Diophantine m-tuples and elliptic curves

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Do you see any pattern ?

$$\{1, 3, 8\}$$

How about this set ?

$$\{1, 3, 8, 120\}$$

Introduction of Diophantine m-tuples

A Diophantine tuple (over $\mathbb{N}$) is a set $\mathcal{D}$ of positive integers such that for any distinct elements a and b in $\mathcal{D}$, we have

$$ab + 1 = \square.$$

The previous example $\{1, 3, 8, 120\}$ is by Fermat :

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and this is the only way of extending $\{1, 3, 8\}$!

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Can you find any Diophantine quintuple ?
No ! (He - Togbé - Ziegler, 2019)

Diophantus of Alexandria found the first set of positive rationals having this property.

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

Generalizations

A Diophantine tuple over F is a subset $\mathcal{D}$ of a field F such that for any distinct elements a and b in $\mathcal{D}$, we have

$$ab + 1 = \square.$$

$F = \mathbb{Q}$, we have a quintuple

$$\left\{\frac{243}{560}, \frac{1147}{5040}, \frac{1100}{63}, \frac{7820}{567}, \frac{95}{112}\right\}$$

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but there is no known example of size 7.

Why is it hard to extend Diophantine tuples ?

Given a Diophantine triple {a,b,c} over a field F, extending it to a Diophantine quadruple {a,b,c,x} over F means to find $x \in F$ such that

$$ax + 1 = s^2$$
$$bx + 1 = t^2$$
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$$(ax + 1)(bx + 1)(cx + 1) = (str)^2$$

Hence, extending a Diophantine triple to a quadruple is as hard as finding such $x \in F$!

Why elliptic curves ?

An elliptic curve (over $\mathbb{Q}$) is the set of solutions to an equation of the form

