Combinatorial models in the representation theory of quantum affine Lie algebras

Carly Briggs (Bennington College), Cristian Lenart (State Univ. of New York at Albany)
Adam Schultze (State Univ. of New York at Albany)

Abstract

We give an explicit description of the unique crystal isomorphism between two combinatorial models for tensor products of Kirillov-Reshetikhin crystals: the tableau model and the quantum alcove model.

Crystal Bases

Main idea: use colored directed graphs to encode certain representations V of the quantum group $U_q(\mathfrak{g})$ as $q \to 0$ ($\mathfrak{g}$ complex semisimple or affine Lie algebra).
Kashiwara (crystal) operators are modified versions of the Chevalley generators (indexed by the simple roots α_i): $\tilde{e}_i, \tilde{f}_i$.
V has a *crystal basis* $\boldsymbol{B}$

$$\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \sqcup 0,$$

$$\tilde{f}_i(b) = b' \Leftrightarrow \tilde{e}_i(b') = b \Leftrightarrow b \xrightarrow{i} b'.$$

Crystal graph: directed graph on B with edges colored $i \leftrightarrow a_i$.

Kirillov-Reshetikhin (KR) crystals

Correspond to certain *finite*-dimensional representations (not highest weight) or affine Lie algebras $\hat{\mathfrak{g}}$. Consider the untwisted affine types $\mathbf{A}_{n-1}^{(1)} - \mathbf{G}_2^{(1)}$. The corresponding crystals have edges (associated to crystal operators) $\tilde{f}_0, \tilde{f}_1, \ldots$.

Labeled by $p \times q$ rectangles, and denoted $\mathbf{B}^{p,q}$.

Definition. Given a composition $\mathbf{p} = (p_1, p_2, \ldots)$, let

$$\mathbf{B}^{\mathbf{p}} = \mathbf{B}^{p_1,1} \otimes \mathbf{B}^{p_2,1} \otimes \ldots.$$

The crystal operators are defined on $\mathbf{B}^{\mathbf{p}}$ by a tensor product rule.

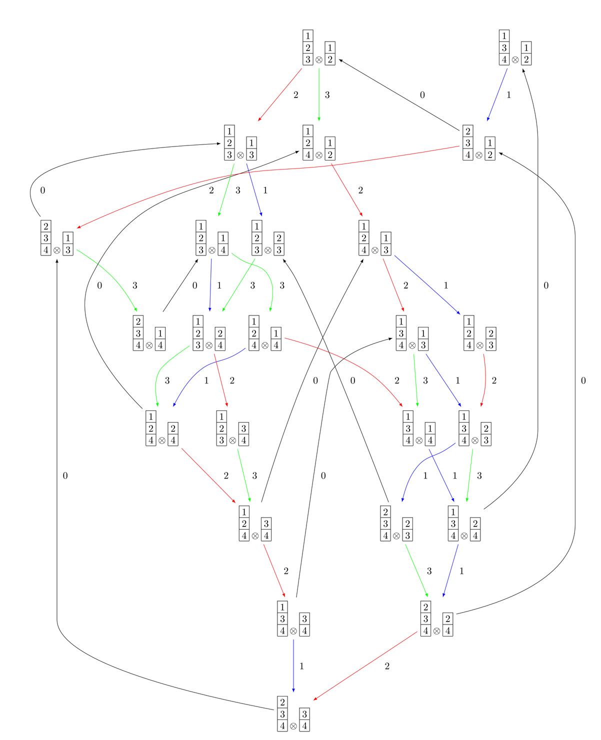
The Tableau Model

With the removal of the $\tilde{f}_0$ arrows, in types A_{n-1} and C_n, we have $\mathbf{B}^{k,1} \cong \mathbf{B}(\omega_k)$ and in types C_n and D_n, we have

$$\mathbf{B}^{k,1} \cong \mathbf{B}(\omega_k) \sqcup \mathbf{B}(\omega_{k-2}) \sqcup \mathbf{B}(\omega_{k-4}) \sqcup \ldots$$

where each $B(\omega_k)$ is given by KN columns of height k. These are strictly increasing fillings of the columns with entries $1 < 2 < \ldots < n$ in type A_{n-1}. With some additional conditions, they are fillings with entries $1 < \ldots < n < \overline{n} < \ldots < \overline{1}$ in type C_n. Types B_n and D_n are similar.

