On positivity for flag manifolds and Hessenberg spaces

A story of combinatorics, algebra, and geometry

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Women in Algebra and Combinatorics

ENUMERATIVE GEOMETRY

Given two lines in $\mathbb{R}^2$, how many times do they intersect?
Answer: $0, 1$ or ∞.
Given 2 circles in the $\mathbb{R}^2$, how many common tangents do they have?
Answer: $0, 1, 2, 3, 4$ or ∞.

What about generically?

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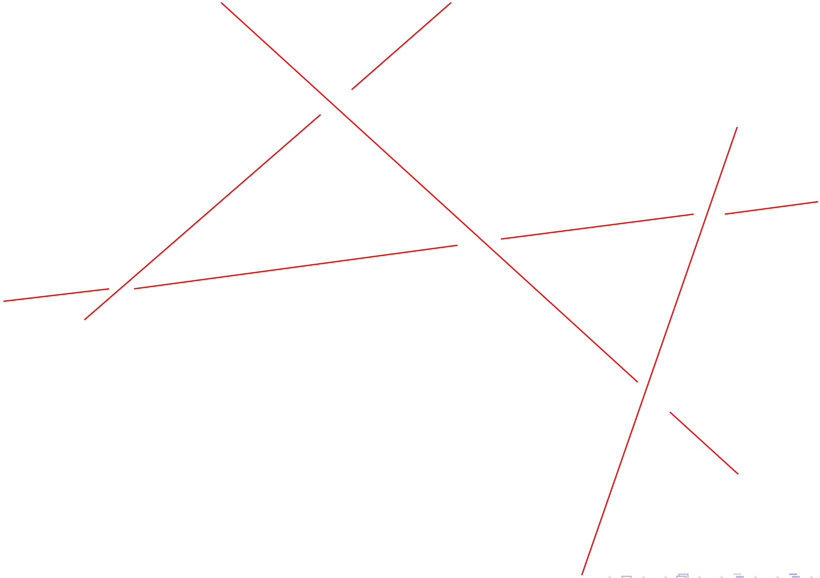
What about generically?

A QUESTION POSED BY SCHUBERT

Given four (red) lines, how many (blue) lines intersect all 4?



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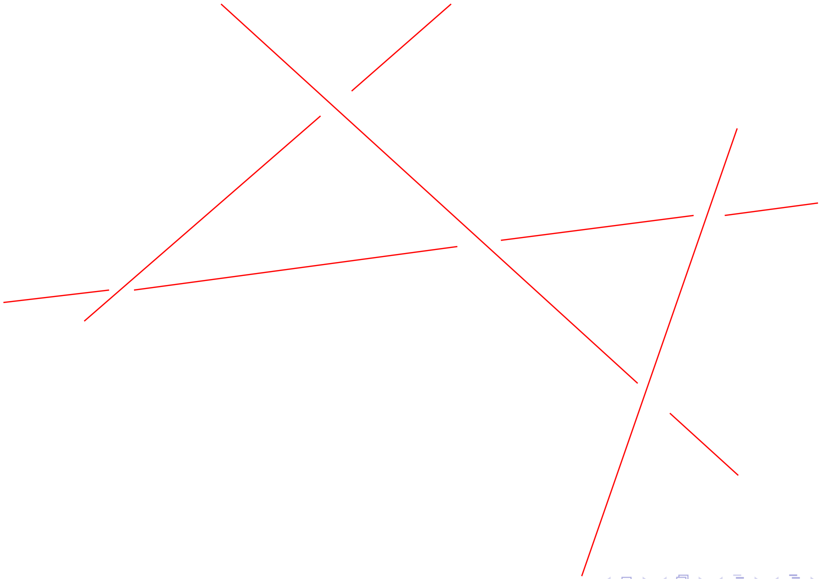
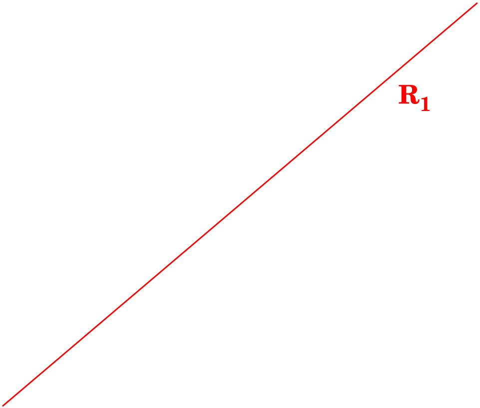


