The isomorphism problem for cominuscule Schubert varieties

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- $X = G/P$ flag variety
- Fix Borel $B \subset G$
- Schubert varieties $:=$ B-orbit closures in X

  (B'-Schubert varieties are translates of B-Schubert varieties)

Question

Given Schubert varieties $\Omega \subseteq X$ and $\Omega' \subseteq X'$, when is $\Omega \cong \Omega'$ as varieties? Is there a combinatorial classification of their isomorphism classes?

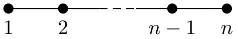
Devlin–Martin–Reiner '07 Classified a class of smooth Schubert varieties in SL_n / P.

Richmond–Slofstra '21 Classified Schubert varieties in G/B using Cartan equivalence.

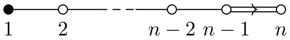
Țarigradschi–X '22 For Grassmannian Schubert varieties,

$$X_\lambda \cong Y_\mu \iff \lambda = \mu \text{ or } \mu^T.$$

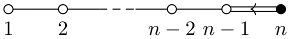
Richmond–Țarigradschi–X '23 Classified Schubert varieties in cominuscule G/P.



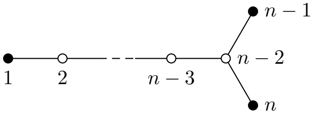
$A_n/P_k = \mathrm{Gr}(k, n+1)$ Grassmannian



$B_n/P_1 = Q^{2n-1}$ Odd quadric

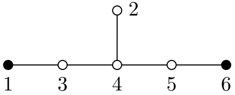


$C_n/P_n = \mathrm{LG}(n, 2n)$ Lagrangian Gr

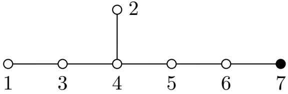


$D_n/P_1 = Q^{2n-2}$ Even quadric

$D_n/P_{n-1} \cong D_n/P_n = \mathrm{OG}(n, 2n)$ Orthogonal Gr



$E_6/P_1 \cong E_6/P_6$ Cayley plane



E_7/P_7 Freudenthal variety

- Let X be the cominuscule flag variety determined by $(\mathcal{D}, \gamma)$,

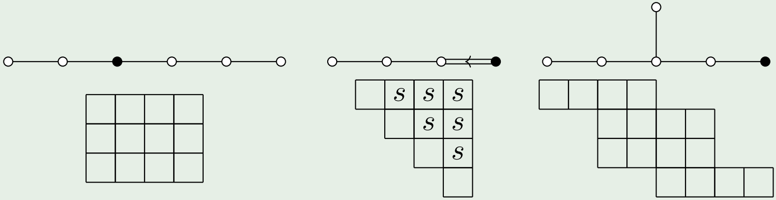
$$\mathcal{P}_X := \{\alpha \in R : \alpha \geq \gamma\}.$$

Proctor Schubert varieties in X are indexed by lower order ideals in $\mathcal{P}_X$.

- Label roots in $\mathcal{P}_X$ by long/short.

Example

$\mathrm{Gr}(3,7)$ $\qquad$ $\mathrm{LG}(4,8)$ $\qquad$ E_6/P_6



Theorem (Richmond–Țarigradschi–X '23)

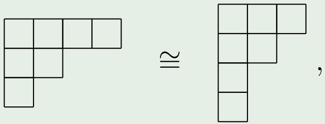
Given lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$,

$$X_\lambda \cong Y_\mu \text{ as varieties} \quad \Longleftrightarrow \quad \lambda \cong \mu \text{ as labeled posets.}$$

Example

In type A (i.e., Grassmannians),

$$\lambda \cong \mu \text{ as labeled posets} \quad \Longleftrightarrow \quad \lambda = \mu \text{ or } \lambda = \mu^T$$



recovering the classification of Grassmannian Schubert varieties in Țarigradschi–X '22.

Example

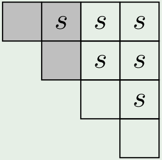
The projective plane $\mathbb{P}^2 \quad \not\cong \quad$ the Schubert divisor in Q^3:

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \quad \not\cong \quad \begin{array}{|c|c|}\hline & s \\ \hline \end{array}\,.$$

Geometrically, the latter is singular.

Example

The quadric Q^3 embeds in $\mathrm{LG}(n, 2n)$ $(n \geq 3)$ as a Schubert variety:



Fulton–MacPherson–Sottile–Sturmfels $\{[X_\mu] : \mu \subseteq \lambda\} = \{$minimal elements in the extremal rays of the effective cone in $A_*(X_\lambda)\}$.

Mathieu $\mathrm{Pic}(X_\lambda)$ is freely generated by a unique effective class D_λ.

