The Kronecker Coefficients

Rosa Orellana
Dartmouth College
joint work with Mike Zabrocki

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Three Problems in Classical Combinatorial Representation Theory

Restriction Problem: Given a polynomial representation of the GL_n, give a combinatorial description of the coefficients when restricted to S_n?

$$Res_{S_n}^{GL_n} V^{\lambda} \cong \bigoplus_{\mu} r_{\lambda,\mu} \mathbb{S}^{\mu}$$

Kronecker Problem: Given two representation of the S_n, give a combinatorial description of the coefficients when we tensor these representations?

$$\mathbb{S}^{\lambda} \otimes \mathbb{S}^{\mu} \cong \bigoplus_{\nu} g(\lambda, \mu, \nu) \mathbb{S}^{\nu}$$

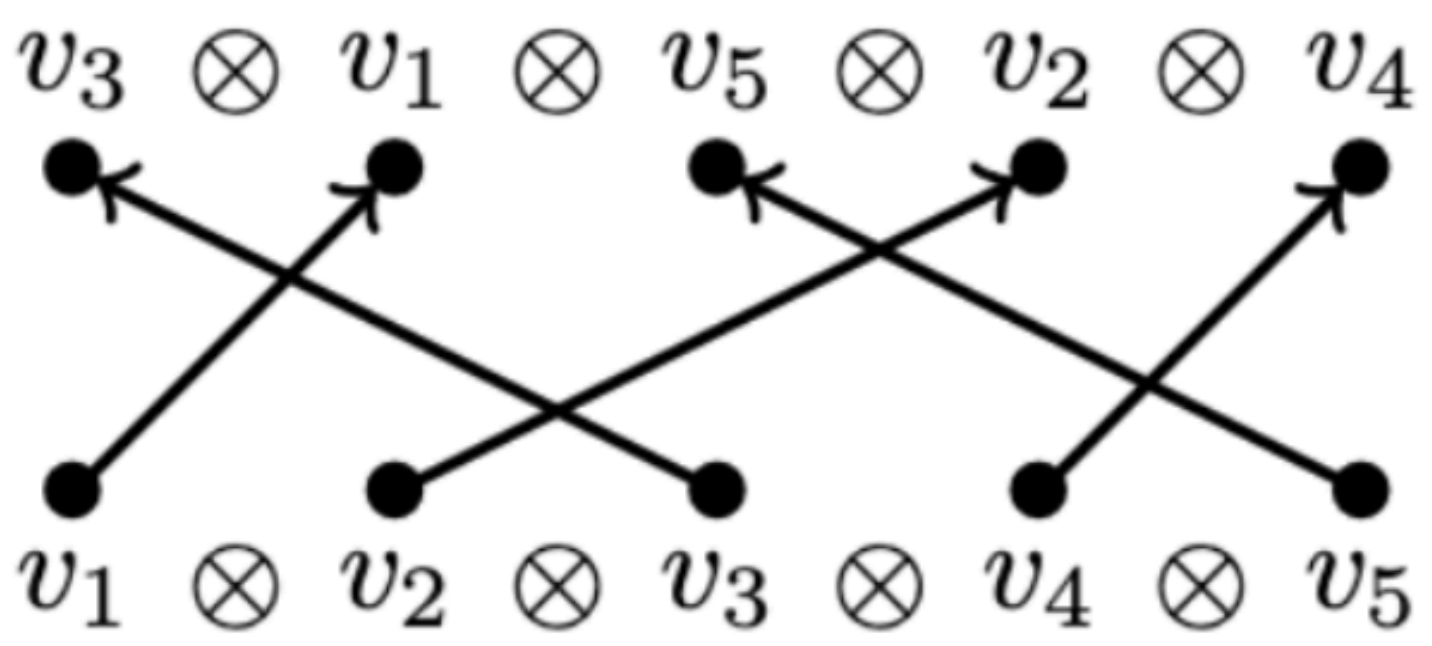
Plethysm Problem: Given two polynomial representation of the GL_n, give a combinatorial description of the coefficients when we compose these representations?

Schur-Weyl Duality

1. $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

2. S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



3. These actions commute!

Centralizer relationship produces

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \mathbb{S}^\lambda \otimes V^\lambda \qquad \text{as a } S_k \times \mathrm{GL}_n \text{ bimodule}$$

Consequences of Schur-Weyl Duality

I. **Tensoring and restriction/induction correspond:**

$$[V^\lambda \otimes V^\mu : V^\nu] = c^\nu_{\lambda,\mu} = [\mathbb{S}^\nu \downarrow^{S_{r+t}}_{S_r \times S_t} : \mathbb{S}^\lambda \times \mathbb{S}^\mu]$$

where $c^\nu_{\lambda,\mu}$ is the Littlewood-Richardson coefficient.

