The plethystic inverse of the odd Lie modules

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Background and notation

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For the symmetric group it corresponds to forming representations of wreath products $S_m[S_n]$, and inducing up to S_{mn}.

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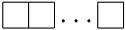
- The homogeneous symmetric function h_n of degree n is the Frobenius characteristic of the trivial representation of S_n, indexed by the partition $(n) =$ .
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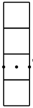
- The elementary symmetric function e_n of degree n is the Frobenius characteristic of the sign representation of S_n, indexed by the partition $(1^n) =$ . It is also the character of $GL(V)$ acting on the nth exterior power $\bigwedge^n(V)$.
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$(1 - p_1)^{-1} = \sum_{n\geq 0} p_1^n$ is the Gl(V)-character on the full tensor algebra $T(V)$.

If $F = \sum_{j \geq 1} f_j$ where each f_j is the Frobenius characteristic ch of an S_j-module W_j, and λ is the partition of n with m_i parts equal to i, then:

$$H_\lambda[F] = \mathrm{ch}\,(\bigotimes_i \mathbf{1}_{S_{m_i}}[W_i])\uparrow^{S_n} = \prod_i h_{m_i}[f_i],$$

$$E_\lambda[F] = \mathrm{ch}\,(\bigotimes_i \mathbf{sgn}_{S_{m_i}}[W_i])\uparrow^{S_n} = \prod_i e_{m_i}[f_i].$$

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The modules are induced from the subgroup $\prod_i S_{m_i}[S_i]$, the normaliser in S_n of the direct product of m_i copies of $S_i, i \geq 1$.

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The involution ω in the ring of symmetric functions, corresponding to tensoring with the sign representation, is defined by

$$\omega(h_n) = e_n.$$

$$Reg_G := 1\uparrow_e^G = \sum_{\chi \text{ irreducible repn of } G} (\dim \chi)\chi.$$

For the symmetric group S_n :

Theorem (Reg0)

$$Reg_{S_n} = \sum_{\lambda \vdash n} f^\lambda \chi^\lambda,$$

with Frobenius characteristic

$$\mathrm{ch}\, Reg_{S_n} := \sum_{\lambda \vdash n} f^\lambda s_\lambda,$$

where λ is an integer partition of n,
$f^\lambda = |\{\text{standard Young tableaux of shape } \lambda\}|$, and
s_λ is the Schur function indexed by λ, so $s_\lambda = \mathrm{ch}\,\chi^\lambda$.

The regular representation of S_n — (I)

Let C_n be the cyclic subroup of S_n generated by the long cycle $\sigma = (1\,2\,\ldots\,n)$, let ω_n be a primitive nth root of unity.
For $1 \leq k \leq n$, $\sigma \mapsto \omega_n^k$ yields a representation of C_n, and these are all the distinct irreducibles, so

$$Reg_{C_n} = \sum_{k=1}^{n} (\omega_n^k).$$

This also gives a decomposition of the regular representation of S_n :

Theorem (Reg1)

$$Reg_{S_n} = \sum_{k=1}^{n} (\omega_n^k)\uparrow_{C_n}^{S_n}$$

The summands in Reg1

Definition

$Lie_n := \mathrm{ch}\,\omega_n \uparrow_{C_n}^{S_n}.$

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$Conj_n := \mathrm{ch}\,\omega_n^n \uparrow_{C_n}^{S_n} = \mathrm{ch}\,1 \uparrow_{C_n}^{S_n}.$

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$Conj_n$ is the permutation representation of S_n by conjugation on the class of n-cycles, since the stabiliser of an n-cycle is the cyclic group C_n.

Let $Lie := \sum_{n\geq 1} Lie_n = \sum_{n\geq 1} \mathrm{ch}\, \omega_n \uparrow_{C_n}^{S_n}$. Then

Theorem (Thrall 1942)

$$H[Lie] = (1 - p_1)^{-1}.$$

The regular representation of S_n — (II)

The plethystic identity $H[Lie] = (1 - p_1)^{-1}$ is equivalent to

Theorem (Reg2 : Thrall 1942)

$$\mathrm{ch}\, Reg_{S_n} = \sum_{\lambda \vdash n} H_\lambda[Lie] = \sum_{\lambda \vdash n} h_{m_1}[Lie_1] h_{m_2}[Lie_2] \ldots,$$

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$$\mathrm{ch}\, Reg_{S_3} = H_{(3)}[Lie] + H_{(2,1)}[Lie] + H_{(1^3)}[Lie]$$

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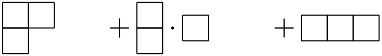
So $Reg_{S_3} =$ 

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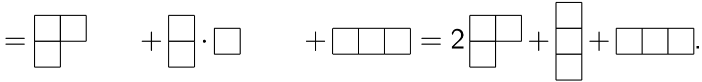
So $Reg_{S_3} =$ 

Plethystic Inverses

Definition

We say symmetric functions f and g are plethystic inverses if

$$f[g] = g[f] = p_1.$$

Theorem (INV1: Cadogan 1971)

$$\sum_{n\geq 1} h_n[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n)] = p_1.$$

Plethystic inverse identities often correspond to *acyclic complexes of modules*.

