Castelnuovo-Mumford regularity of ladder determinantal ideals via Grothendieck polynomials

Colleen Robichaux
UCLA

joint work with Jenna Rajchgot and Anna Weigandt
NEWAC50AWM
April 28, 2023

Kazhdan-Lusztig varieties of Woo–Yong '06

We can identify $\mathcal{F}l_n(\mathbb{C})$ with $\mathrm{GL}_n(\mathbb{C})/B$, where $B \subset \mathrm{GL}_n(\mathbb{C})$ is the Borel subgroup.

B acts on $\mathcal{F}l_n(\mathbb{C})$ with finitely many orbits X_w° called **Schubert cells**. The **Schubert varieties** X_w are closures of these orbits.

Of particular interest is the **Kazhdan-Lusztig variety**

$$\mathcal{N}_{v,w} = X_w \cap \Omega_v^\circ,$$

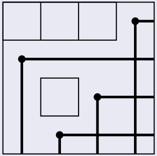
where Ω_v° is the opposite Schubert cell.

Kazhdan-Lusztig varieties

Kazhdan-Lusztig variety $\mathcal{N}_{v,w}$ has defining ideal

$$I_{v,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathbf{z}_{i\times j}(v)\rangle.$$

Example: $w = 4132$, $v = 4231$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathbf{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_{v,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31}\rangle$$

Matrix Schubert varieties and determinantal varieties are all examples of KL varieties.

Minimal free resolution

Consider the coordinate ring S/I. The **minimal free resolution**

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{l,j}} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{0,j}} \to S/I \to 0.$$

The K-**polynomial** of S/I

$$\mathcal{K}(S/I;\mathbf{t}) := \sum_{j \in \mathbb{Z}, i \geq 0} (-1)^i \beta_{i,j} t^j.$$

The **Castelnuovo-Mumford regularity** of S/I

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

Proposition

For Cohen-Macaulay S/I

$$\operatorname{reg}(S/I) = \deg \mathcal{K}(S/I;\mathbf{t}) - \operatorname{codim}_S I.$$

Matrix Schubert varieties

Matrix Schubert varieties $\overline{X}_w$ are special cases of $\mathcal{N}_{v,w'}$.

Combining results of Fulton '92 and Knutson–Miller '05,

Theorem

$$\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w(x_1,\ldots,x_n)) - \ell(w),$$

where $\mathfrak{G}_w(x_1,\ldots,x_n)$ is the Grothendieck polynomial and $\ell(w)$ is the Coxeter length of w.

Problem

Give an easily computable formula for $\deg(\mathfrak{G}_w(x_1,\ldots,x_n))$, where $w \in S_n$.

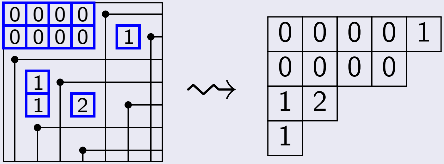
Finding the degree of $\mathfrak{G}_v$ vexillary

Theorem [Rajchgot-R.-Weigandt '23]

Suppose $v \in S_n$ vexillary. Then

$$\deg(\mathfrak{G}_v) = \ell(v) + \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



gives $\deg(\mathfrak{G}_v) = \ell(v) + (3+1) = 12 + 4 = 16.$

Pechenik–Speyer–Weigandt '21 give a result for general $w \in S_n$.

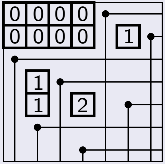
Finding the regularity of $\overline{X}_v$ vexillary

Theorem [Rajchgot-R.-Weigandt '23]

Suppose $v \in S_n$ vexillary. Then

$$\mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \deg(\mathfrak{G}_v) - \ell(v) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



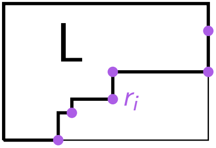
$\rightsquigarrow$

0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \deg(\mathfrak{G}_v) - \ell(v) = 3 + 1 = 4$.

Application: one-sided mixed ladder determinantal ideals

Consider a matrix $X = (x_{ij})$ of indeterminates. Let L denote the submatrix of X defined by choosing SE corners. $I(L)$ is the ideal generated by the NW r_i minors of L. This defines the one-sided mixed ladder determinantal variety $X(L)$.



