A Deletion-Contraction Relation for the Chromatic Symmetric Function

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University of Albany Discrete Math 2-Day

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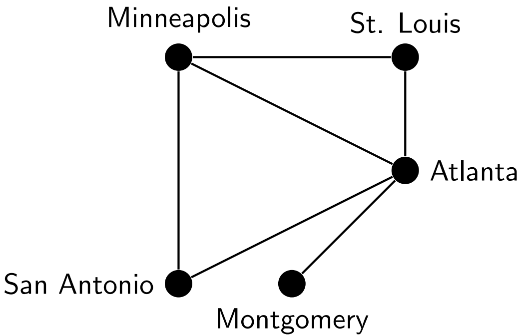
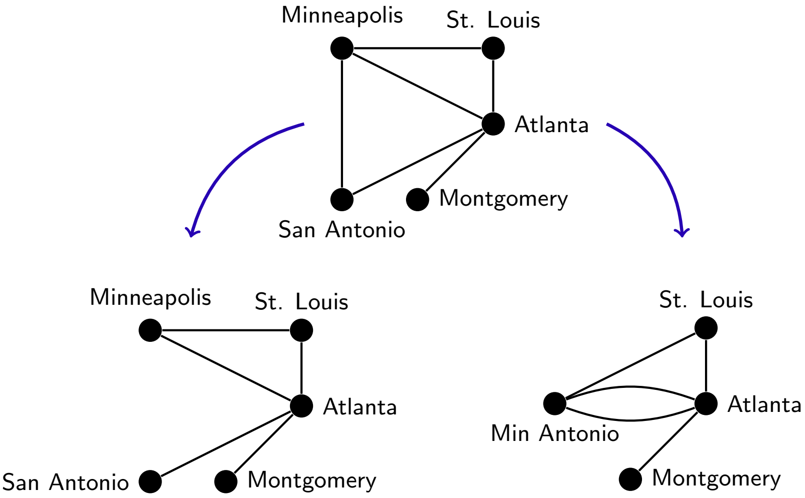
A Graph



The Graph $\Delta = (V, E)$: Airports and Flights

Edge Deletion and Contraction in Graphs



Deletion: $\Delta \backslash$(Min-San) **Contraction:** $\Delta /$(Min-San)

A Deletion-Contraction Relation for χ_G

Definition (Birkhoff)

The **chromatic polynomial** $\chi_G(x)$ is defined by letting $\chi_G(n)$ be the number of n-colorings of G for all $n \in \mathbb{N}$.

Theorem (Folklore)

For every graph $G = (V, E)$ *and any edge* $e \in E$,

$$\chi_G(x) = \chi_{G \setminus e}(x) - \chi_{G/e}(x).$$

The Chromatic Symmetric Function

Let $G = (V, E)$ be a graph.

Definition (Stanley (1995))

$$X_G(x_1, x_2, \dots) = \sum_{\text{col. } \kappa} \prod_{v \in V} x_{\kappa(v)}$$

This function is a power series in $\mathbb{R}[[x_1, x_2, \dots]]$. It is called a **symmetric function** because for every permutation π of $\mathbb{N}$,

$$f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots).$$

This function is a generalization of the chromatic polynomial since

$$X_G(\underbrace{1, 1, \dots, 1}_{n \text{ 1s}}, 0, 0, \dots) = \chi_G(n)$$

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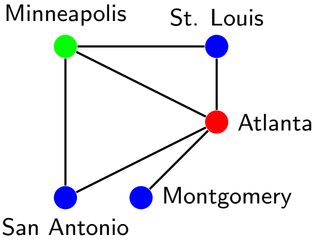
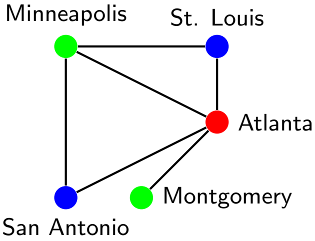
Computing X_Δ

Let blue $= 1$, green $= 2$, red $= 3$.

$$X_\Delta = x_1^3 x_2 x_3 + \cdots + x_1^2 x_2^2 x_3 + \ldots$$



$x_1^3 x_2 x_3$



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Vertex-Weighted X_G

Let $G = (V, E)$ be a graph.

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Let $w : V \to \mathbb{N}$.