As D_λ is pulled back from X, its intersection product with $[X_\mu]$ is computed by the Chevalley formula (Fulton–Woodward):

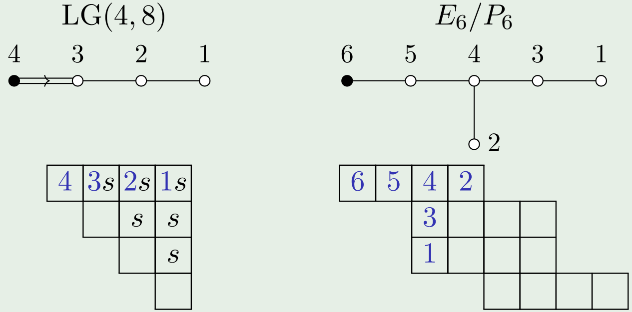
$$D_\lambda \cdot [X_\mu] \quad = \sum_{\alpha \text{ maximal}} \frac{(\gamma,\gamma)}{(\alpha,\alpha)}\,[X_{\mu\setminus\{\alpha\}}].$$

- Coefficient of $[X_\nu] \neq 0 \iff \mu$ covers ν in Bruhat order $\subseteq$, recovering the Bruhat interval $\{\mu : \mu \subseteq \lambda\}$ and the poset λ as its sub-poset of join-irreducible elements.
- If μ is join-irreducible with (unique) maximal box α, then

$$D_\lambda \cdot [X_\mu] = \frac{(\gamma,\gamma)}{(\alpha,\alpha)}\,[X_{\mu\setminus\{\alpha\}}].$$

- $\mathcal{D}$ “sits in” $\mathcal{P}_X$ as a lower order ideal $\lambda_{\mathcal{D}}$.

Example



- By intersecting with $\lambda_{\mathcal{D}}$, each non-empty lower order ideal $\lambda \subseteq \mathcal{P}_X$ determines a connected Dynkin sub-diagram $\mathcal{D}_\lambda$ and γ is a vertex.

Richmond–Slofstra

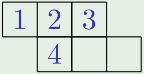
$$X_\lambda \hookrightarrow \text{“minimal” comin. } X' \text{ given by } (\mathcal{D}_\lambda, \gamma).$$

- The graph $\mathcal{D}_\lambda$ depends only on the labeled poset structure of λ. Moreover, every isomorphism $\lambda \to \mu$ induces

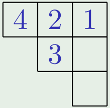
$$\begin{tikzcd} X' \arrow[r, "\cong"] & Y' \\ X_\lambda \arrow[u, hook] \arrow[r, "\cong"] & Y_\mu \arrow[u, hook] \end{tikzcd}.$$

Example

$\mathcal{P}_{Q^6}$ =



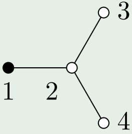
$\cong$



= $\mathcal{P}_{\mathrm{OG}(4,8)}$

$$\Longrightarrow \qquad Q^6 \cong \mathrm{OG}(4,8).$$

It comes from



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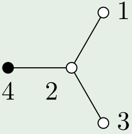

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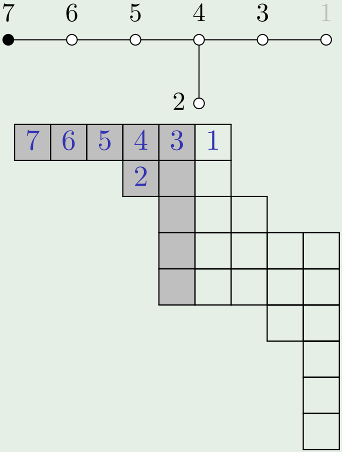
Example

$$X = E_7/P_7 \qquad\qquad X_\lambda \cong X' \cong Q^{10}$$



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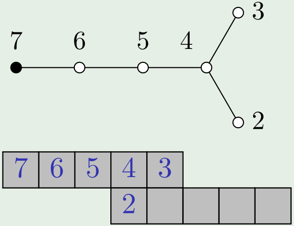
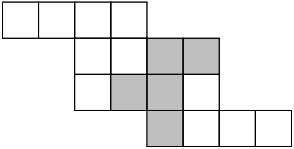
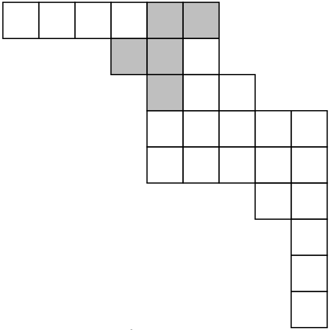


What about Richardson varieties?



$\cong$



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Thank you!