Visualizing the Four Line Problem: the First Line



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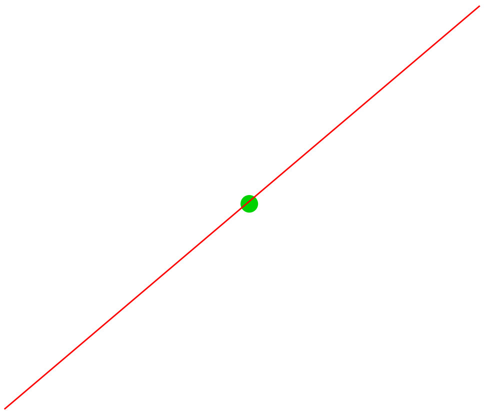
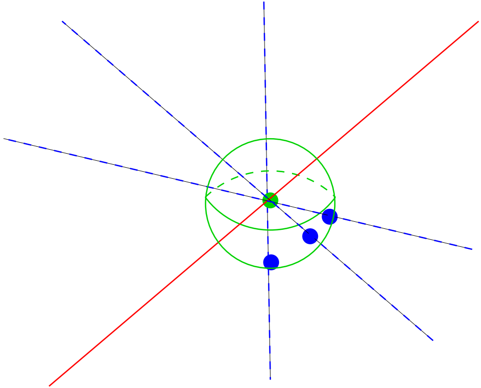


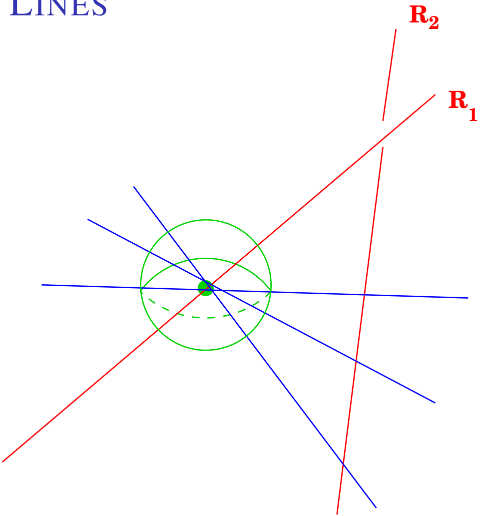
The lines intersecting R_1 could intersect it at any point.

LINES INTERSECTING R_1 IN $\mathbb{R}^3$



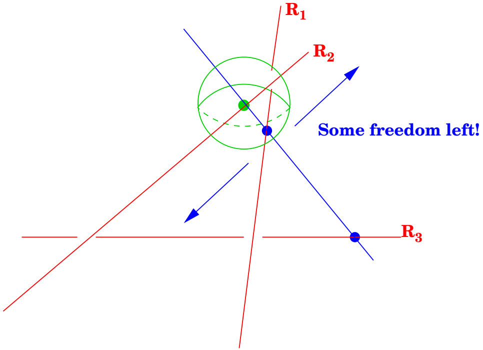
The set of all lines intersecting R_1 is 3-dimensional.

Visualizing the Four Line Problem: the First and Second Lines



The set of (blue) lines intersecting both R_1 and R_2 is two dimensional.

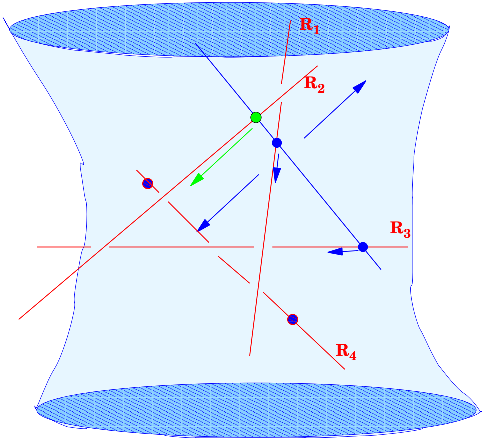
Visualizing the Four Line Problem: the First, Second and Third Lines



As you sweep out the possible (blue) lines intersecting these (red) lines, you obtain a quadratic surface.