Definition (C.-Spirkl (2019))

$$X_{(G,w)}(x_1, x_2, \dots) = \sum_{\text{col. } \kappa} \prod_{v \in V} x_{\kappa(v)}^{w(v)}$$

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A Deletion-Contraction Relation

- $X_{(G,w)}(\underbrace{1, 1, \dots, 1}_{n \text{ 1s}}, 0, 0, \dots) = \chi_G(n)$
- $X_{(G,w)}$ is homogeneous of degree $\sum_{v \in V} w(v)$

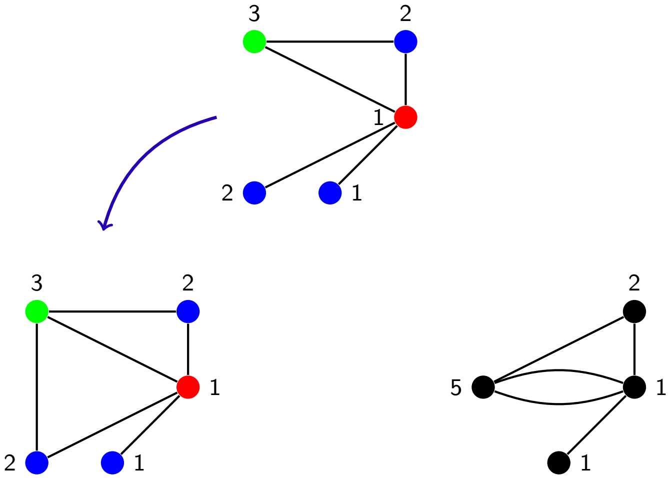
Theorem (C.-Spirkl (2019))

Let (G, w) be a vertex-weighted graph, and let e be any edge of G. Then

$$X_{(G,w)} = X_{(G \backslash e, w)} - X_{(G/e, w/e)}.$$

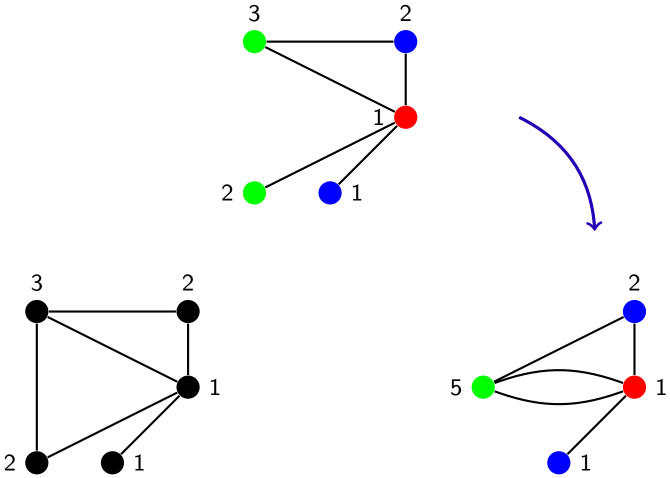
Here w/e means that when the edge e is contracted, the weights of the contracted vertices are added.

A Deletion-Contraction Relation



$$X_{(G\backslash e,w)} = X_{(G,w)} + X_{(G/e,w/e)}$$

A Deletion-Contraction Relation



$$X_{(G \backslash e, w)} = X_{(G,w)} + X_{(G/e, w/e)}$$

Acyclic Orientations

In a directed graph G, a **sink** is a vertex with no out-edges.

Theorem (Stanley (1995))

Let $G = (V, E)$, and let $X_G = \sum c_\lambda e_\lambda$, where $\{e_\lambda\}$ is the basis of elementary symmetric functions. Then the number of acyclic orientations of G is

$$\sum c_\lambda.$$

The number of acyclic orientations of G with exactly k sinks is

$$\sum_{\substack{\lambda \text{ has} \\ k \text{ parts}}} c_\lambda.$$

- Analogue of the formula $(-1)^m \chi_G(-1)$ for acyclic orientations

Acyclic Orientations

For an acyclic orientation γ of (G, w), let $Sink(\gamma)$ be the set of sink vertices, and $sink(\gamma) = |Sink(\gamma)|$.

Define a **sink map** of γ to be a map $S : V \to 2^{\mathbb{N}}$ such that $S(v) \subseteq [w(v)]$ and $S(v) \neq \emptyset$ iff $v \in Sink(\gamma)$.

Theorem (C.-Spirkl (2019))

Let $n = |V|$, $d = \sum_{v \in V} w(v)$, *and* $X_{(G,w)} = \sum_{\lambda \vdash d} c_\lambda e_\lambda$. *Then*

$$\sum_{\substack{\lambda \text{ has} \\ k \text{ parts}}} c_\lambda = (-1)^{d-n} \sum_{(\gamma, S)} (-1)^{k - sink(\gamma)}$$

where the sum is over (γ, S) *such that* γ *is an acyclic orientation of* G, S *is a sink map of* γ, *and* $sw(G, \gamma, S) = \sum_{v \in Sink(\gamma)} |S(v)| = k$.

