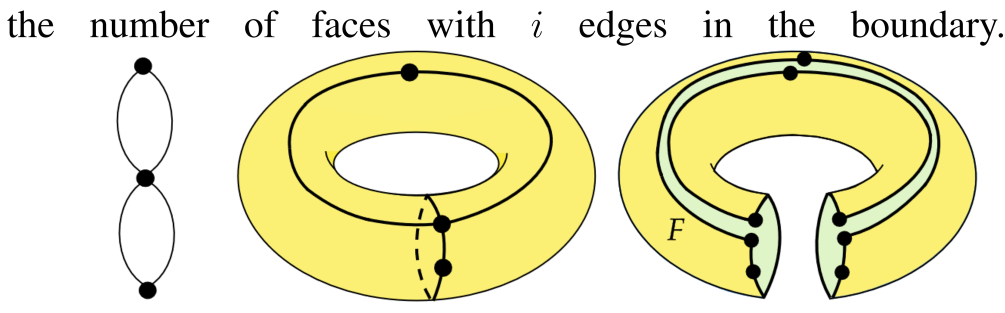

Fig.1: A Graph Γ. Fig.2: A torus graphs G with underlying graph Γ. Fig.3: F is a face in G.

Sparsity and Tightness

- Given Γ as above and a positive integer k, let $\gamma_k(\Gamma) = k|V| - |E|$.
- Let l, k be nonnegative integers with $l \leq k$. We say that Γ is *(k, l)-sparse* if, for every nonempty subgraph Ω of Γ, $\gamma_k(\Omega) \geq l$. If Γ is (k, l)-sparse and $\gamma_k(\Gamma) = l$ then we say that Γ is *(k, l)-tight*.

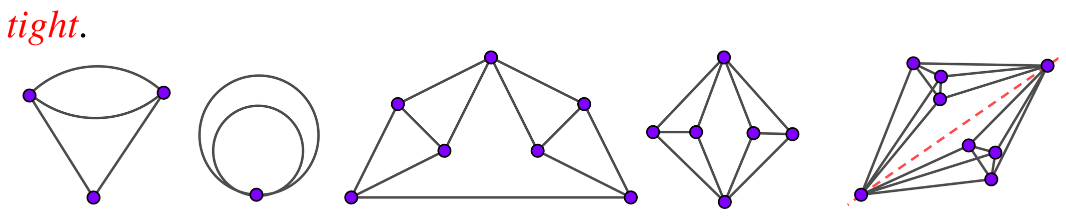

Fig.4: (2,2)-tight graph. Fig.5: (2,0)-tight graph. Fig.6: (2,3)-tight graph. Fig.7: (2,2)-tight graph. Fig.8: (3,6)-tight graph.

Topological Inductive Operations

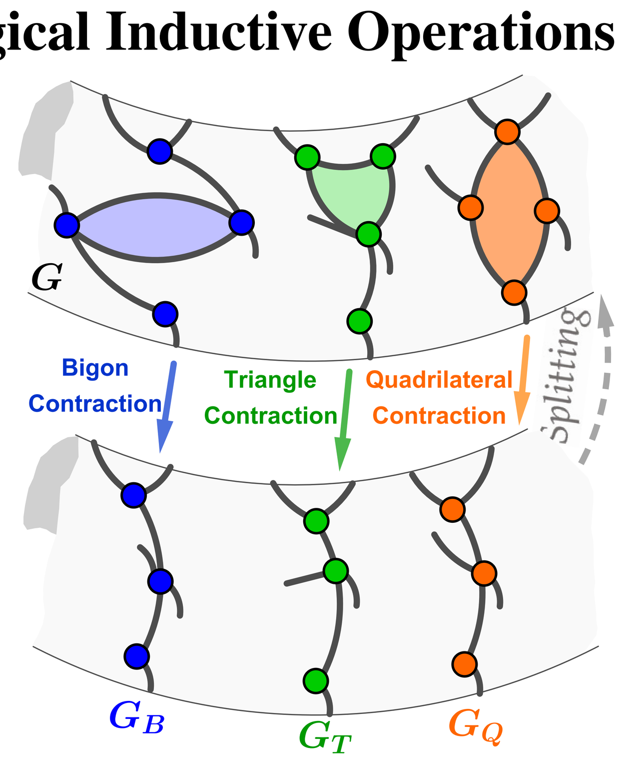

Fig. 9: Bigon, Triangle and quadrilateral contraction operations and their inverses.

