Bipartite walks and the Hamiltonians

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Outline

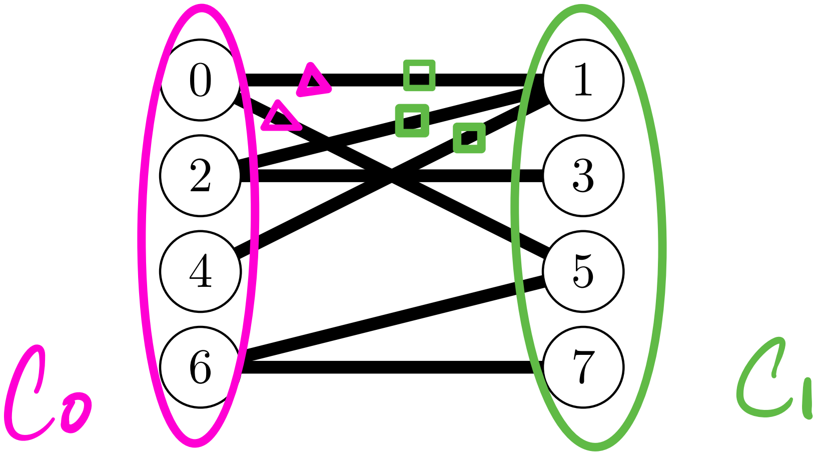
Discrete Quantum Walk

- A discrete quantum walk is determined by a unitary matrix U, the transition matrix of the walk.
- Use a vector e_i to denote the initial state of the quantum walker, then the probability of our walker is at state e_j at k-th step is

$$\left|\left(U^k\right)_{i,j}\right|^2.$$

- Grover's Algorithm: Grover showed how an implementation of this setup could be used to enable quantum computers to search a database faster than any known classical algorithms.

What is a bipartite walk?



- A bipartite graph with color class $C_0 = \{0, 2, 4, 6\}$ and $C_1 = \{1, 3, 5, 7\}$.
- Two edge partitions: two edges are in the same cell in π_0, then they have the same end in the C_0. Partition π_1 is defined in the same fashion.

$$\pi_0 = \{\{(0,1),(0,5)\},\{(2,1),(2,3),\},\{(4,1)\},\{(6,5),(6,7)\}\},$$
$$\pi_1 = \{\{(0,1),(2,1),(4,1)\},\{(2,3),\},\{(0,5),(6,5)\},\{(6,7)\}\}.$$

Two characteristic matrices for π_0, π_1

- P_0, P_1 are the characteristic matrices of π_0, π_1 respectively

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

edges of G. →

↑ cells of π_0

- $\hat{P}_0, \hat{P}_1$ are the normalized P_0, P_1

$$\hat{P}_0 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{P}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Orthogonal Projections from Two Partitions

- Projections: $P = \hat{P}_0 \hat{P}_0^T, \quad Q = \hat{P}_1 \hat{P}_1^T$

$$P = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Transition matrix of a bipartite walk

Transition matrix: $U = (2P - I)(2Q - I)$
The transition matrix of the bipartite walk on G is

$$U = \begin{pmatrix} 0 & -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now, we know what is discrete quantum walk and a discrete quantuam walk model, bipartite walks. So what is next...

Continuous Quantum Walk

Definition

Let G be a graph. A continuous-time quantum walk on G is described by its transition matrix, i.e.,

$$U(t) = \exp\left(itH\right),$$

where H denotes some suitable Hermitian matrix associated to G.

Some common choices of H are adjacency matrix of G, or the Laplacian of G.

Definition

For every unitary matrix U, there exists a Hermitian matrix H such that

$$U = \exp(iH).$$

Such H is called a Hamiltonian of U.

Let $e^{i\theta_r}$ be an eigenvalue of U with eigenmatrix E_r, then the spectral decomposition of U is

$$U = \sum_r e^{i\theta_r} E_r.$$

We have that

$$H = -i \sum_r \log\left(e^{i\theta_r}\right) E_r = \sum_r \theta_r E_r.$$

the Hamiltonian of bipartite walks

Definition

If the angle satisfies that

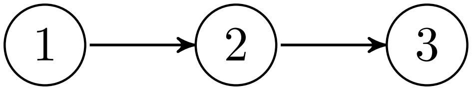
$$-\pi < \theta \leq \pi,$$

then we call H the principal Hamiltonian of U.