Acyclic Orientations: Proof Sketch

- Induction on $|E|$; want to show

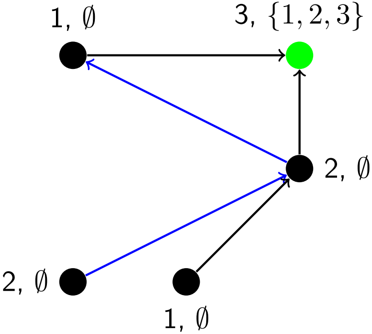
$$\sum_{sw(G\backslash e,\gamma,S)=k} (-1)^{sink(\gamma)} = \sum_{sw(G,\gamma,S)=k} (-1)^{sink(\gamma)} - \sum_{sw(G/e,\gamma,S)=k} (-1)^{sink(\gamma)}.$$

- Fix γ_0, an acyclic orientation of $G\backslash e$.
- Fix $S_0 : V \to 2^{\mathbb{N}}$ with $S_0(v) \subseteq [w(v)]$ for all v.
- Get two (γ, S) for G (both orientations of e), and one (γ, S) in G/e (with $S(v^*) = S(v_1) \cup \{w(v_1) + i : i \in S(v_2)\}$).
- Only count (γ, S) if γ is acyclic, and S is a sink map for γ.
- Want to show: LHS = RHS for terms arising from γ_0 and S_0.

Acyclic Orientations: Proof Sketch

- Here: $k = 3$.

Case 1: There is a directed path between the endpoints of e. Then regardless of the map S_0, contraction fails, and one orientation of adding e fails. The other is valid with S_0 if and only if the original on $G\backslash e$ is.



Valid term for $\Delta\backslash e$ with 1 sink

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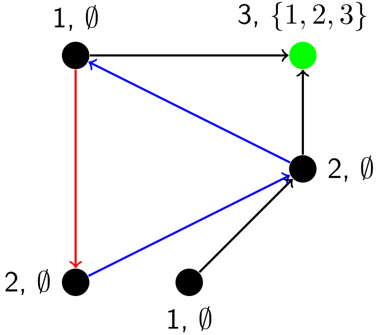

Invalid first term for Δ

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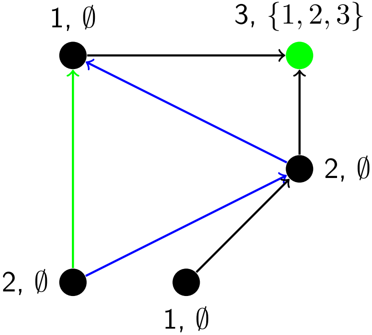

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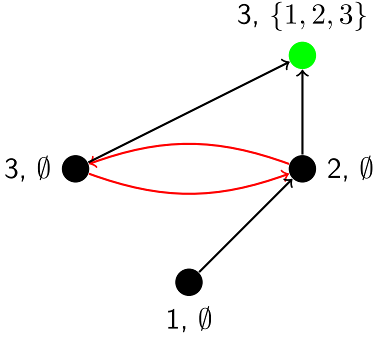


Invalid term for Δ/e

Acyclic Orientations: Proof Sketch

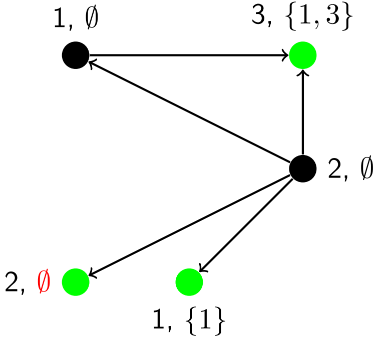
- Here: $k = 3$.

We now divide into cases based on whether one, both, or neither of the endpoints of e is a sink with respect to γ. All of these cases have fairly similar approaches, so we will go through just one of them, the case in which exactly one endpoint is a sink.

Acyclic Orientations: Proof Sketch

- Here: $k = 3$.

Subcase: S_0(San Antonio) is empty (must have S_0(Minneapolis) empty).