Plethystic Identities: A meta theorem

Fix $\psi : \mathbb{N}_+ \to \mathbb{R}$.

Let $f_n := \frac{1}{n}\sum_{d|n} \psi(d) p_d^{\frac{n}{d}}$, and $f_n(t) := \frac{1}{n}\sum_{d|n}\psi(d) t^{\frac{n}{d}}$.

Let $F := \sum_{n\geq 1} f_n$, $\quad F^{alt} := \sum_{n\geq 1}(-1)^{n-1} f_n$ (symmetric functions).

Theorem (S 2017)

$$H[F] = \prod_{m\geq 1}(1-p_m)^{-f_m(1)} \quad (1)$$

$$\iff E[F] = \prod_{m\geq 1}(1-p_m)^{f_m(-1)} \quad (2)$$

$$\iff H[\omega(F)^{alt}] = \prod_{m\geq 1}(1+p_m)^{f_m(1)} \quad (3)$$

$$\iff E[\omega(F)^{alt}] = \prod_{m\geq 1}(1+p_m)^{-f_m(-1)} \quad (4)$$

Specialisations of the Meta Theorem I

Thrall's theorem is equivalent to Cadogan's computation of the plethystic inverse of $\sum_{n\geq 1} h_n$, and also to:

Theorem (INV2 : Plethystic inverse of *Lie*)

(Orlik-Solomon, Lehrer-Solomon 1986)

$$\sum_{n\geq 1} (-1)^{n-1} e_n[Lie] = p_1$$

Corresponds to an acyclic complex on the *Orlik-Solomon algebra* for the cohomology of the complement of the type A arrangement.

Specialisations of the Meta Theorem II

Recall $E = \sum_{n\geq 0} e_n$.

Definition (S 2018)

Let k_n be the 2-adic valuation of the positive integer n. Define

$$Lie_n^{(2)} := \omega_n^{2^{k_n}} \uparrow_{C_n}^{S_n}.$$

Notice: $Lie_n^{(2)} = \begin{cases} Lie_n, & n \text{ odd}, \\ Conj_n, & n \text{ a power of 2.} \end{cases}$.

Theorem (Reg3: S 2018)

$$E[\sum_{n\geq 1} Lie_n^{(2)}] = (1-p_1)^{-1}.$$

$$\mathrm{ch}\, Reg_{S_n} = \sum_{\lambda\vdash n} E_\lambda[Lie_n^{(2)}] = \sum_{\lambda\vdash n} e_{m_1}[Lie_1^{(2)}] e_{m_2}[Lie_2^{(2)}] \ldots$$

The inverses of $E - 1 = \sum_{n\geq 1} e_n$ and $\sum_{n\geq 1} Lie_n^{(2)}$

Theorem (S 2018)

$$(INV3) \qquad \sum_{n\geq 1} e_n[\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n^{(2)})] = p_1.$$

and

$$(INV4) \qquad \sum_{n\geq 1}(-1)^{n-1} h_n[\sum_{n\geq 1} Lie_n^{(2)}] = p_1.$$

Question: Is there a more conceptual explanation for (Reg3), (Inv3), (Inv4)? An acyclic complex?

The odd *Lie* modules – a first appearance (?)

Let J_n be the degree n multilinear component of the *free Jordan algebra on n generators*. Instead of the Lie bracket, we have the bracket

$$[x, y] = x \otimes y + y \otimes x.$$

By Schur-Weyl duality:
View J_n as an S_n-module, with Frobenius characteristic η_n. Set $\eta_0 = 1$.

Theorem (Calderbank-Hanlon-S 1994)

The Frobenius characteristic of the S_n-module on the free Jordan algebra satisfies

$$H[\sum_{n\geq 1} Lie_{2n-1}] = \sum_{n\geq 0} \eta_n.$$

Recall that $E := \sum_{n\geq 0} e_n$, $H := \sum_{n\geq 0} h_n$, Define

$$E_{odd} := \sum_{n\geq 0} e_{2n+1}, \ E_{even} := \sum_{n\geq 0} e_{2n};$$

$$H_{odd} := \sum_{n\geq 0} h_{2n+1}, \ H_{even} := \sum_{n\geq 0} h_{2n}.$$

A quotient of symmetric functions I

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Consider

$$\frac{E_{odd}}{E_{even}} \qquad \text{and} \qquad \frac{H_{odd}}{H_{even}}.$$

A quotient of symmetric functions II

Lemma (S 2020)

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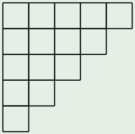
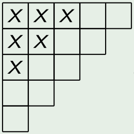
Let $\delta_n = (n-1, n-2, \ldots, 1)$, $n \geq 2$. (Set $\delta_1 = \emptyset$.) δ_n is the staircase shape. We will need to look at the skew-shape δ_n / δ_{n-2}.