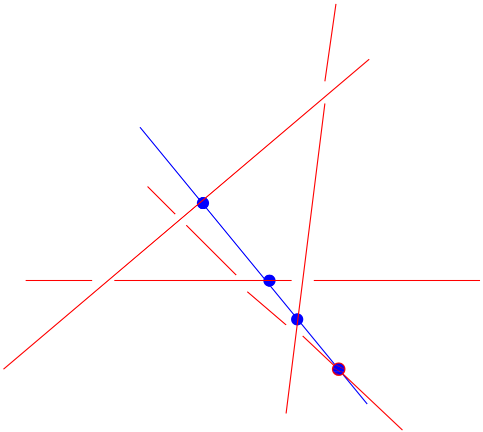
Visualizing the Four Line Problem: All Four Lines

Finally put in the fourth line: it intersects the quadratic surfaces in two points. Each of these is a blue line intersecting all four red lines.

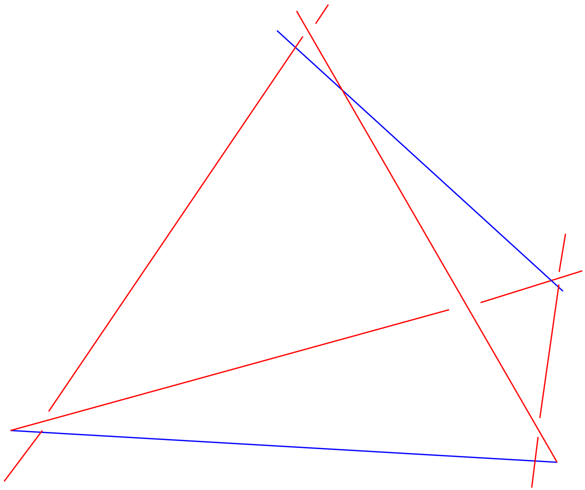


Visualizing the Four Line Problem: All Four Lines

Moving the first line we drew down to intersect all four lines.



Visualizing the Four Line Problem



Modern Language

- Work in projective space because we want to count parallel lines as intersecting at infinity;
- Increase the dimension and talk about 2-dimensional planes through 0 rather than affine lines;
- Work over $\mathbb{C}$ because it is algebraically closed
- Rather than lines in $\mathbb{R}^3$, we consider planes through 0 in $\mathbb{C}^4$.

Definition

The Grassmannian of 2 planes on $\mathbb{C}^4$ is denoted $Gr(2, 4)$ and is given as a set by $\{V_2 \subset \mathbb{C}^4\}$, where V_2 is a 2-dimensional subspace of $\mathbb{C}^4$.

THE GRASSMANNIAN $Gr(k, n)$

DEFINITION

The Grassmannian of k planes on $\mathbb{C}^n$ is denoted $Gr(k, n)$ and is given as a set by $\{V_k \subset \mathbb{C}^n\}$, where V_k is a k-dimensional subspace of $\mathbb{C}^n$.

The space of (all possible) blue lines through R_1 forms a three-dimensional subvariety X_1 of $Gr(2, 4)$. Similarly, the space of lines through each of R_2, R_3 and R_4 form three-dimensional subvarieties X_2, X_3, X_4, respectively.

The only blue lines that intersect R_1, R_2, R_3 and R_4 are in the intersection

$$X_1 \cap X_2 \cap X_3 \cap X_4.$$

So the original question we posed was really about the number of points in this intersection.

Can we find formulas for the number of intersections for questions like these?

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Hilbert's 15th Problem

"To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him." – *Bulletin of the AMS, 1900*

The search for *positivity* and *nonnegative formulas* over the past 150 years inspired work in many diverse fields within mathematics: combinatorics, singular homology, cohomology, Chow cohomology, equivariant cohomology, K-theory, quantum cohomology, intersection theory, and representation theory.

What is Positivity?

Multiple meanings, depending on the community involved.
A "positive" answer to a (combinatorial, geometric, algebraic) question is:

1. Existence of a positive answer;
2. Existence of a non-negative answer;
3. Existence of a *combinatorial* formula for the answer;
4. Finding objects to count, where the number of objects answer the question;
5. Finding a bijection from your set to another set, whose cardinality is known;
6. Converting a geometric problem into an algebraic one, and introducing a notion of algebraic positivity;
7. Finding a combinatorial formula for certain polynomials (or other algebraic objects), whose coefficients are positive (or nonnegative).