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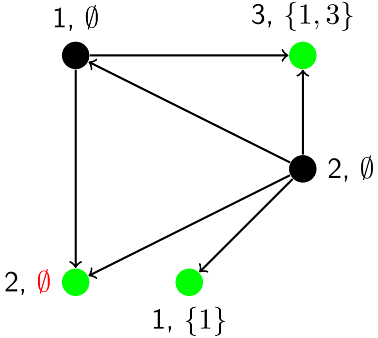

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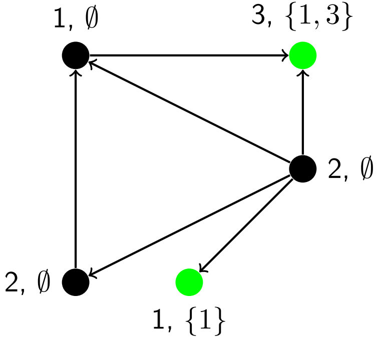

Valid second term for Δ with 2 sinks

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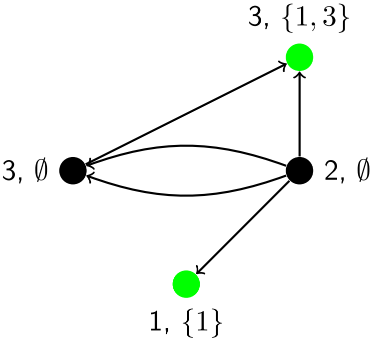

Valid term for Δ/e with 2 sinks

Acyclic Orientations: Proof Sketch

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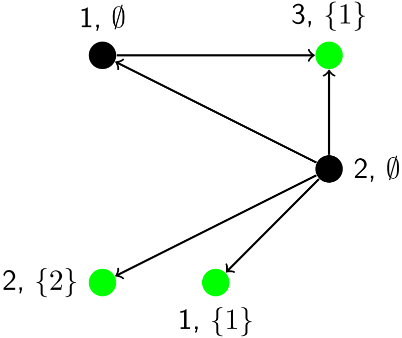
Subcase: S_0(San Antonio) is nonempty (must have S_0(Minn.) empty).



Valid term for $\Delta\backslash e$ with 3 sinks

Acyclic Orientations: Proof Sketch

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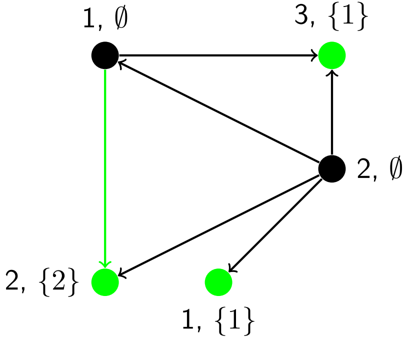

Valid first term for Δ with 3 sinks

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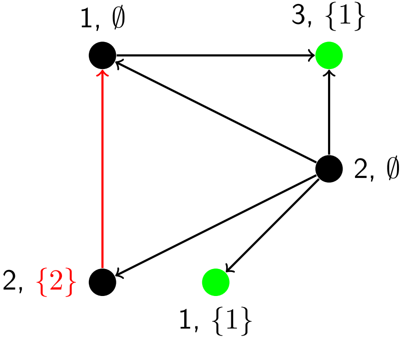

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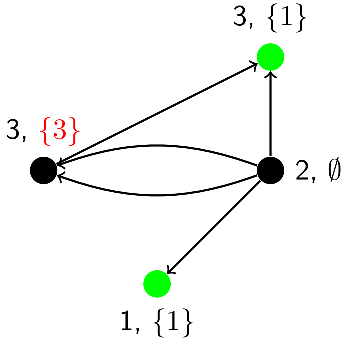

Invalid term for Δ/e

Other Results

Let (G, w) be a vertex-weighted graph with n vertices and total weight d.

Theorem (Stanley (1995), C.-Spirkl(2019))

$$X_{(G,w)} = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(G,w,S)}$$

where $\lambda(G, w, S)$ is the partition of the total weights of the connected components of (V, S).

Theorem (Stanley (1995), C.-Spirkl(2019))

$$\sum_{\substack{(\gamma,\kappa) \\ u \to_\gamma v \implies \kappa(u) \leq \kappa(v)}} \prod_{v \in V(G)} x_{\kappa(v)}^{w(v)} = (-1)^{d-n} \omega(X_{(G,w)})$$

where the sum ranges over all acyclic orientations γ of G and κ is a (not necessarily proper) coloring of G.

The End

This talk is based on the paper "A Deletion-Contraction Relation for the Chromatic Symmetric Function" joint with Sophie Spirkl, `https://arxiv.org/abs/1910.11859`.

Thank you!