- G_B is a $(2,l)$-sparse if and only if G is $(2,l)$-sparse.
- If G is a $(2,2)$-sparse and T is a triangular face then there is some contraction of T that yields a $(2,2)$-tight graph.
- Not every quadrilateral contraction preserves $(2,2)$-sparsity in a torus graph.

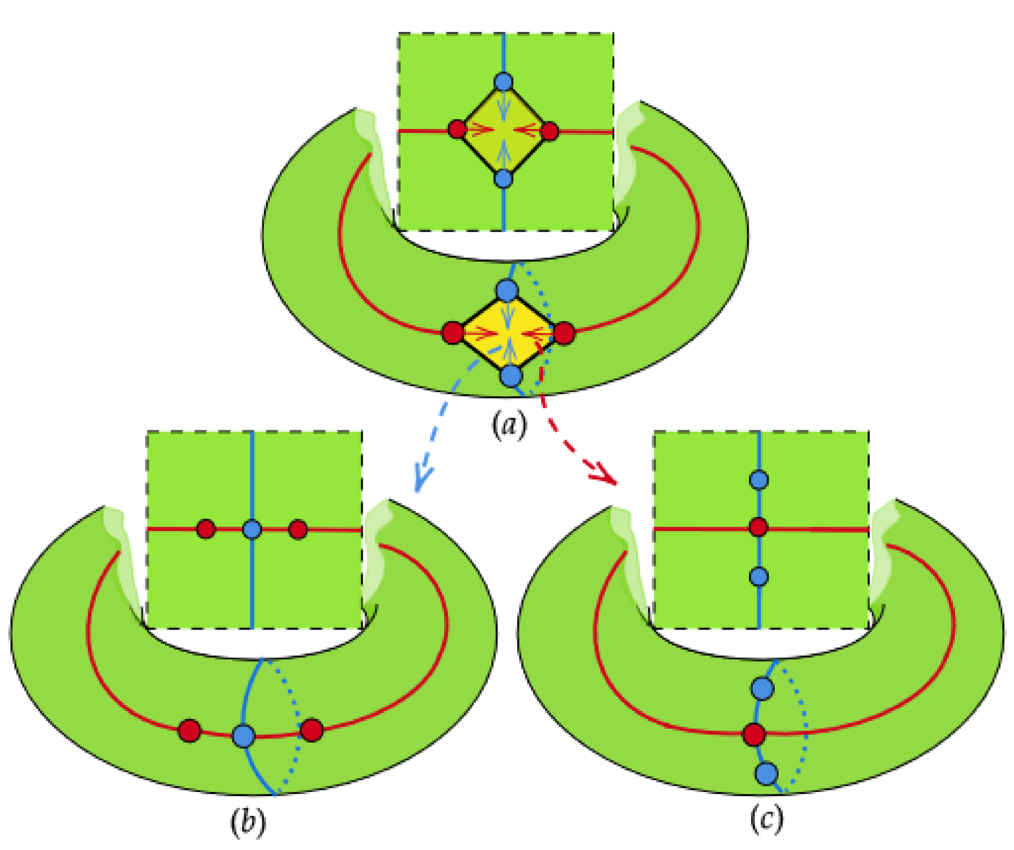

Fig. 10: (a) A (2,2)-tight torus graph. (b) Not a (2,2)-tight torus graph. (c) Not a (2,2)-tight torus graph.

■ Every (2,2)-tight plane graph can be reduced into a single vertex by a sequence of bigon contraction or edge contraction.

Irreducible $(2,2)$-Tight Torus Graphs

- A $(2,2)$-tight torus graph G is irreducible if G has no bigon,triangle or a contractible quadrilateral.

■ G has at most two quadrilateral faces.

Our main result in this project is the following theorem.

Theorem: G has at most 8 vertices. In particular there are finitely many isomorphism classes of such graphs.