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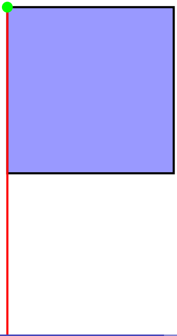
Flag varieties

$Gl(n,\mathbb{C})/B$, B upper triangular matrices:
Identify with set $\{V_\bullet\}$ of nested vector spaces:

$$\{\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\}$$

$$\dim(V_i) = i.$$



G complex semi-simple Lie group, with Lie algebra $\mathfrak{g}$

B choice of Borel, with Lie algebra $\mathfrak{b}$, and B^- opposite Borel

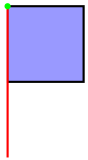
$T = B \cap B^-$ a maximal torus, with Lie algebra $\mathfrak{t}$

G/B the *flag variety*.

The Flag Variety

$Gl(n,\mathbb{C})/B$, B upper triangular matrices

$$V_\bullet = (\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$$



S_n indexes coordinate flags: Given a basis $e_1, \ldots, e_n$, consider flags

$$\langle e_{w(1)}\rangle \subseteq \langle e_{w(1)}, e_{w(2)}\rangle \subseteq \cdots \subseteq \mathbb{C}^n$$

for $w \in S_n$

Represent a flag

$$\begin{pmatrix} | & | & \ldots & | \\ v_1 & v_2 & \ldots & v_n \\ | & | & \ldots & | \end{pmatrix}$$

where $\langle v_1\rangle = V_1$, second column v_2 with $\langle v_1, v_2\rangle = V_2$, etc.
For every $w \in S_n$, there is a Schubert *cell* BwB/B and *Schubert variety* $\overline{BwB/B}$.

Schubert varieties

$W = N(T)/T$ the Weyl group. (or $W = S_n$, the permutation group)

If w is a represented by a permutation matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, then

BwB/B consists of matrices you can obtain by sweeping rows upward and then clearing out entries to the right of the 1's.

$$\begin{pmatrix} \star & 1 & \star & \star \\ \star & 0 & 1 & \star \\ \star & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \star & 1 & 0 & 0 \\ \star & 0 & 1 & 0 \\ \star & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$X_v = \overline{BvB/B}$ Schubert varieties, closure of B-orbits in G/B, for $v \in W$.

Positivity for the Cohomology of the Flag Variety

$X_v = \overline{BvB/B}$ Schubert varieties, closure of B-orbits in G/B, for $v \in W$.
$X^v = \overline{B^- vB/B}$ Schubert varieties, closure of B-orbits in G/B, for $v \in W$.

- $\{[X_v] : v \in W\}$ form a basis for the homology of G/B.
- $\{[X^v] : v \in W\}$ form a basis for the homology of G/B.
- There is a basis $\{\sigma_w : w \in W\}$ for $H^*(G/B)$ as a free module module over $\mathbb{Z}$
- $\{\sigma_w : w \in W\}$ dual *basis* to $\{[X_v] : v \in W\}$ under taking the cap product and integrating (Poincaré duality for bases on G/B)
- The structure constants $c_{uv}^w \in \mathbb{Z}$ defined by

$$\sigma_u \sigma_v = \sum_w c_{uv}^w \sigma_w$$

have *nonnegative coefficients*.

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Why is Schubert calculus positive?

$$\sigma_u \sigma_v = \sum_w c_{uv}^w \sigma_w.$$

- Multiplication in cohomology corresponds to intersecting the varieties they represent, placed in general position
  Poincaré duality tells us that we may pick off each coefficient by pairing with the right homology class, which is another intersection
- The complex group G acts on G/B transitively, so we can make the intersection transverse.
- The group action preserves the complex structure, so all intersection points contribute a *positive* number.

$$c_{uv}^w = \#\left(gX^u \cap X^v \cap X_w\right)$$

Equivariant Cohomology of the Flag Variety

$X_v = \overline{BvB/B}$ Schubert varieties, closure of B-orbits in G/B, for $v \in W$.
$X^v = \overline{B^- vB/B}$ Schubert varieties, closure of B-orbits in G/B, for $v \in W$.
Δ set of simple positive roots Φ^+ set of positive simple roots for G

- H_T^* is a contravariant functor, like H^*, but it takes into account the T action.
- $H_T^*(pt) = S(\mathfrak{t}^*)$ is naturally given by polynomials in Φ^+.
- $\{[X_v] : v \in W\}$ and $\{[X^v] : v \in W\}$ form two bases for the equivariant homology of G/B.
- $H_T^*(G/B)$ is a free module module over $H_T^*(pt)$
- There is a basis $\{\sigma_w : w \in W\}$ for $H_T^*(G/B)$ that is the Poincaré dual class to $\{[X_v] : v \in W\}$ under a nondegenerate pairing.
- Each element σ_w is Poincaré dual to $[X^v]$ in topological sense.
- The structure constants $c_{uv}^w \in H_T^*(pt)$ defined by

$$\sigma_u \sigma_v = \sum_w c_{uv}^w \sigma_w$$

  are polynomials in $\alpha \in \Phi^+$ with *nonnegative coefficients* (Graham, '99)