Definition

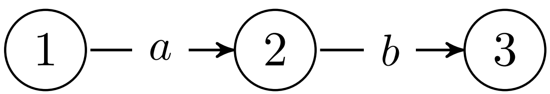
Given an oriented graph $\overrightarrow{G}$ of n vertices, then the skew-adjacency matrix of $A\left(\overrightarrow{G}\right)$ is a skew-symmetric $n \times n$ matrix such that

$$A\left(\overrightarrow{G}\right)_{i,j} = \begin{cases} 1, & \text{if } i \to j \\ -1, & \text{if } i \leftarrow j \\ 0, & \text{otherwise.} \end{cases}$$



$$A(\overrightarrow{H}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Weighted oriented graph



$$A(\overrightarrow{H}) = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & b \\ 0 & -b & 0 \end{pmatrix}$$

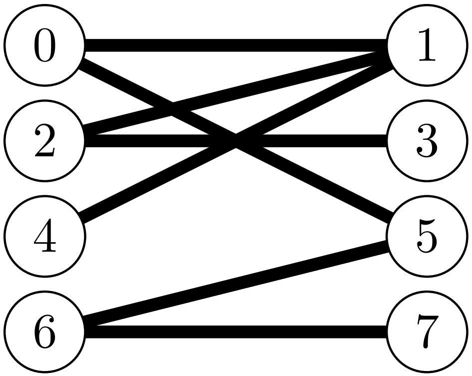
Definition

A Hamiltonian H of the form $H = iS$ with a real skew-symmetric S corresponds to the skew-adjacency matrix of a weighted oriented graph, which we call H-digraph.

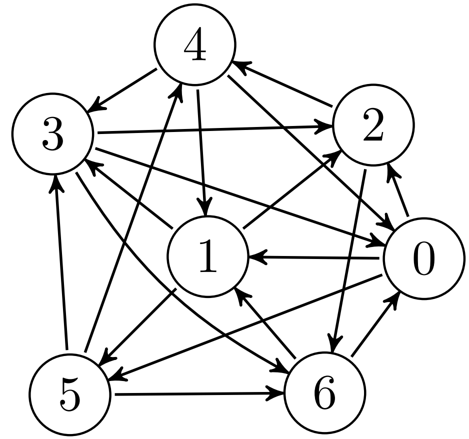
When $H = iS$ with skew-symmetric S, a discrete walk with transition matrix $U = \exp(iH)$ is equivalent to a continuous walk on H-digraph over integer time, i.e.

$$U^k = \exp\left(ikH\right),$$

for non-negative integer k.



(a) G



(b) H-digraph from the bipartite walk on G

$$H = i\begin{pmatrix} 0.0 & 0.76 & 0.08 & -0.92 & -0.31 & 1.07 & -0.68 \\ -0.76 & 0.0 & 0.31 & 0.92 & -0.08 & 0.68 & -1.07 \\ -0.08 & -0.31 & 0.0 & -0.99 & 1.31 & 0.0 & 0.08 \\ 0.92 & -0.92 & 0.99 & 0.0 & -0.99 & -0.61 & 0.61 \\ 0.31 & 0.08 & -1.31 & 0.99 & 0.0 & -0.08 & 0.0 \\ -1.07 & -0.68 & 0.0 & 0.61 & 0.08 & 0.0 & 1.07 \\ 0.68 & 1.07 & -0.08 & -0.61 & 0.0 & -1.07 & 0.0 \end{pmatrix}$$

Invertible U and the skew-symmetric Hamiltonian H

Theorem (Tina Chen)

Let U be the transition matrix of the bipartite walk on a bipartite graph G. Let H be the Hamiltonian of U. Then

$$H = iS_0 + \pi E_{-1},$$

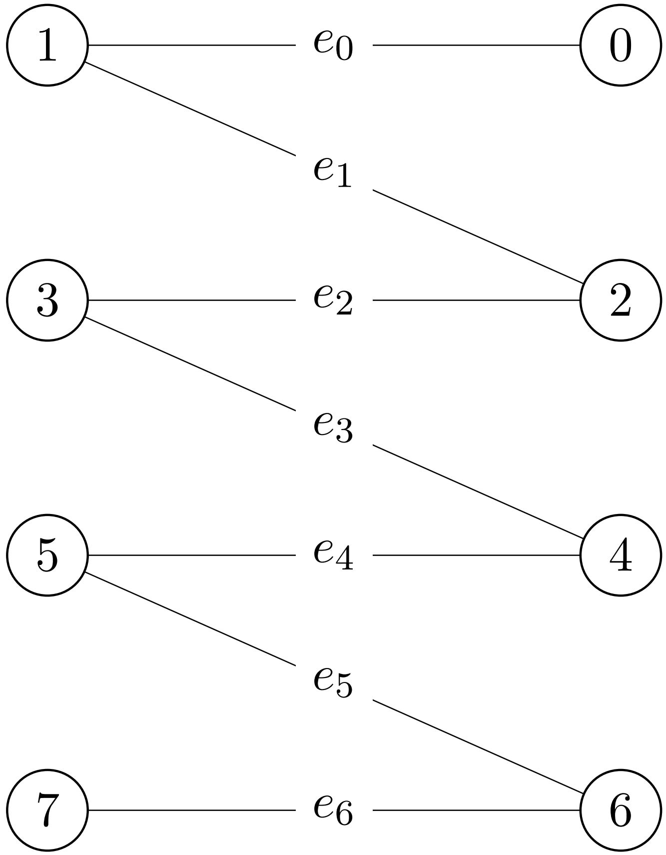
where S_0 is a real skew-symmetric matrix and E_{-1} is the (-1)-eigenmatrix of U. □

Note that E_{-1} is a real matrix.