II. **Frobenius Formula:** The character of

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} \mathbb{S}^\lambda \otimes V^\lambda$$

at $(\sigma, g) \in S_k \times GL_n$ where

- g has eigenvalues $x_1, x_2, \ldots, x_n$
- σ has cycle type μ

$$p_\mu(x_1, \ldots, x_n) = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda(x_1, \ldots, x_n)$$

The Classical Schur-Weyl Duality

Induction: $\mathbb{S}^\lambda \times \mathbb{S}^\mu \uparrow_{S_n \times S_m}^{S_{n+m}}$

Specht modules: $\mathbb{S}^\lambda$

Representations of Symmetric group

Schur Functor
Schur-Weyl Duality

Combinatorics:

RSK algorithm

Littlewood-Richardson Rule

Tableaux filled with numbers

Frobenius Characteristic Map

Poly. Reps. of General Linear Group

"universal"
(formal) character

symmetric functions

Irreps: V^λ

Tensor products: $V^\lambda \otimes V^\mu$

Schur functions: s_λ

$s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^\nu s_\nu$

Symmetric Functions and characters of GLn

Let A be a matrix in GLn with eigenvalues $x_1, x_2, \cdots, x_n$ and $V = \mathbb{C}^n$

Representation	Character
irrep indexed by λ: V^λ	Schur function: $s_\lambda(x_1, \ldots, x_n)$
$Sym^{\lambda_1} V \otimes \cdots \otimes Sym^{\lambda_\ell} V$	homogeneous: $h_\lambda(x_1, \ldots, x_n)$
$V^{\otimes k}$ as a $S_k \times GL_n$ rep.	Power: $p_\mu(x_1, \ldots, x_n)$
$V^\lambda \otimes V^\mu$	$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$

The Kronecker Coefficients

Let $\mathbb{S}^\lambda$ and $\mathbb{S}^\mu$ be irreducible representations of the symmetric group. Then,

$$\mathbb{S}^\lambda \otimes \mathbb{S}^\mu \cong \bigoplus_\nu g(\lambda,\mu,\nu)\mathbb{S}^\nu$$

The multiplicities $g(\lambda,\mu,\nu)$ are called Kronecker coefficients.

Example:

$$\begin{aligned}\mathbb{S}^{3,2,1,1} \otimes \mathbb{S}^{4,2,1} \quad = \quad & \mathbb{S}^{6,1} \oplus 3\mathbb{S}^{5,2} \oplus 3\mathbb{S}^{5,1,1} \oplus 3\mathbb{S}^{4,3} \oplus 8\mathbb{S}^{4,2,1} \\ & \oplus 5\mathbb{S}^{4,1,1,1} \oplus 5\mathbb{S}3,3,1 \oplus 5\mathbb{S}^{3,2,2} \oplus 9\mathbb{S}^{3,2,1,1} \\ & \oplus 4\mathbb{S}^{3,1,1,1,1} \oplus 4\mathbb{S}^{2,2,2,1} \oplus 4\mathbb{S}^{2,2,1,1,1} \\ & \oplus 2\mathbb{S}^{2,1,1,1,1,1} \oplus \mathbb{S}^{1,1,1,1,1,1,1}\end{aligned}$$

Open Problem: Find a set of objects depending on three partitions λ, μ and ν that contains $g(\lambda,\mu,\nu)$ elements.