Type A_4 Crystal Graph of $\mathbf{B}^{3,1} \otimes \mathbf{B}^{2,1}$



The Quantum Alcove Model for $\mathbf{B}^{\mathbf{p}}$

The main ingredient is the Weyl group $\mathbf{W} = \langle s_\alpha : \alpha \in \Phi \rangle$.
The *quantum Bruhat graph* on $\mathbf{W}$ is the directed graph with labeled edges $w \xrightarrow{\alpha} ws_\alpha$, where
$l(ws_\alpha) = l(w) + 1$ (Bruhat graph), or
$l(ws_\alpha) = l(w) + 1 - 2\langle \rho, \alpha^\vee \rangle$.

Definition. Given a dominant weight $\lambda = \omega_{p_1} + \ldots + \omega_{p_r}$, we associate with it a sequence of roots, called a $\lambda-chain$ (many choices possible):

$$\Gamma = (\beta_1, \beta_2, \ldots, \beta_m).$$

Let $r_i := s_{\beta_i}$. We consider subsets of positions in Γ

$$J = (j_1 < j_2 < \ldots < j_s) \subseteq \{1, \ldots, m\}.$$

Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is *admissible* if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s}.$$

Theorem [LNSSS, 2016]: The collection of all admissible subsets, $A(\Gamma)$,is a combinatorial model for $\mathbf{B}^{\mathbf{p}}$.

The Two Realizations

- The Tableaux model is simpler and has less structure.
- The Quantum Alcove model has extra structure which makes it easier to do several computations (energy function, combinatorial R-Matrix, charge statistic…)

Relating the Two Models

We build a forgetful map $fill : \mathcal{A}(\Gamma) \to Tableau(\lambda)$ where $\lambda = \omega_{p_1} + \ldots \omega_{p_r}$.

Definition: For any $k = 1, \ldots, n-1$ we define $\Gamma(k)$ to be the following chain of roots:

$$\begin{gathered}((k, k+1), (k, k+2), \ldots, (k, n) \ldots \\ (2, k+1), (2, k+2), \ldots, (2, n) \\ (1, k+1), (1, k+2), \ldots, (1, n))\end{gathered}$$

Definition: We construct a λ-*chain* as a concatenation $\Gamma := \Gamma^{\mu_1} \ldots \Gamma^1$ where $\Gamma^j = \Gamma(p_j)$.

Example Consider $n = 4$ and $\lambda = (3, 2, 1, 0)$. Then the associated λ-chain is $\Gamma = \Gamma^3 \Gamma^2 \Gamma^1 =$

$$((3,4),(2,4),(1,4)|(2,3),(2,4),(1,3),(1,4)|(1,2),(1,3),(1,4)).$$

Example $J = \{1, 2, 4, 5, 8\} \in \mathcal{A}(\Gamma)$.

$$(\underline{(3,4)},\underline{(2,4)},(1,4)|\underline{(2,3)},\underline{(2,4)},(1,3),(1,4)|\underline{(1,2)},(1,3),(1,4))$$

We get the corresponding path in the Bruhat order/quantum Bruhat graph

$$id = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline \\ \hline 4\\\hline\end{array} \xrightarrow{3,4} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline \\ \hline 3\\\hline\end{array} \xrightarrow{2,4} \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline \\ \hline 2\\\hline\end{array} \Big| \begin{array}{|c|}\hline 1\\\hline 3\\\hline \\ \hline 4\\\hline 2\\\hline\end{array} \xrightarrow{2,3} \begin{array}{|c|}\hline 1\\\hline 4\\\hline \\ \hline 3\\\hline 2\\\hline\end{array} \xrightarrow{2,4} \begin{array}{|c|}\hline 1\\\hline 2\\\hline \\ \hline 3\\\hline 4\\\hline\end{array} \Big| \begin{array}{|c|}\hline 1\\\hline \\ \hline 2\\\hline 3\\\hline 4\\\hline\end{array} \xrightarrow{1,2} \begin{array}{|c|}\hline 2\\\hline \\ \hline 1\\\hline 3\\\hline 4\\\hline\end{array} = end(J)\,.$$

This gives us $fill(J) =$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2\\\hline 3 & 2 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array}\,.$$

The Reverse Map in Type A_{n-1}

Consider the tableau in $\bigotimes_{i=1}^r B^{p_i,1}$ from the previous example

$$f(T) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2\\\hline 3 & 2 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array}\,.$$

Use entries of columns i and $i-1$ viewed as sets to build the desired sub-list of Γ^i where the zero column is the size n column of strictly increasing entries.
This is done with two algorithms: Reorder and Greedy
The resulting bijection is a crystal isomorphism [LL,2015].