Corollary (Tina Chen)

Let S be a real skew-symmetric matrix and the Hamiltonian H of the form $H = iS$ if and only if $A(G)$ is invertible. □

Example: Bipartite Walk over P_8



$$U = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Permutation (0135642) in S_7

$$U^7 = I.$$

Bipartite Walks on Paths

Theorem (Tina Chen)

The transition matrix of the bipartite walk on P_n corresponds to a permutation of the form

$$(e_0, e_1, e_3, \cdots, e_{n-3}, e_{n-2}, e_{n-4}, \cdots, e_2)\,. \qquad \square$$

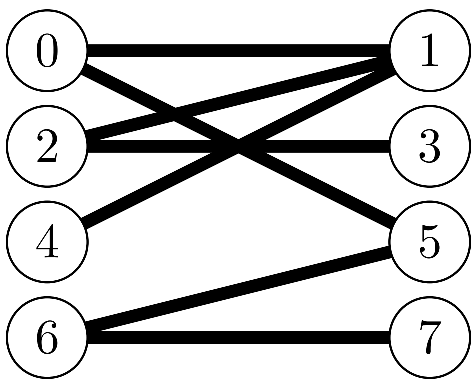
Theorem (Tina Chen)

For an even $n \geq 4$, the H-digraph obtained from the bipartite walk on P_n is an oriented K_{n-1}. $\square$

Perfect State Transfer (PST)

Let U be the transition matrix of a quantum walk defined over graph G, then we say there is perfect state transfer from state a to state b at time t if

$$\left|\left(U^t\right)_{a,b}\right|^2 = 1, \text{ or } |U(t)_{a,b}|^2 = 1.$$



$$U = \begin{pmatrix} 0 & -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Universal PST

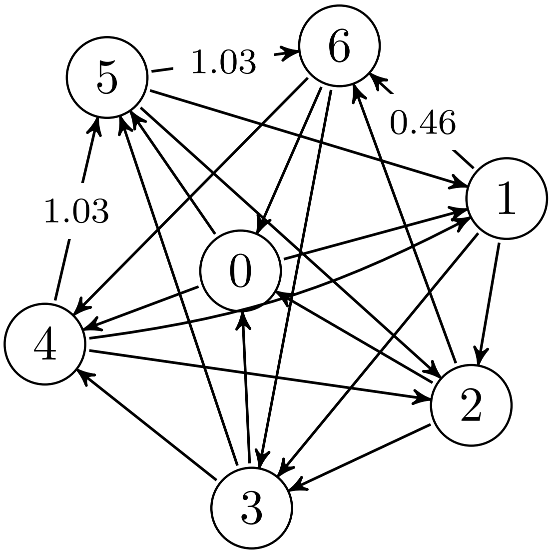
Definition

A graph G has universal perfect state transfer if it has perfect state transfer between every pair of its vertices.

The transition matrix of bipartite walk on P_8 is

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \exp(iH), \quad U^7 = I.$$

the Hamiltonian and H-digraph coming from P_8



$$i\begin{pmatrix} 0.0 & 1.03 & -1.03 & -0.57 & 0.57 & 0.46 & -0.46 \\ -1.03 & 0.0 & 0.57 & 1.03 & -0.46 & -0.57 & 0.46 \\ 1.03 & -0.57 & 0.0 & 0.46 & -1.03 & -0.46 & 0.57 \\ 0.57 & -1.03 & -0.46 & 0.0 & 0.46 & 1.03 & -0.57 \\ -0.57 & 0.46 & 1.03 & -0.46 & 0.0 & 0.57 & -1.03 \\ -0.46 & 0.57 & 0.46 & -1.03 & -0.57 & 0.0 & 1.03 \\ 0.46 & -0.46 & -0.57 & 0.57 & 1.03 & -1.03 & 0.0 \end{pmatrix}$$

For distinct $s, t \in \{0, \cdots, n-2\}$, we define

$$\alpha = \begin{cases} \frac{t-s}{2}, & \text{if both } s, t \text{ are odd;} \\ \frac{s+t+1}{2}, & \text{if } s \text{ is even and } t \text{ is odd;} \\ \frac{-t-s-1}{2}, & \text{if } s \text{ is odd and } t \text{ is even;} \\ \frac{s-t}{2}, & \text{if both } s, t \text{ are even.} \end{cases} .$$

Corollary (Tina Chen)

Let n be an even integer. The edge (s, t) of K_{n-1} is assigned with weight

$$\frac{2}{n-1} \sum_{r=1}^{\frac{n}{2}-1} \frac{2\pi r}{(n-1)} \sin\left(\frac{2\pi r}{n-1}\alpha\right)$$

for all distinct $s, t \in \{0, \cdots, n-2\}$. Let A be the weighted adjacency matrix of the resulting K_{n-1}. Then the continuous walk $\exp(iA)$ has universal PST and PST between any two states happens within time $t \leq n-1$. □

Thank you!