These are KL-varieties $\mathcal{N}_{u_\rho, w_\gamma}$ for $u_\rho, w_\gamma \in S_n$ Grassmannian

$$\mathbb{C}[X(L)] \cong \mathbb{C}[\mathcal{N}_{u_\rho, w_\gamma}] \cong \mathbb{C}[\overline{X}_v].$$

Corollary [Rajchgot–R.–Weigandt '23]

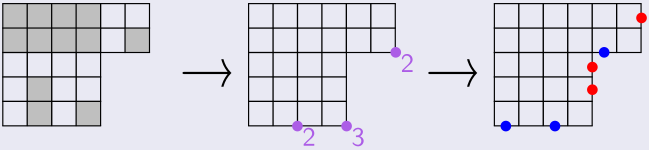
$$\operatorname{reg}(\mathbb{C}[X(L)]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(v)|_{\geq i}).$$

One-sided ladders and lattice paths

To each one-sided ladder, we can associate families of northeast-oriented lattice paths.

The marked points on horizontal edges determine starting points of the paths and the marked points on vertical edges determine ending points of the paths.

Example: $v = 5713624$ corresponds to L below.



This correspondence uses the combinatorics of *excited Young diagrams*.

CM-regularity of one-sided ladders and lattice paths

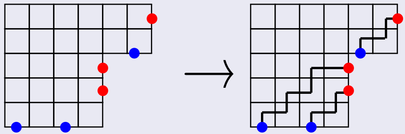
Following work of Krattenhaler–Prohaska '99 and Ghorpade '02 combined with Knutson–Miller Yong '09 and Matsumura '17, $\mathcal{K}(X(L);\mathbf{t})$ is determinantal in terms of these lattice paths.

Theorem

For a one-sided ladder L

$$\operatorname{reg}(\mathbb{C}[X(L)]) = \max_{P \in NILP(L)} \#\{\text{elbows } \lrcorner \text{ in } P\}.$$

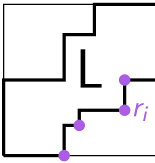
Example: $P \in NILP(L)$ with maximal number of NE elbows



gives $\operatorname{reg}(\mathbb{C}[X(L)]) = 4 = \operatorname{reg}(\mathbb{C}[\overline{X}_v])$.

Generalization: two-sided mixed ladder determinantal ideals

Consider a matrix $X = (x_{ij})$ of indeterminates. Let L denote the submatrix of X defined by choosing SE and NW corners. $I(L)$ is the ideal generated by the NW r_i minors of L. This defines the two-sided mixed ladder determinantal variety $\tilde{X}(L)$.



Further, for any such variety, we can construct $u, w \in S_n$ 321-avoiding such that $\tilde{X}(L) \cong \mathcal{N}_{u,w}$. In current work, we provide an algorithm to compute $\mathrm{reg}(\mathbb{C}[\tilde{X}(L)])$.

Conclusions

- We can express $\operatorname{reg}(\mathbb{C}[\overline{X}_w])$ in terms of the degree of the K-polynomial and the codimension of I_w.
- Use that $\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg \mathfrak{G}_w - \ell(w)$.
- For v vexillary, we obtain an easily computable formula for $\deg \mathfrak{G}_v$, and thus for $\operatorname{reg}(\mathbb{C}[\overline{X}_v])$.
- By relating $\mathcal{N}_{u_\rho, w_\gamma}$ to $\overline{X}_v$, we obtain formulas for regularities of one-sided ladders. We then extend this construction to compute regularities of two-sided ladders.

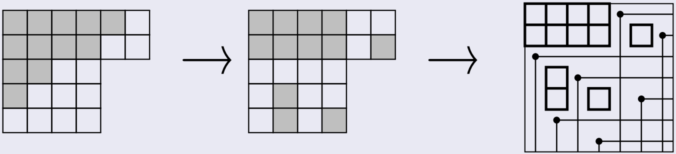
Computing CM-regularity of certain KL varieties

Theorem [Rajchgot-R.-Weigandt '22+]

For $u_\rho, w_\gamma \in S_n$ Grassmannian with descent k, $(u_\rho, w_\gamma) \mapsto v$ vexillary such that

$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $u_{(6,6,4,4,4)}, w_{(5,4,2,1,0)} \mapsto v = 5713624$



gives $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_v]) = 4$.