An insertion algorithm for diagram algebras

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Reference

[COSSZ'20] L. Colmenarejo, R. Orellana, F. Saliola, A. Schilling, M. Zabrocki, **An insertion algorithm on multiset partitions with applications to diagram algebras**, *Journal of Algebra*, Volume 557, 2020, Pages 97-128, ISSN 0021-8693.

https://authors.elsevier.com/c/1ayBr4~FP4Mqn
(Free download before June 12, 2020)

Insertion algorithms

General idea: RSK algorithm establishes a correspondence between the initial input and pairs of combinatorial objects, called tableaux, subject to certain constraints.

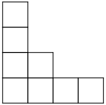
Input Two-line array with $\begin{pmatrix} a_1 & a_2 & \cdots & a_l \\ b_1 & b_2 & \cdots & b_l \end{pmatrix}$

Output Two tableaux of the same shape (P, Q)

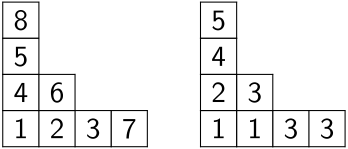
- **R** Robinson (1938): insertion of permutations of S_n
- **S** Schensted (1961): insertion of words of length n in $[k]$
- **K** Knuth (1970): insertion of generalized permutations over $[n]$ and $[k]$ of length ℓ

Partitions and tableaux

- A partition of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$, is a sequence of positive integers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell > 0$ whose sum is n. For instance, $\lambda = (4, 2, 1, 1)$ is a partition of 8 with length 4.
- To each shape λ, we associate a diagram.



- A Young tableau of shape λ is a filling of the diagram of λ.
  - Numbers / Sets of numbers
  - Standard / semistandard



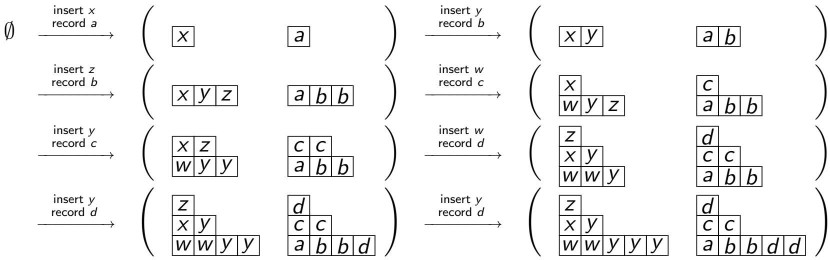
RSK Insertion Procedure

Input

Alphabets: $A = \{a < b < c < d\}$ and $B = \{w < x < y < z\}$

Generalized permutation: $\begin{pmatrix} a & b & b & c & c & d & d & d \\ x & y & z & w & y & w & y & y \end{pmatrix}$

Output



Why are these algorithms interesting?

- Enumerative results
  - Cauchy Formulas
- Connection with representation theory
  - Irreducible representations
  - Dimensions
- Insertion algorithms for partial orders
  - Chromatic polynomials

Classical RSK

R $\{\text{Permutations of } S_n\} \longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$

Enumerative Results

$$n! = \sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2$$

$$f^{\lambda} = \#\mathrm{SYT}(\lambda)$$

Representation Theory

$$\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} \left(S^{\lambda}\right)^* \otimes S^{\lambda}$$

$$(S^{\lambda})^* = \mathrm{Hom}_{S_n}(S^{\lambda}, \mathbb{C})$$

S $\{\text{Words of length } n \text{ in } [k]\} \longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SSYT}_{[k]}(\lambda) \times \mathrm{SYT}(\lambda)$

$$k^n = \sum_{\lambda \vdash n} \#\mathrm{SSYT}_{[k]}(\lambda) \cdot f^{\lambda}$$

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} W_k^{\lambda} \otimes S^{\lambda}$$

K $\left\{ \begin{array}{c} \text{gen.perm. length } \ell \\ \text{from } [n] \text{ to } [k] \end{array} \right\} \longrightarrow \bigcup_{\lambda \vdash \ell} \mathrm{SSYT}_{[k]}(\lambda) \times \mathrm{SSYT}_{[n]}(\lambda)$

$$\prod_{i=1}^{n} \binom{k + \alpha_i - 1}{\alpha_i}$$

$$\mathbb{C}[X] = \bigoplus_{\lambda} W_k^{\lambda} \otimes (W_n^{\lambda})^*$$

RSK from a new perspective

Generalized permutation from [6] to [5]