An approach to Kronecker

Schur-Weyl Duality

$$S_k \times GL_n$$

Think of S_n as
permutation matrices
Contained in GL_n

Restricted Shur-Weyl Duality

$$P_k(n) \times S_n$$

RSK algorithm
Standard tableaux
Semistandard tableaux
Littlewood-Richardson rule

Bowman, De Visscher
Zabrocki
Colmenarejo, Saliola, Schilling, and Zabrocki

Restricting Schur-Weyl Duality

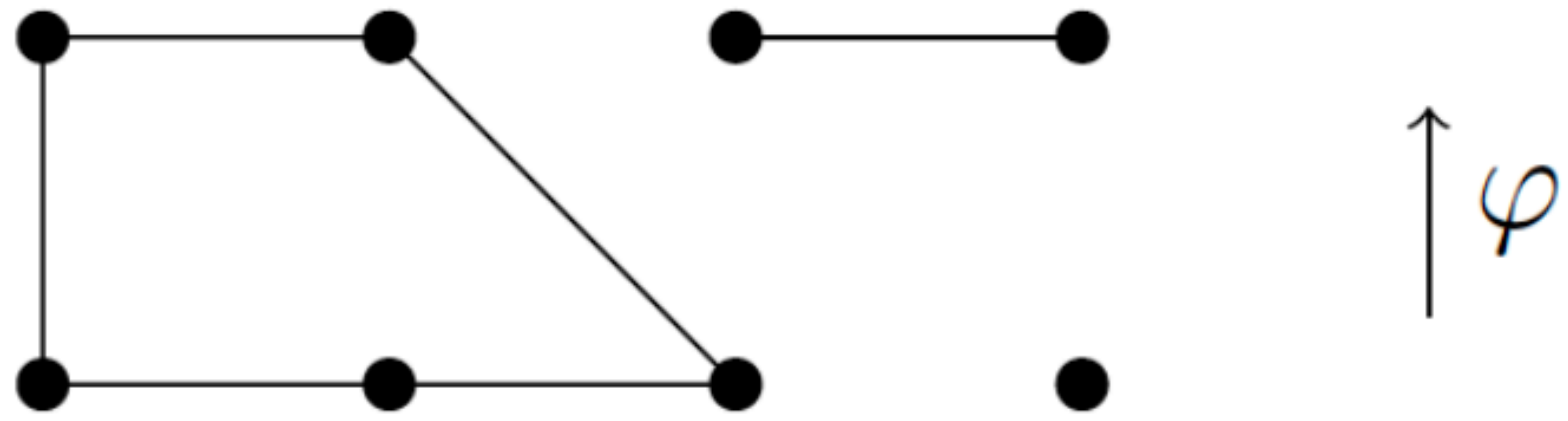
Think of $S_n \subseteq GL_n$ as the subgroup of permutation matrices acting diagonally on $V^{\otimes k}$

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sigma v_1 \otimes \sigma v_2 \otimes \cdots \otimes \sigma v_k$$

What commutes with this action?

Permutation of the factors, but a lot more!

$$\delta_{a=b=c}\ (v_a \otimes v_a) \otimes \left(\sum_{i=1}^{n} v_i \otimes v_i \right)$$



$$v_a \otimes v_b \otimes v_c \otimes v_d$$

The partition algebra!

The Partition Algebra

Fix $k \in \mathbb{Z}_{>0}$, and let

$$[k] = \{1, \ldots, k\} \qquad \text{and} \qquad [k'] = \{1', \ldots, k'\}.$$

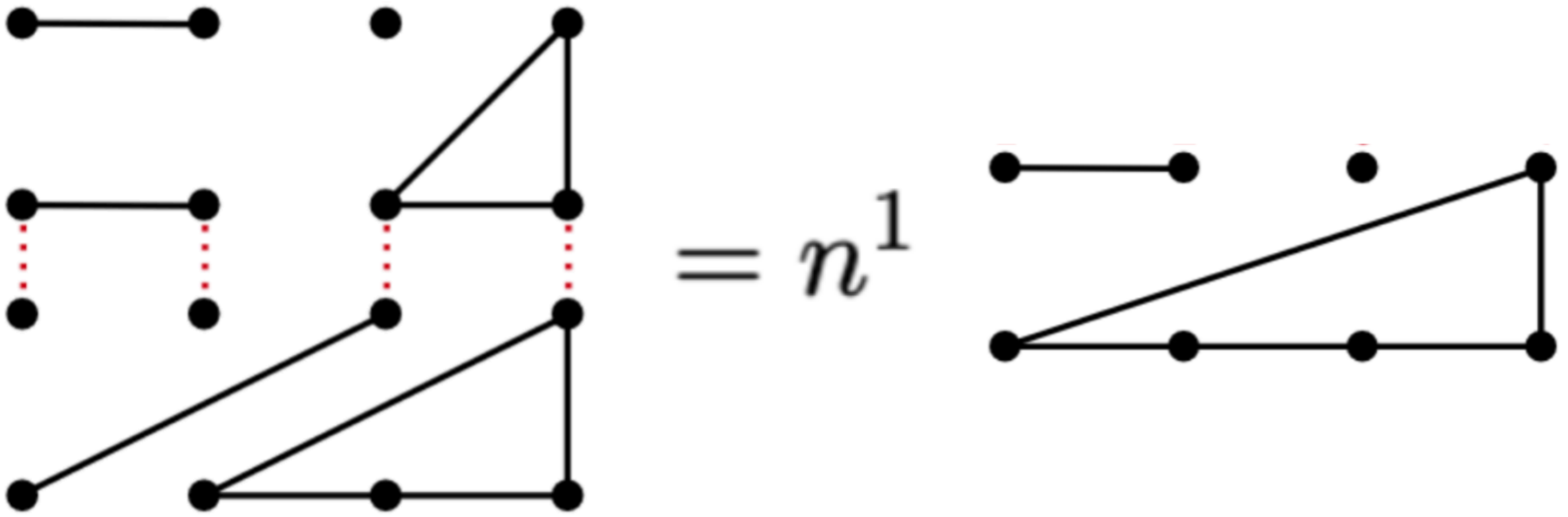
We're interested in set partitions of $[k] \cup [k']$. Either as sets of sets