Theorem: There are 116 irreducible $(2,2)$-tight torus graphs.

■ Every $(2,2)$-tight torus graph can be constructed from one of the 116 irreducible graphs in Figure 11 by a sequence of the inverse of bigon, triangle or quadrilateral contractions.

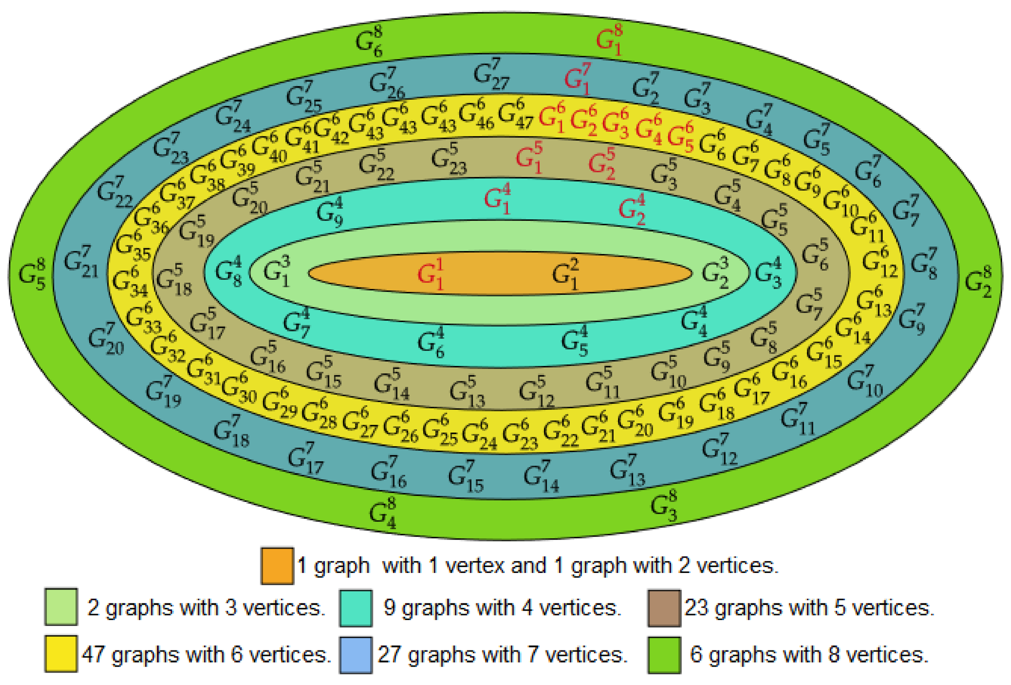

Fig. 11: All the 116 irreducible (2,2)-tight torus graphs.

Examples of Irreducible $(2,2)$-Tight Torus Graphs

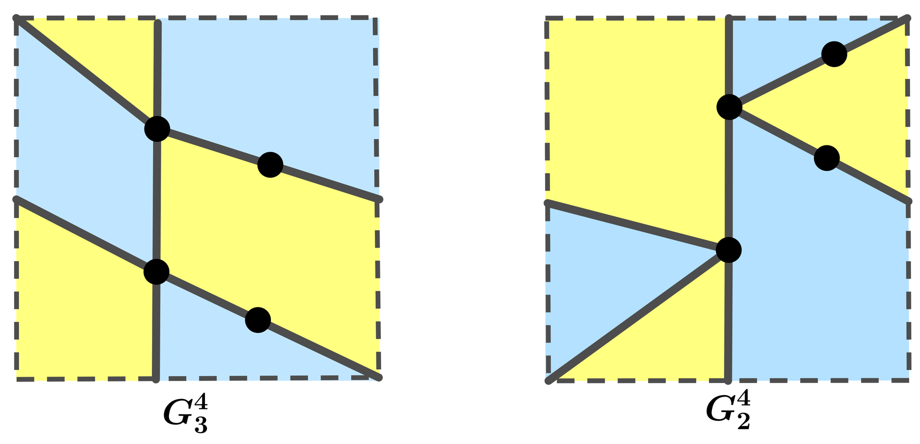

Fig.12: G_3^4 and G_4^4 are two irreducible (2,2)-tight torus graphs with the same underlying abstract graph.