Positivity for equivariant cohomology

The structure constants $c_{uv}^{w} \in H_T^*(pt)$ defined by

$$\sigma_u \sigma_v = \sum_w c_{uv}^{w} \sigma_w$$

are polynomials in $\alpha \in \Phi^+$ with *nonnegative coefficients*.
Does this generalize to other subvarieties of G/B? (and what combinatorial formulas can be found for the coefficients?)

A family of subschemes of G/B

$G = Gl(n, \mathbb{C})$ (or any reductive Lie group) $\mathfrak{g}$ its Lie algebra
B upper triangular matrices (or any Borel subgroup) $\mathfrak{b}$ its Lie algebra
T invertible diagonal matrices (or the maximal torus of B)

Additional data:

$$\mathrm{x} \in \mathfrak{g}$$

$$H \subseteq \mathfrak{g} \ \mathfrak{b}\text{-invariant subspace containing } \mathfrak{b}$$

Definition

The Hessenberg variety associated to X, H is

$$\mathcal{H}ess(\mathrm{x}, H) = \{gB \in G/B : Ad(g^{-1})\mathrm{x} \in H\}.$$

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Many manifestations: the flag variety itself, Springer fibers, the Peterson variety, the toric variety associated to Weyl chambers
Many fields: combinatorics, geometric representation theory, hyperplane arrangements, algebraic geometry, quantum cohomology
Many open questions: Which properties of G/B are preserved by which Hessenberg varieties (specifying H, x)?

- Group action properties
- Cohomological properties
- Geometric properties

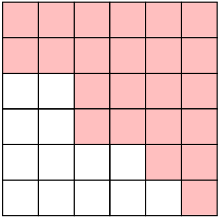
$\mathcal{H}ess(\mathsf{x}, H) = \{gB \in G/B : Ad(g^{-1})\mathsf{x} \in H\}$.

A REPHRASING FOR TYPE A

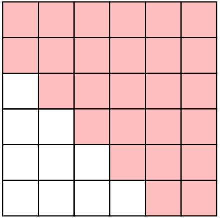
Let $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with $h(i) \geq i$ and $h(i) \leq h(i+1)$

$$H \leftrightarrow h$$

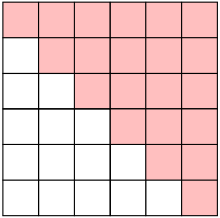
$h = (2, 2, 4, 4, 5, 6)$



$h_0 = (2, 3, 4, 5, 6, 6)$



$h_{\mathfrak{b}} = (1, 2, 3, 4, 5, 6)$



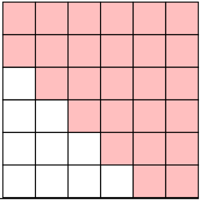
For type $G = Gl(n, \mathbb{C})$: Identify G/B with $\{V_\bullet : V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n, \dim_{\mathbb{C}} V_i = i\}$. For $\mathsf{x} \in \mathfrak{g}$,

$$\mathcal{H}ess(\mathsf{x}, h) = \{V_\bullet : xV_i \subseteq V_{h(i)}\}.$$

Some examples: permutohedron, Peterson

$H_0 = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$

$h_0 = (2, 3, 4, \ldots, n-1, n, n)$



$$\mathsf{n} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\mathcal{H}ess(\mathsf{n}, H_0)$ is a Peterson variety

$$\mathsf{s} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

$\mathcal{H}ess(\mathsf{s}, H_0)$ is a permutohedral variety

$a_i \neq a_j$ for $i \neq j$

The Peterson variety

The Peterson variety *Pet* is

$$Pet = \{V_\bullet : XV_i \subset V_{i+1}\},$$

where X is $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$

Pet has a $\mathbb{C}^*$ action, via diagonal matrices with entries $(t^n, t^{n-1}, \dots, t)$.