$$d = \{\{1, 2, 1'\}, \{3\}, \{2', 3', 4', 4\}\}$$

or as diagrams (considering connected components)

The Partition Algebra

Multiplying diagrams:



The partition algebra $P_k(n)$ is the $\mathbb{C}$-span of the partition diagrams with this product.

Nice facts:

($*$) Associative algebra with identity $1 = \{\{1, 1'\}, \ldots, \{k, k'\}\}$.

($*$) $\dim(P_k(n)) =$ the Bell number $B(2k)$.

The Partition Algebra and Kronecker

Theorem: (Jones 1994)

$$V^{\otimes k} \cong \bigoplus_{\lambda} L^{\bar{\lambda}} \otimes \mathbb{S}^{\lambda} \qquad \text{as a } P_k(n) \times S_n \text{ representation}$$

Theorem: (Bowman, De Visscher and Orellana, 2015) For any partitions λ, μ, and ν of n, then

$$[\mathbb{S}^{\lambda} \otimes \mathbb{S}^{\mu} : \mathbb{S}^{\nu}] = g(\lambda, \mu, \nu) = \left[L(\bar{\nu}) \downarrow^{P_{n_1+n_2}}_{P_{n_1} \times P_{n_2}} : L(\bar{\lambda}) \times L(\bar{\mu}) \right]$$

where $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_\ell)$.

In comparison to: $[V^{\lambda} \otimes V^{\mu} : V^{\nu}] = c^{\nu}_{\lambda,\mu} = [\mathbb{S}^{\nu} \downarrow^{S_{r+t}}_{S_r \times S_t} : \mathbb{S}^{\lambda} \times \mathbb{S}^{\mu}]$

The character

$$V^{\otimes k} \cong \bigoplus_{\lambda} L^{\bar{\lambda}} \otimes \mathbb{S}^{\lambda} \quad \text{as a } P_k(n) \times S_n \text{ representation}$$

The character at an element (d_μ, σ) in $P_k(n) \times S_n$ where σ has eigenvalues $x_1, x_2, \ldots, x_n$ is

$$p_\mu(x_1, \ldots, x_n) = \sum_{\lambda} \chi^{\bar{\lambda}}_{P_k(n)}(d_\mu)\chi^{\lambda}(\sigma)$$

$$p_\mu(x_1, \ldots, x_n) = \sum_{\lambda} \chi^{\bar{\lambda}}_{P_k(n)}(d_\mu)\tilde{s}_{\lambda}(x_1, \ldots, x_n)$$

Note: This is in comparison with the Frobenius formula which arises from classical Schur-Weyl duality between the general linear group and the symmetric group:

$$p_\mu(x_1, \ldots, x_n) = \sum_{\lambda \vdash k} \chi^{\lambda}(\mu) s_{\lambda}(x_1, \ldots, x_n)$$

A new basis of symmetric function $\{\tilde{s}_\lambda\}$

Ξ_μ are eigenvalues corresponding to a permutation matrix of cycle type μ .

$$\tilde{s}_\lambda(\Xi_\mu) = \chi^{(n-|\lambda|,\lambda)}(\mu)$$

Example: $\tilde{s}_{(1)}(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 1$

Representing Matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

eigenvalues: 1,1,1 1, -1, 1 $1, \xi, \xi^2$

$$\tilde{s}_{(1)}(1,1,1) = 2 \quad \tilde{s}_{(1)}(1,-1,1) = 0 \quad \tilde{s}_{(1)}(1,\xi,\xi^2) = -1$$

These values correspond to the character $\chi^{(2,1)}$.