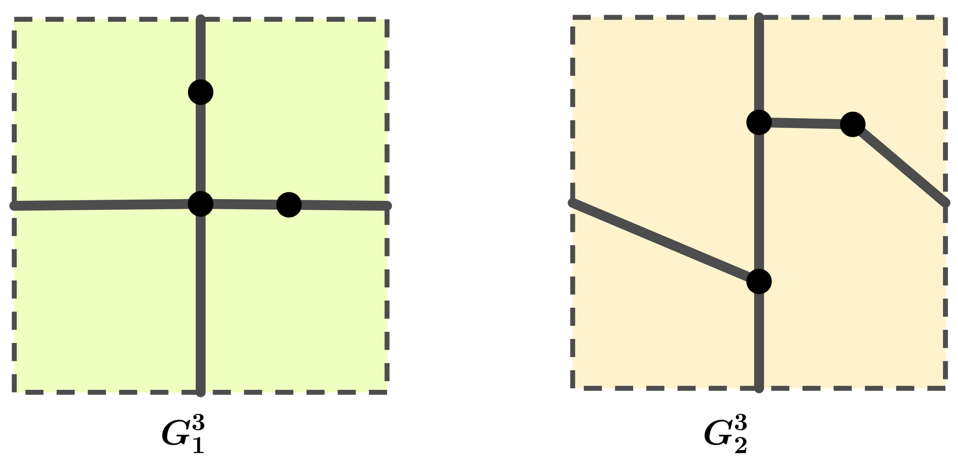

Fig.13: G_1^2 and G_2^3 are the two irreducible (2,2)-tight torus graphs with three vertices and one ocatgonal face.

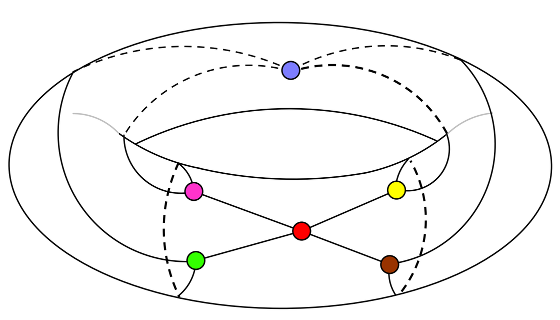

Fig.14: G_1^6 in toroidal represenation

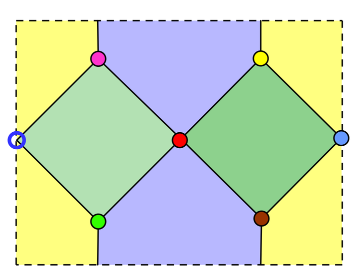

Fig. 15: G_1^6 as flat torus

Contacts of Circular Arcs Representation

- A CCA (Contacts of Circular Arcs) representation of a surface graph G is a configuration of circular arcs embedded in the surface so that the graph induced by the contacts between the arcs is isomorphic to G.
- Every $(2,2)$-tight plane graph admits a CCA rep. in the Euclidean plane.

We present and prove the following main theorem.

Theorem: Every $(2,2)$-tight torus graph admits a CCA rep. in the flat torus.

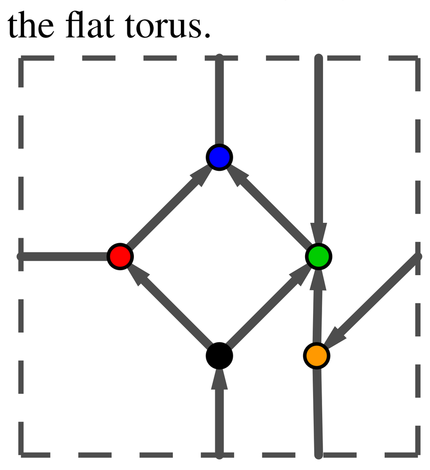

Fig.16: Oriented G_2^5.