$$E : y^2 = x^3 + Ax^2 + Bx + C, \quad \text{where } A, B, C \in \mathbb{Q}.$$

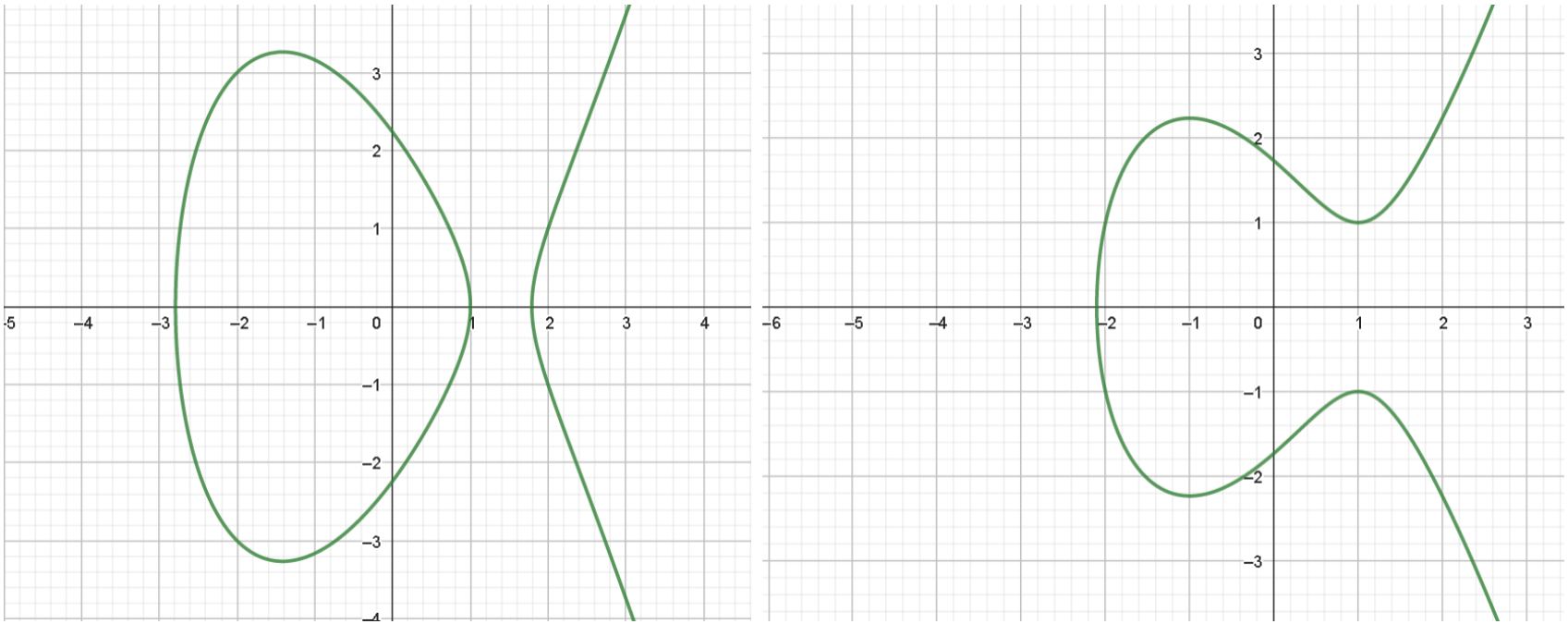


Figure – Graph of $y^2 = x^3 - 6x + 5$ and $y^2 = x^3 - 3x + 3$.

Structure of elliptic curves

The rational points on elliptic curves form an abelian group!

$$E(\mathbb{Q}) = \{(X, Y) \in \mathbb{Q} \times \mathbb{Q} : Y^2 = X^3 + AX^2 + BX + C\} \cup \{\mathcal{O}\}$$

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r \quad \text{(Mordell, 1922)}$$

They play crucial roles in modern number theory.

The classification of $E(\mathbb{Q})_{\text{tors}}$ (Mazur, 1977)

The group $E(\mathbb{Q})$ contains at most 16 points of finite order.

The rank of elliptic curves : how large ?

There are conjectures on the rank r of elliptic curves

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$$

Current Record : There is an elliptic curve with rank either 28 or 29 (Elkies), and 28 subject to GRH.

Siegel's Theorem

Moreover, we have $\#E(\mathbb{Z}) < \infty$.

Therefore, there is no infinite Diophantine m-tuples.

Similarly,

Generalized Diophantine m-tuples

Fix a natural number $k \geq 2$. A set of natural numbers $S = \{a_1, a_2, \ldots, a_m\}$ is said to satisfy property $D_k(n)$ if $a_i a_j + n$ is a k-th power for all $1 \leq i < j \leq m$.

We analogously define the following quantity for each n,

$$M_k(n) = \sup\{|S| : S \text{ satisfies property } D_k(n)\}$$
$$M_k(n; L) = \sup\{|S \cap [n^L, \infty)| : S \text{ satisfies property } D_k(n)\}$$

For $k \geq 3$ and $m \geq 3$, the theorem of Faltings implies that a superelliptic curve of the form

$$y^k = (a_1x + n)(a_2x + n)(a_3x + n)(a_4x + n) \cdots (a_mx + n)$$

has only finitely many rational points, and hence finitely many integral points. Therefore, a set S satisfying property $D_k(n)$ must be finite.

We produce sharper bounds on $M_k(n)$ under the Paley graph conjecture, namely,

Paley graph conjecture

Let $\epsilon > 0$, $S, T \subseteq \mathbb{F}_p$ for an odd prime p with $|S|, |T| > p^{\epsilon}$, and χ be any non-trivial multiplicative character modulo p. Then, there is $\delta = \delta(\epsilon)$ for which the inequality

$$\left| \sum_{a \in S,\, b \in T} \chi(a+b) \right| \leq p^{-\delta} |S||T|$$

holds for primes p larger than some constant $C(\epsilon)$.

The conjecture is known for the case $|S| > p^{1/2+\epsilon}$ and $|T| > p^{\epsilon}$.

Theorem (Dixit - K. - Murty)

Let $k \geq 3$ be a positive integer. Then, the following holds as $n \to \infty$.

(a) For $L \geq 3$,

$$M_k(n, L) \ll 1,$$

where the implied constant depends on k and L, but is independent of n.