The $\{\tilde{s}_\lambda\}$ basis

$\{\tilde{s}_\lambda\}$ is a "new" basis for symmetric functions:

- When evaluated at roots of unity (eigenvalues of permutation matrices) we get the irreducible characters of the symmetric group.
- The stable Kronecker coefficients are the structure coefficients.

Compared to:

The Schur functions $\{s_\lambda\}$ form a basis of symmetric functions:

- When evaluated at the eigenvalues of a matrix A, we get the value of the irreducible characters of GL_n.
- The Littlewood-Richardson coefficients are the structure coefficients.

Structure coefficients of $\{\tilde{s}_\lambda\}$

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_\nu \overline{g}(\lambda, \mu, \nu) \tilde{s}_\nu$$

where $\overline{g}_(\lambda, \mu, \nu)$ are the "stable" Kronecker coefficients.

Example:

$$\tilde{s}_{(2)}\tilde{s}_{(1)} = \tilde{s}_{(1)} + \tilde{s}_{(1,1)} + \tilde{s}_{(2)} + \tilde{s}_{(2,1)} + \tilde{s}_{(3)}$$

which corresponds to

$$\mathbb{S}^{(n-2,2)} \otimes \mathbb{S}^{(n-1,1)} = \mathbb{S}^{(n-1,1)} + \mathbb{S}^{(n-2,1,1)} + \mathbb{S}^{(n-2,2)} + \mathbb{S}^{(n-3,2,1)} + \mathbb{S}^{(n-3,3)}$$

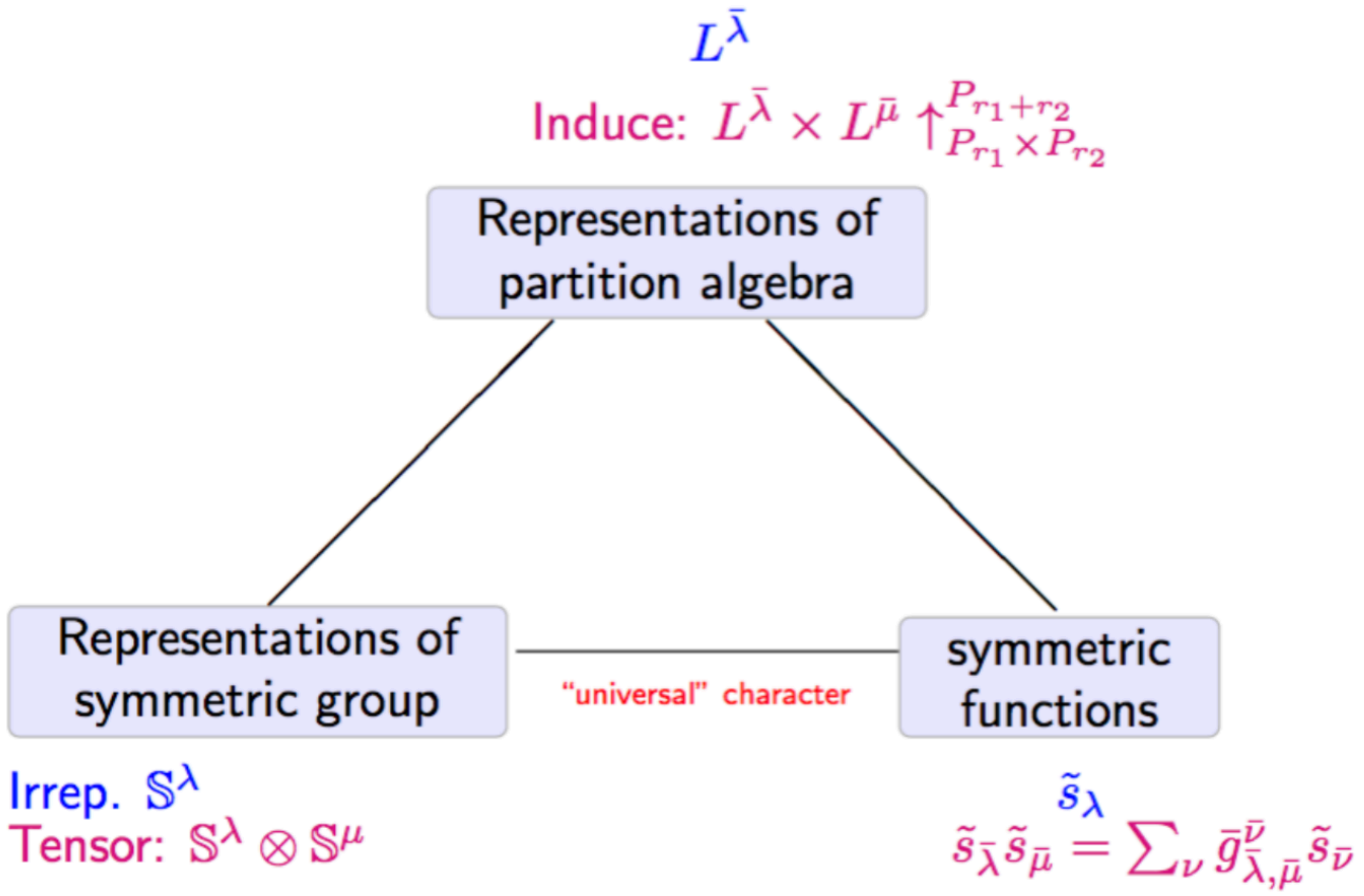
for $n \geq 6$.