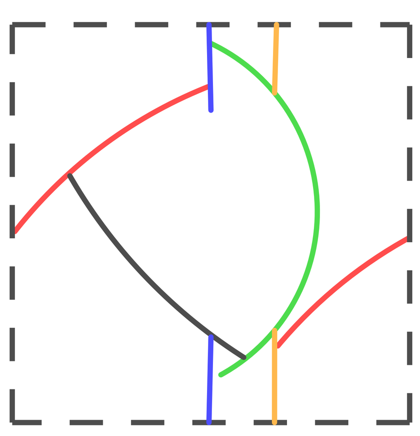

Fig.17: CCA rep. of G_2^5.

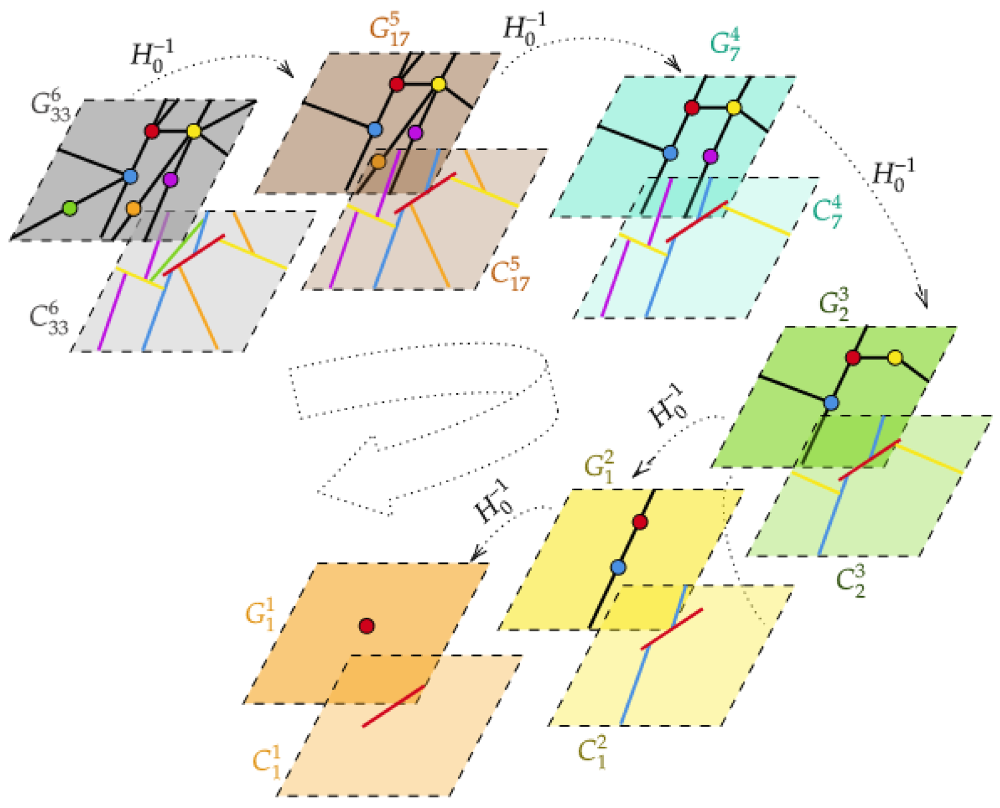

Fig. 18: A sequence of bivalent vertex deletions.

The strategy for proving the previous theorem is to show first that each of irreducible graph has a CCA rep.. This can be done by using the following theorem.

■ Every $(2,2)$-tight torus graph can be reduced to one of the eight graphs colored with red in figure 11 by a sequence of bigon , triangle , quadrilateral contractions or bivalent vertex deletions.

Then we show that the three inductive moves preserve the CCA rep.