The Reorder Rule

First, let us consider the circular order

$$a \preceq_a a+1 \preceq_a \ldots \preceq_a n \preceq_a 1 \preceq_a \ldots \preceq_a a-1\,.$$

We write all chains in $\preceq_a$ starting with a, so the subscript a can be dropped.
Let C and C' be two columns. We fix the entries in C and wish to reorder those in C'.
For each $1 \leq i \leq \#C'$, we have

$$a_i = C'(i) = min\{C'(l) : i \leq l \leq \#C'\}$$

where the minimum is taken with respect to the circle order on $[n]$ starting with $C(i)$.

Example: If $C = \begin{array}{|c|}\hline 2\\\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$ and $C' = \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$. Then $reorder_C(C') = \begin{array}{|c|}\hline 3\\\hline 1\\\hline 4\\\hline\end{array}$.

The Greedy Algorithm

We now rebuild the desired sublist of Γ_i by going through Γ_i root by root.

For root (j_1, j_2) if $C[j_1] \prec C[j_2] \prec \hat{C}'[j_1]$ and $C \xrightarrow{(j_1,j_2)} \hat{C}'$ is in the corresponding QBG, then apply it. Otherwise skip. Continue.

So for our example, we have $\Gamma_1 = ((3,4),(2,4),(1,4))$ and get

$$C = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array} \xrightarrow{(3,4)} \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline 3\\\hline\end{array} \xrightarrow{(2,4)} \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 2\\\hline\end{array}$$

The Type C_n Map

- The filling map is similar.
- The inverse map has one major change. Many KN columns have both i and $\overline{\imath}$ in them, so we use the splitting algorithm [Lecouvey] to bijectively make two columns with no $i, \overline{\imath}$ pairs in either.
- Then similar reorder and greedy algorithms work.
- So now the reverse map is made up of a process of Split, Reorder, and Greedy.
- Example:

$$\begin{array}{|c|}\hline 4\\\hline 5\\\hline 5\\\hline 4\\\hline 3\\\hline\end{array} \xrightarrow{split} \begin{array}{|c|c|}\hline 4 & 1\\\hline 5 & 2\\\hline 3 & 5\\\hline 2 & 4\\\hline 1 & 3\\\hline\end{array}$$

The $\Gamma(k)$ in type C_n comes in two parts. We use the first to get a chain from the left split to the reordered right split and the second to get a chain from the right split to the next column's left split.

The Type B_n Map

- There is a similar filling map
- For the reverse, similar to C_n, we need a splitting map.
- Recall that we now have columns of length $k - 2l$, so we need to Extend back to length k [Briggs].
- Further, the greedy algorithm and reorder algorithm no longer work.
- There is a configuration of two columns CC' that we call being blocked-off.
- Modify greedy and reorder to avoid block-off pattern.

Definition: We say that columns $L = (l_1, l_2, \ldots, l_k), R' = (r_1, r_2, \ldots, r_k)$ are *blocked off at i by $b = r_i$* iff $0 < b \geq |l_i|$ and

$$\{1, 2, \ldots, b\} \subset \{|l_1|, |l_2|, \ldots, |l_i|\}$$

and

$$\{1, 2, \ldots, b\} \subset \{|r_1|, |r_2|, \ldots, |r_i|\}$$

and $|\{j : 1 \leq j \leq i, l_j < 0, r_j > 0\}|$ is odd.

Further Work

- The map in type D_n similar to type B_n, but there is a second pattern to be avoided in Reorder and Greedy.
- The bijections for types B_n and D_n given here are actually crystal isomorphisms.

Bibliography

- C. Lenart, A. Lubovsky, *J. Algebraic Combin.*, 2015
- C. Lenart, S. Naito, D. Sagaki, A. Schilling, M. Shimozono, *Int. Math. Res. Not.*, 2016