(b) Unconditionally,

$$M_k(n) \ll_k \log n.$$

(c) Assuming the Paley graph conjecture, for any $\epsilon > 0$,

$$M_k(n) \ll_{k,\epsilon} (\log n)^{\epsilon}.$$

Over finite fields ?

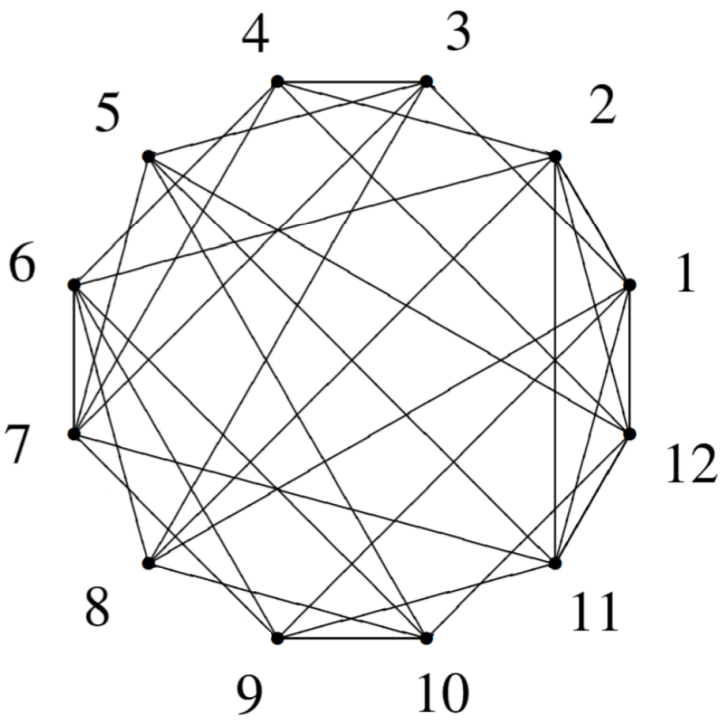
Finite fields are important to test whether there is an integer solution. When $F = \mathbb{F}_q$, Diophantine tuples can be studied using graphs.

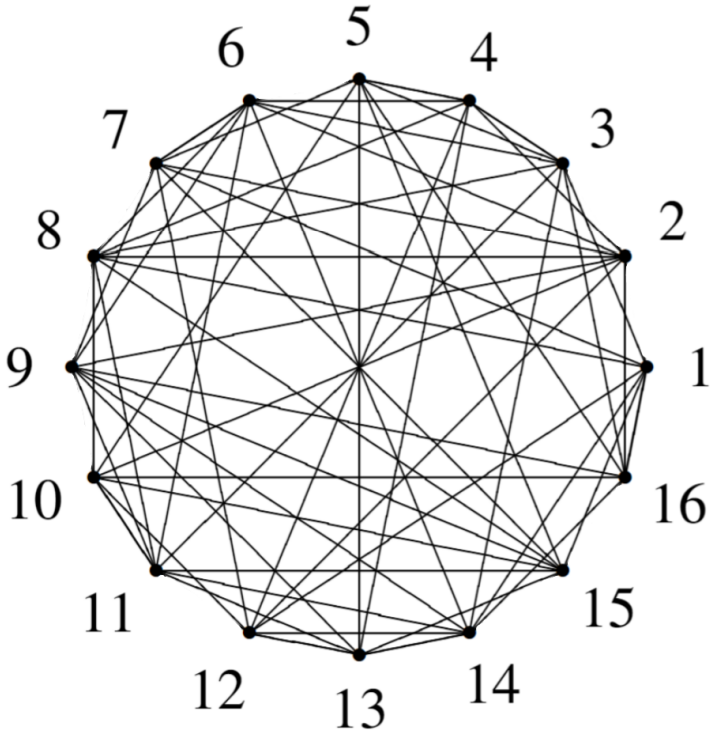
Diophantine graph D_q

The Diophantine graph D_q is the graph whose vertex set is $\mathbb{F}_q^*$, and two vertices x and y are adjacent if and only if $xy + 1$ is a square in $\mathbb{F}_q$.