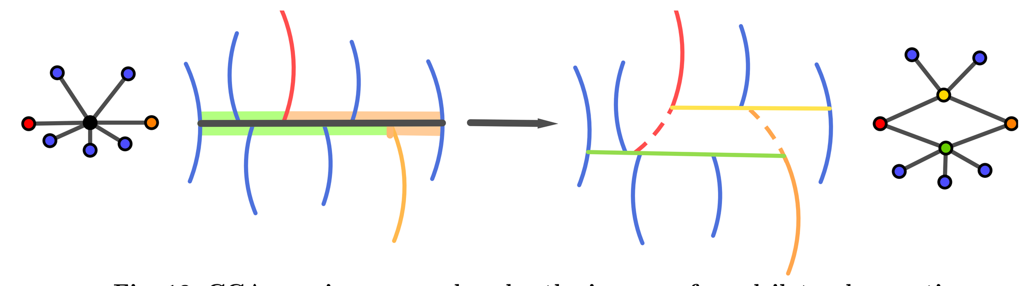

Fig. 19: CCA rep. is preserved under the inverse of quadrilateral operation

■ A torus graph G has a CCA rep. in the flat torus if and only if the universal cover of G has a doubly periodic CCA rep. in the plane.

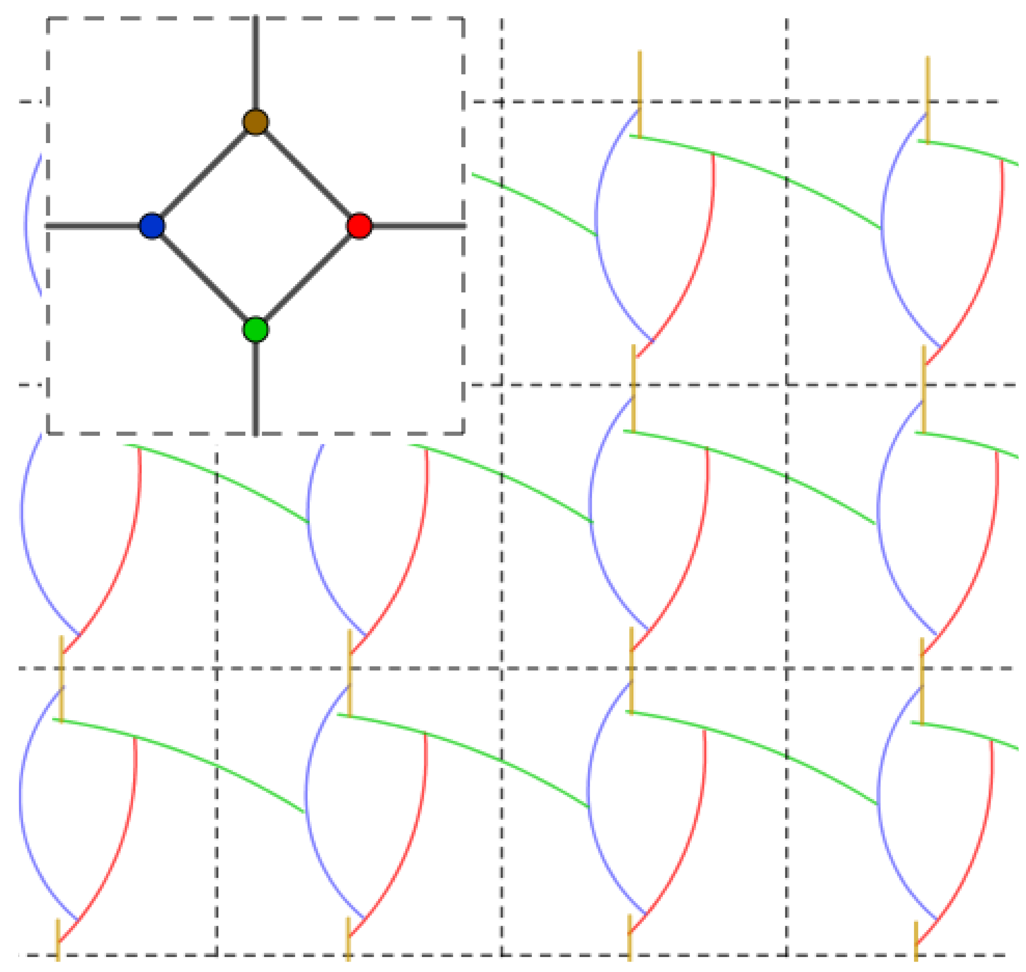

Fig. 20 : Doubly periodic CCA rep. of the torus graph G_1^4.

Key Symbols of the Poster

- Known def. • Known result. ■ New theorem. • New lemma.
- New def. • Example.

Acknowledgement

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