Symmetric functions and characters of S_n

Let $\sigma \in S_n$ of cycle type μ with eigenvalues Ξ_μ.

Representation	Character
irrep indexed by λ: $\mathbb{S}^\lambda$	irreducible character basis: $\tilde{s}_\lambda(\Xi_\mu)$
$\mathbf{1} \uparrow_{S_\lambda}^{S_n}$	induced trivial character basis: $\tilde{h}_\lambda(\Xi_\mu)$
$V^{\otimes k}$ as an $P_k(n) \times S_n$-module	power $p_\mu(\Xi_\mu)$
$\mathbb{S}^{(n-\lvert\lambda\rvert,\lambda)} \otimes \mathbb{S}^{n-\lvert\mu\rvert,\mu)}$	$\tilde{s}_\lambda \tilde{s}_\mu = \sum_\nu g(\lambda,\mu,\nu)\tilde{s}_\nu$

Schur-Weyl duality between partition algebra and the symmetric group



Note: The combinatorial objects governing this picture are **multiset tableaux.**

Products of symmetric functions

The objective for the rest of the talk!

- A rule for multiplying $h_\mu \tilde{s}_\lambda$
- A rule for multiplying $\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_k} \tilde{s}_\lambda$
- A rule for multiplying $\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_k} \tilde{s}_\lambda$
- A rule for multiplying $\tilde{h}_\mu \tilde{s}_\lambda$

Multisets

- A multiset is a collection of objects where the objects can be repeated. Example: $\{\!\{1, 1, 2, 3, 3, 4\}\!\}$
- Our multiset will contain barred and unbarred numbers: Ordered $\bar{1} < \bar{2} < \cdots < 1 < 2 < \cdots$
- Given two multisets M_1 and M_2 we want to order them using the reverse lexicographic order
  For example, $\{\!\{\bar{5}, 1, 1, 1, 2, 2, 3, 4\}\!\} < \{\!\{\bar{2}, 1, 1, 2, 3, 3, 4\}\!\}$.
- The content of a tableau T is defined as the multiset which contains $a_i^{m_i}$ where $a_i \in \{\bar{1}, \bar{2}, \ldots, 1, 2, \ldots\}$ occurs m_i times in T.
- The shape of a tableau T is the sequence obtained by reading the lengths of each row in T. We denote by $sh(T)$ the shape of T. All of our tableaux will be of shape $(r, \gamma)/(\gamma_1)$ for a partition γ and some integer $r \geq \gamma_1$.

Multiset Tableaux

- Let α and β be compositions and γ a partition
- MCT$_\gamma(\alpha, \beta)$ contains tableaux T such that
  - are column strict
  - have shape $(r, \gamma)/(\gamma_1)$
  - have content $\{\{\overline{1}^{\alpha_1}, \overline{2},^{\alpha_2}, \ldots, \overline{\ell}^{\alpha_\ell}, 1^{\beta_1}, 2^{\beta_2}, \ldots, k^{\beta_k}\}\}$
  - have at most one barred entry in each cell
  - cells in the first row cannot be filled with multisets containing only barred entries.