In particular, the clique number of D_q gives the largest length of Diophantine tuples over $\mathbb{F}_q$.

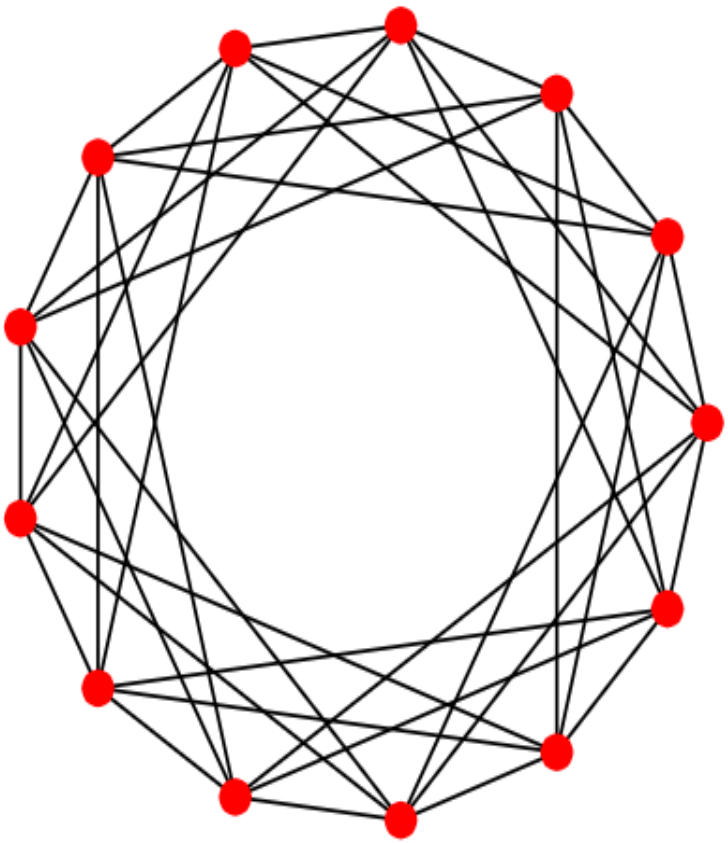
D_{13} and D_{17}

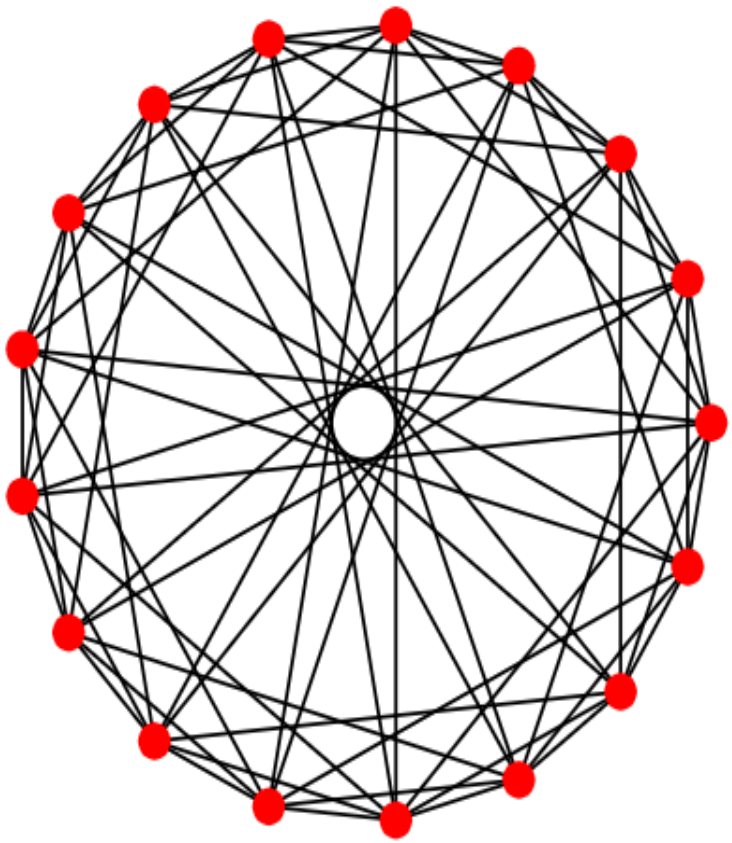




They share many similar properties with Paley graphs! Paley graphs have vertices $\mathbb{F}_q$ and edges (a, b) iff $a - b \in (\mathbb{F}_q^*)^2$