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 6 & 6 & 6 \\ 1 & 5 & 5 & 2 & 3 & 1 & 3 & 5 & 5 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Multiset perspective

$$\begin{pmatrix} \{\!\{\}\!\} & \{\!\{2,6\}\!\} & \{\!\{2,3,6\}\!\} & \{\!\{1,1,3,3\}\!\} & \{\!\{1,3,4,6\}\!\} \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}$$

Classic RSK insertion (graded lexicographic order)

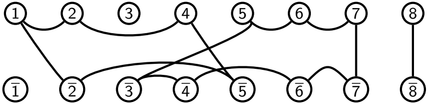
$$\left(\begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & & \\ \hline 1 & 3 & 5 \\ \hline \end{array} \;, \; \begin{array}{|c|c|c|} \hline 1346 & & \\ \hline 26 & & \\ \hline & 236 & 1133 \\ \hline \end{array} \right)$$

Diagram Algebras

Partition algebra: $P_k(n) = \text{span}_{\mathbb{C}}\{\pi \mid \pi \vdash [k] \cup [\overline{k}]\}$, where $[k] \cup [\overline{k}] = \{1, 2, \ldots, k\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{k}\}$ are two disjoint sets.

Digram representation

$$\pi = \{\{1, 2, 4, \overline{2}, \overline{5}\}, \{3\}, \{5, 6, 7, \overline{3}, \overline{4}, \overline{6}, \overline{7}\}, \{8, \overline{8}\}, \{\overline{1}\}\}$$



The dependency on n arises when we multiply the set partitions. We consider the case $n \geq 2k$, so the algebra is semisimple.

Partition elements as arrays

$\pi = \{\pi_1, \pi_2, \ldots, \pi_r\}$ set partition of $[k] \cup [\overline{k}]$.

Blocks are ordered by last letter order.

- $\pi_{j_1}, \pi_{j_2}, \ldots, \pi_{j_p}$ propagating blocks of π ordered as $\pi^+_{j_1} < \cdots < \pi^+_{j_p}$, where $\pi^+_j = \pi_j \cap [k]$ and $\pi^-_j = \pi_j \cap [\overline{k}]$
- $\sigma_{i_1}, \ldots, \sigma_{i_a} \subseteq [k]$ non-propagating blocks in $[k]$ ordered as $\sigma_{i_1} < \cdots < \sigma_{i_a}$
- $\tau_{i_1}, \ldots, \tau_{i_b} \subseteq [\overline{k}]$ non-propagating blocks in $[\overline{k}]$ ordered as $\tau_{i_1} < \cdots < \tau_{i_b}$

$$\pi = \{\{2,3,4,\overline{4},\overline{5}\}, \{5,\overline{2},\overline{3}\}, \{1,6,\overline{7},\overline{8}\}, \{7,8\}, \{9,\overline{6}\}, \{\overline{1}\}, \{\overline{9}\}\}$$

$$\begin{pmatrix} \pi^+_{j_1} & \pi^+_{j_2} & \cdots & \pi^+_{j_p} \\ \pi^-_{j_1} & \pi^-_{j_2} & \cdots & \pi^-_{j_p} \end{pmatrix} = \begin{pmatrix} \{2,3,4\} & \{5\} & \{1,6\} & \{9\} \\ \{\overline{4},\overline{5}\} & \{\overline{2},\overline{3}\} & \{\overline{7},\overline{8}\} & \{\overline{6}\} \end{pmatrix}$$

The correspondence

$$(P,Q) = \textbf{RSK}\begin{pmatrix} \pi^+_{j_1} & \pi^+_{j_2} & \cdots & \pi^+_{j_p} \\ \pi^-_{j_1} & \pi^-_{j_2} & \cdots & \pi^-_{j_p} \end{pmatrix}$$

T $= P$ by adjoining row containing $n-p-b$ empty cells followed by $\tau_{i_1}, \ldots, \tau_{i_b}$

S $= Q$ by adjoining row containing $n-p-a$ empty cells followed by $\sigma_{i_1}, \ldots, \sigma_{i_a}$

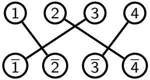
Apply RSK:

$$\begin{pmatrix} \{2,3,4\} & \{5\} & \{1,6\} & \{9\} \\ \{\overline{4},\overline{5}\} & \{\overline{2},\overline{3}\} & \{\overline{7},\overline{8}\} & \{\overline{6}\} \end{pmatrix} \longmapsto \quad P = \begin{array}{|c|c|}\hline \overline{45} & \overline{78} \\ \hline \overline{23} & \overline{6} \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|}\hline 5 & 9 \\ \hline 234 & 16 \\ \hline \end{array}$$