Example: MCT$_{(3,3,3,2,1)}((6,4,2), (12,4))$

$$sh(T) = (4,3,3,3,2,1)/(3)$$

$$\overline{sh(T)} = (3,3,3,2,1)$$

$$T = \begin{array}{|c|c|c|c|}
\cline{1-1}
\bar{1}12 \\
\cline{1-2}
\bar{1}2 & \bar{3}12 \\
\cline{1-3}
11 & \bar{1}11 & \bar{2}2 \\
\cline{1-3}
\bar{2} & \bar{2}1 & \bar{1}11 \\
\cline{1-3}
\bar{1} & \bar{1}1 & \bar{2}1 \\
\hline
 & & & \bar{3}1 \\
\hline
\end{array}$$

Reading the entries of a multiset tableau

- For any tableaux T, let $read(T)$ be the word of entries in the tableau from bottom row to top row and from right to left in the rows.
- If S is a multiset of non-barred entries, then let $read(T|_S)$ represent the reading word of the barred entries in the cells with $\{\{\bar{j}, S\}\}$ as a label and let $read(T|\_)$ represent the reading word of the cells which have only a barred entry, i.e., when $S = \varnothing$.
- The multiset with no barred entries do not contribute to the reading word.
- The reading word is the concatenation of all the reading words for all the multisets.

Example:

$$
\begin{aligned}
read(T|\_) &= \bar{1}\bar{2}, \\
read(T|_1) &= \bar{3}\bar{2}\bar{1}\bar{2}, \\
read(T|_{1^2}) &= \bar{1}\bar{1} \\
read(T|_2) &= \bar{2}\bar{1}, \\
read(T|_{12}) &= \bar{3}\bar{1}.
\end{aligned}
$$

$$
T = \begin{array}{|c|c|c|c|}
\hline
\bar{1}12 & & & \\
\hline
\bar{1}2 & \bar{3}12 & & \\
\hline
11 & \bar{1}11 & \bar{2}2 & \\
\hline
\bar{2} & \bar{2}1 & \bar{1}11 & \\
\hline
\bar{1} & \bar{1}1 & \bar{2}1 & \\
\hline
 & & & \bar{3}1 \\
\hline
\end{array}
$$

T has reading word

$$read(T|\_)read(T|_1)read(T|_{1^2})read(T|_2)read(T|_{12}) = \bar{1}\bar{2}.\bar{3}\bar{2}\bar{1}\bar{2}.\bar{1}\bar{1}.\bar{2}\bar{1}.\bar{3}\bar{1}$$

The rule for $\tilde{s}_\gamma$ in $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} \tilde{s}_\lambda$

Theorem: Let λ and γ be partitions and α any composition,

$$h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} \tilde{s}_\lambda = \sum_{\gamma} d^{\gamma}_{\alpha,\lambda} \tilde{s}_\gamma,$$

The coefficient $d^{\gamma}_{\alpha,\lambda}$ is equal to the number of $T \in \mathrm{MCT}_\gamma(\lambda, \alpha)$ such that T is a lattice tableaux.

Example: The coefficient of $\tilde{s}_4$ in $h_2 h_1 \tilde{s}_{22}$ is equal to 8.

$$\begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{1}1 & \bar{2}1 & \bar{2}2 \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{1-4} \bar{1} & \bar{1} & \bar{2}1 & 2 & \multicolumn{1}{c}{} \\ \hline & & & & \bar{2}1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{1-4} \bar{1} & \bar{1} & 1 & \bar{2}1 & \multicolumn{1}{c}{} \\ \hline & & & & \bar{2}2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{1-4} \bar{1} & \bar{1} & 1 & \bar{2}2 & \multicolumn{1}{c}{} \\ \hline & & & & \bar{2}1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}12 \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{1-4} \bar{1} & \bar{1} & \bar{2}1 & \bar{2}2 & \multicolumn{1}{c}{} \\ \hline & & & & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{1-4} \bar{1} & \bar{1} & \bar{2}1 & \bar{2}1 & \multicolumn{1}{c}{} \\ \hline & & & & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{1} & \bar{2}11 & \bar{2}2 \\ \hline & & & \\ \hline \end{array}$$

The rule for $\tilde{s}_\gamma$ in $\tilde{h}_{\alpha_1}\tilde{h}_{\alpha_2}\cdots\tilde{h}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda$

Theorem: Let λ and γ be partitions and α any composition,

$$\tilde{h}_{\alpha_1}\tilde{h}_{\alpha_2}\cdots\tilde{h}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda = \sum_\gamma m^\gamma_{\alpha,\lambda}\tilde{s}_\gamma,$$

The coefficient $m^\gamma_{\alpha,\lambda}$ is equal to the number of $T \in \mathrm{MCT}_\gamma(\lambda,\alpha)$ such that T is a lattice tableaux, and such that the entries of the tableaux are sets (no repeated entries).