P_{13} and P_{17}





Theorem (K. - Yip - Yoo)

The degrees of vertices of D_q are given as follows.

(1) If $q \equiv 1 \pmod 4$, there are $(q-1)/2$ vertices of degree $(q-1)/2$, and $(q-1)/2$ vertices of degree $(q-3)/2$, with

$$|E(D_q)| = \frac{q^2 - 3q + 2}{4}.$$

(2) If $q \equiv 3 \pmod 4$, there are $(q+1)/2$ vertices of degree $(q-1)/2$, and $(q-3)/2$ vertices of degree $(q-3)/2$

$$|E(D_q)| = \frac{q^2 - 3q + 4}{4}.$$

This implies that, for any prime power q, the graph D_q is almost-regular with diameter 2, and hence connected.

Moreover, we obtained a nontrivial lower bound on the clique number of D_q.

Theorem (K. - Yip - Yoo)

If $q \equiv 1 \pmod 4$, we have

$$\omega(D_q) \geq \frac{p}{p-1}\left\{\frac{\frac{1}{2}\log q - 2\log\log q}{\log 2} + 1\right\}.$$

Furthermore, we expect to improve the lower bound (à la Alon and Solymosi) to

Work in progress (K. - Yip - Yoo)

$\max\{\omega(D_q), \omega(\overline{D_q})\} \geq \log_{3.1} q$.

The conjectural bound is $\omega(D_q) \geq \log_2 q$.

q	$\omega(P_q)$	$\omega(\overline{D_q})$	$\omega(D_q)$	q	$\omega(P_q)$	$\omega(\overline{D_q})$	$\omega(D_q)$	q	$\omega(P_q)$	$\omega(\overline{D_q})$	$\omega(D_q)$
3	*	*	*	83	*	7	8	199	*	9	9
5	2	2	2	89	5	7	8	211	*	9	9
7	*	2	3	97	6	7	8	223	7	10	10
9	3	3	3	101	5	8	8	227	*	10	10
11	*	3	4	103	*	7	8	229	9	10	10
13	3	4	4	107	*	8	8	233	7	10	10
17	3	4	4	109	6	8	8	239	*	10	10
19	*	4	4	113	7	8	8	241	7	10	10
23	*	4	5	121	11	9	10	243	*	11	12
25	5	5	5	125	7	8	8	251	*	10	11
27	*	6	5	127	*	8	8	257	7	10	10
29	4	5	5	131	*	9	9	263	*	11	11
31	*	5	5	137	7	9	9	269	8	10	10
37	4	5	6	139	*	8	9	271	*	10	10
41	5	6	6	149	7	9	9	277	8	10	10
43	*	6	6	151	*	9	9	281	7	10	10
47	*	6	6	157	7	9	9	283	*	10	11
49	7	6	7	163	*	9	9	289	17	11	16
53	5	6	6	167	*	9	9	293	8	10	11
59	*	7	7	169	13	9	12	307	*	11	11
61	5	6	7	173	8	9	9	311	*	11	11
67	*	7	7	179	*	9	9	313	8	11	11
71	*	8	7	181	7	9	10	317	9	11	11
73	5	7	7	191	*	9	9	331	*	11	11
79	*	7	7	193	7	9	9	337	9	11	11
81	9	7	8	197	8	9	10	343	*	11	11

TABLE 3.1. Clique numbers for the Paley graph P_q, the Diophantine graph D_q, and the complement the Diophantine graph $\overline{D_q}$ up to $q = 343$

Thank you !