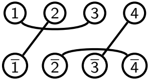
Adjoin new rows:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \overline{45} & \overline{78} \\ \cline{1-2} \overline{23} & \overline{6} \\ \hline & & & & & & & & & & & & \overline{1} & \overline{9} \\ \hline \end{array} \qquad S = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 5 & 9 \\ \cline{1-2} 234 & 16 \\ \hline & & & & & & & & & & & & & 78 \\ \hline \end{array}$$

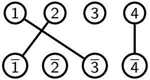
Subalgebras of the Partition Algebra



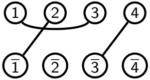
permutation



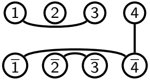
perfect matching



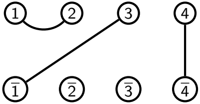
partial permutation



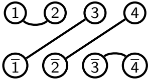
matching



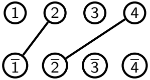
planar



planar matching



planar perfect matching



planar partial permutation

Subalgebras of the Partition Algebra

Subalgebra	Diagrams	Dimension
Partition algebra $P_k(n)$	all diagrams	$B(2k)$
Group algebra $\mathbb{C}S_k$	permutations	$k!$
Brauer algebra $B_k(n)$	perfect matchings	$(2k-1)!!$
Rook algebra $R_k(n)$	partial permutations	$\sum_{i=0}^{k} \binom{k}{i}^2 i!$
Rook-Brauer algebra $RB_k(n)$	matchings	$\sum_{i=0}^{k} \binom{2k}{2i} (2i-1)!!$
Temperley–Lieb algebra $TL_k(n)$	planar perfect matchings	$\frac{1}{k+1}\binom{2k}{k}$
Motzkin algebra $M_k(n)$	planar matchings	$\sum_{i=0}^{k} \frac{1}{i+1}\binom{2i}{i}\binom{2k}{2i}$
Planar rook algebra $PR_k(n)$	planar partial permutations	$\binom{2k}{k}$
Planar algebra $PP_k(n)$	planar diagrams	$\frac{1}{2k+1}\binom{4k}{2k}$

Irreducible Representations

A_k	Index set for irreducibles	Dimension of irreducible $V^{\lambda}_{A_k}$
$P_k(n)$	$\{\lambda \mid \lambda \vdash m, 0 \leqslant m \leqslant k\}$	$f^{\lambda} \sum_{i=\lvert\lambda\rvert}^{k} \binom{k}{i} \left\{ {i \atop \lvert\lambda\rvert} \right\} B(k-i)$
$\mathbb{C}S_k$	$\{\lambda \mid \lambda \vdash k\}$	f^{λ}
$B_k(n)$	$\{\lambda \mid \lambda \vdash k-2r, 0 \leqslant 2r \leqslant k\}$	$f^{\lambda} \binom{k}{\lvert\lambda\rvert} (k - \lvert\lambda\rvert - 1)!!$
$R_k(n)$	$\{\lambda \mid \lambda \vdash m, 0 \leqslant m \leqslant k\}$	$f^{\lambda} \binom{k}{\lvert\lambda\rvert}$
$RB_k(n)$	$\{\lambda \mid \lambda \vdash m, 0 \leqslant m \leqslant k\}$	$f^{\lambda} \binom{k}{\lvert\lambda\rvert} \sum_{i=0}^{(k-\lvert\lambda\rvert)/2} \binom{k-\lvert\lambda\rvert}{2i} (2i-1)!!$
$TL_k(n)$	$\{(k-2r) \mid 0 \leqslant 2r \leqslant k\}$	$\binom{k}{(k-m)/2} - \binom{k}{(k-m)/2-1}$
$M_k(n)$	$\{(m) \mid 0 \leqslant m \leqslant k\}$	$\sum_{i=0}^{\lfloor (k-m)/2 \rfloor} \binom{k}{m+2i} \left(\binom{m+2i}{i} - \binom{m+2i}{i-1} \right)$
$PR_k(n)$	$\{(m) \mid 0 \leqslant m \leqslant k\}$	$\binom{k}{m}$
$PP_k(n)$	$\{(m) \mid 0 \leqslant m \leqslant k\}$	$\binom{2k}{k-m} - \binom{2k}{k-m-1}$

Thank you very much!