Example: The same coefficient in $\tilde{h}_2\tilde{h}_1\tilde{s}_{22}$ is equal to 7

$\bar{1}$	$\bar{1}1$	$\bar{2}1$	$\bar{2}2$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	2	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	1	$\bar{2}1$	
				$\bar{2}2$

$\bar{1}$	$\bar{1}$	1	$\bar{2}2$	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}12$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}2$	
				1

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}1$	
				2

The rule for $\tilde{h}_\mu \tilde{s}_\lambda$

Theorem: Let λ, μ and γ be partitions

$$\tilde{h}_\mu \tilde{s}_\lambda = \sum_\gamma a^\gamma_{\mu,\lambda} \tilde{s}_\gamma,$$

The coefficient $a^\gamma_{\mu,\lambda}$ is equal to the number of $T \in \mathrm{MTC}_\gamma(\lambda, \mu)$ such that T is a lattice tableaux and whose entries are sets with at most one non-barred entry (and at most one barred entry).

Example The coefficient of $\tilde{s}_4$ in $\tilde{h}_{21}\tilde{s}_{22}$ is equal to 6 since the tableaux described by the theorem are

$$\begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{1}1 & \bar{2}1 & \bar{2}2 \\ \hline \ & \ & \ & \ \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c} \hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}1 & \\ \hline \ & \ & \ & \ & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c} \hline \bar{1} & \bar{1} & \bar{2}1 & 2 & \\ \hline \ & \ & \ & \ & \multicolumn{1}{c|}{\bar{2}1} \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c} \hline \bar{1} & \bar{1} & 1 & \bar{2}1 & \\ \hline \ & \ & \ & \ & \multicolumn{1}{c|}{\bar{2}2} \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c} \hline \bar{1} & \bar{1} & 1 & \bar{2}2 & \\ \hline \ & \ & \ & \ & \multicolumn{1}{c|}{\bar{2}1} \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c} \hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}2 & \\ \hline \ & \ & \ & \ & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

The rule for $\tilde{s}_\gamma$ in $\tilde{s}_{\alpha_1}\tilde{s}_{\alpha_2}\cdots\tilde{s}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda$

Theorem: Let λ and γ be partitions and α any composition,

$$\tilde{s}_{\alpha_1}\tilde{s}_{\alpha_2}\cdots\tilde{s}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda = \sum_\gamma r^\gamma_{\alpha,\lambda}\tilde{s}_\gamma,$$

The coefficient $r^\gamma_{\alpha,\lambda}$ is equal to the number of $T \in \mathrm{MCT}_\gamma(\lambda, \alpha)$ such that T is a lattice tableaux, the entries of the tableaux are sets (no repeated entries) and only labels of sets of size greater than 1 are allowed in the first row.

Example: The same coefficient in $\tilde{s}_2\tilde{s}_1\tilde{s}_{22}$ is equal to 5

$\bar{1}$	$\bar{1}1$	$\bar{2}1$	$\bar{2}2$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	2	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	1	$\bar{2}1$	
				$\bar{2}2$

$\bar{1}$	$\bar{1}$	1	$\bar{2}2$	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}12$

Final Remarks

- We want $\tilde{s}_\mu \tilde{s}_\lambda$, how close are we ?

$$
\begin{array}{ccccc}
\tilde{s}_{\mu_1}\tilde{s}_{\mu_2}\cdots\tilde{s}_{\mu_\ell}\tilde{s}_\lambda & \leq & \tilde{h}_{\mu_1}\tilde{h}_{\mu_2}\cdots\tilde{h}_{\mu_\ell}\tilde{s}_\lambda & \leq & h_\mu \tilde{s}_\lambda \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\boxed{\tilde{s}_\mu \tilde{s}_\lambda} & \leq & \tilde{h}_\mu \tilde{s}_\lambda & \leq & \boxed{s_\mu \tilde{s}_\lambda}
\end{array}
$$

- The products presented here have applications to construction of Bratteli diagrams of towers of algebras.
- Repeated products occur in Quantum entanglement.

Thank You!

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