Example

For $n = 6$, $\delta_6 =$  and $\delta_6/\delta_4 =$ 

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Theorem (S 2020)

$$\frac{e_1 + e_3 + \dots}{1 + e_2 + e_4 + \dots} = s_{(1)} + \sum_{n\geq 3} (-1)^n s_{\delta_n/\delta_{n-2}}$$
$$= \tanh\Big(\sum_{i\geq 1} \operatorname{arctanh} x_i\Big).$$

Note: The dimension of the S_n-module indexed by the skew shape δ_n/δ_{n-2} is the nth Euler number E_n, which counts the set of alternating (down-up) permutations

$$\{\sigma \in S_n : \sigma(1) > \sigma(2) < \sigma(3) > \cdots\}.$$

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Theorem (INV5: S 2020; conjectured by Richard Stanley)

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Quotients of alternating sums

Define

$$E^{alt}_{odd} := \sum_{n \geq 0} (-1)^n e_{2n+1}, \; E^{alt}_{even} := \sum_{n \geq 0} (-1)^n e_{2n};$$

$$H^{alt}_{odd} := \sum_{n \geq 0} (-1)^n h_{2n+1}, \; H^{alt}_{even} := \sum_{n \geq 0} (-1)^n h_{2n}.$$

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Consider

$$\frac{E_{odd}^{alt}}{E_{even}^{alt}} \quad \text{and} \quad \frac{H_{odd}^{alt}}{H_{even}^{alt}}.$$

A theorem of Carlitz

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Theorem (Carlitz 1973)

Let $\delta_n = (n-1, n-2, \ldots, 1), n \geq 2$. *(Set* $\delta_1 = \emptyset$.*) Then*

$$\frac{e_1 - e_3 + e_5 - \ldots}{1 - e_2 + e_4 - \ldots} = s_{(1)} + \sum_{n \geq 3} s_{\delta_n/\delta_{n-2}}$$

$$= \tan\left(\sum_{i \geq 1} \arctan x_i\right).$$

The *alternating* odd Lie modules

Theorem (INV6: S 2020)

The plethystic inverse of $\sum_{n\geq 0}(-1)^n Lie_{2n+1}$ *is*

$$\frac{e_1 - e_3 + e_5 - \dots}{1 - e_2 + e_4 - \dots}.$$

Decompositions of the regular representation of S_n

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$$= \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n \\ \ell(\lambda)=k}} \omega(H_\lambda[Lie]) \qquad \text{(Eulerian idempotents) (Reg2*)}$$

Gerstenhaber & Schack (1987), Hanlon (1990): Sarah Brauner's talk

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$$Reg_{S_n} = \sum_{k=1}^{n} (\omega_n^k)\uparrow_{C_n}^{S_n} \qquad \text{(Reg1)}$$

$$\text{ch}\, Reg_{S_n} = \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n \\ \ell(\lambda)=k}} H_\lambda[Lie] \qquad \text{(Reg2)}$$

$$= \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n \\ \ell(\lambda)=k}} \omega(H_\lambda[Lie]) \qquad \text{(Eulerian idempotents) (Reg2*)}$$

Gerstenhaber & Schack (1987), Hanlon (1990): Sarah Brauner's talk

$$= \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n \\ \ell(\lambda)=k}} E_\lambda[Lie^{(2)}] \qquad \text{(Reg3)}$$

Decompositions of the regular representation of S_n

$$Reg_{S_n} = \sum_{k=1}^{n} (\omega_n^k) \uparrow_{C_n}^{S_n} \qquad \text{(Reg1)}$$

$$\mathrm{ch}\, Reg_{S_n} = \sum_{k\geq 1} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} H_\lambda[Lie] \qquad \text{(Reg2)}$$

$$= \sum_{k\geq 1} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \omega(H_\lambda[Lie]) \qquad \text{(Eulerian idempotents) (Reg2*)}$$

Gerstenhaber & Schack (1987), Hanlon (1990): Sarah Brauner's talk

$$= \sum_{k\geq 1} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} E_\lambda[Lie^{(2)}] \qquad \text{(Reg3)}$$

$$= \sum_{k\geq 1} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \omega(E_\lambda[Lie^{(2)}]) \qquad \text{(Reg3*)}$$

A new decomposition of the regular representation

Theorem (S, 2020)

Let $Lie_{odd} := \sum_{n\geq 0} Lie_{2n+1}$*. Then the regular representation of* S_n *decomposes as follows:*

$$\mathrm{ch}\, Reg_{S_n} = Hk[Lie_{odd}]|_{\deg n}, \qquad (Reg4)$$

where Hk *is the sum of all hooks:*

$$Hk := s_{(1)} + \sum_{n\geq 2} \sum_{r=0}^{n-1} s_{(n-r,1^r)}.$$

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