For $n = 3$, flags in *Pet* may be represented by elements:

$$Pet = \begin{pmatrix} a & b & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 1 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Positivity in Schubert calculus and Peterson Schubert calculus

$S = Diag(t^n, t^{n-1}, \ldots, t)$ acts on Pet.
Its equivariant cohomology ring $H_S^*(Pet)$ has a linear basis

$$\{p_A : A \subset \{1, \ldots, n-1\}\}$$

indexed by subsets of $\{1, \ldots, n-1\}$ obtained by certain subvarieties.
In cohomology, expand the product in terms of this basis to get coefficients $b_{AB}^C \in \mathbb{Q}[t]$ defined by the relationship:

$$p_A p_B = \sum_C b_{AB}^C p_C.$$

The basis is *positive* for geometric reasons.

Theorem (G.-Mihalcea-Singh)

The coefficients b_{AB}^C are monomials with positive, integral coefficients.

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A positive formula for Peterson Schubert calculus

$$p_A p_B = \sum_C b_{AB}^C p_C.$$

For any set $A \subseteq \{1, \dots, n-1\}$, let

$$\mathcal{H}_A = \text{Largest element of } A$$
$$\mathcal{T}_A = \text{Smallest element of } A$$

Theorem (G.-Gorbutt)

Let $A, B, C \subseteq \{1, \dots, n-1\}$ be nonempty consecutive subsets. If $C \supseteq A \cup B$ and $|C| \leq |A| + |B|$, then

$$b_{A,B}^C = d! \binom{\mathcal{H}_A - \mathcal{T}_B + 1}{d,\ \mathcal{T}_A - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_B} \binom{\mathcal{H}_B - \mathcal{T}_A + 1}{d,\ \mathcal{T}_B - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_A} t^d$$

for $d := |A| + |B| - |C|$.

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for $d := |A| + |B| - |C|$.

A strange binomial identity

Let $m, n, w, x, y, z \in \mathbb{Z}$ with $w + x = y + z$.

Theorem (G.-Gorbutt)

$$\binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{z}$$
$$= \sum_{\substack{0 \le i \le m \\ 0 \le j \le n}} \binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}\binom{w+i+n}{w+i+j}.$$

Proving it required a very fancy bijection of sets.

What about other Hessenberg varieties?

- They are paved by affines
- There is a $\mathbb{C}^*$ action with isolated fixed points
- In some cases, Hessenberg varieties are known to be *GKM* (G-Tymoczko)
- In some cases, the Betti numbers of Hessenberg varieties are known to be palindromic
- In some cases, Hessenberg varieties satisfy hard Lefschetz
- In some cases, the cohomology ring is known (with algebraic rather than linear generators).
- In some cases, Hessenberg varieties themselves are known, and satisfy positivity. (G.-Precup)

Case of the minimal nilpotent orbit

$$\mathcal{H}ess(\mathsf{x}, h) = \{V_\bullet : xV_i \subseteq V_{h(i)}\}.$$

Let

$$\mathsf{n} = E^{1n} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Suppose that H is indecomposable.

- As a variety, $\mathcal{H}ess(\mathsf{n}, h)$ is a union of Schubert varieties (Abe-Crooks)
- As a scheme, $\mathcal{H}ess(\mathsf{n}, h)$ is reduced (G.-Precup)

Matrix Hessenberg varieties

type A only

Let $\mathbf{z} := \{z_{ij} \mid (i,j) \in [n] \times [n]\}$ for each $j \in [n]$. Write matrix $Z = (z_{ij})$ using columns: $\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ with $v_j := \sum_{i=1}^{n} z_{ij} e_i \in M_{n,1}(\mathbb{C}[\mathbf{z}])$.

Given x and h. Consider matrices in $M_n(\mathbb{C})$ satisfying $\mathrm{x} V_i \subseteq V_{h(i)}$ where V_i denotes the first i columns of the matrix.

Let $\mathcal{I}_{\mathrm{x},h,i}$ denote the equations obtained from the rank condition

$$\mathit{rank} \begin{bmatrix} | & | & & | & | & | & & | \\ \mathrm{x}v_1 & \mathrm{x}v_2 & \cdots & \mathrm{x}v_i & v_1 & v_2 & \cdots & v_{h(i)} \\ | & | & & | & | & | & & | \end{bmatrix} \leq h(i).$$

Let $\mathcal{I}_{\mathrm{x},h} = \sum_i \mathcal{I}_{x,h,i}$.

Definition

$\overline{\mathcal{H}ess}(\mathrm{x}, h) := \mathit{Spec}(\mathbb{C}[\mathbf{z}]/\mathcal{I}_{\mathrm{x},h})$ *is the* matrix Hessenberg scheme*.*

On the corners of the Hessenberg space

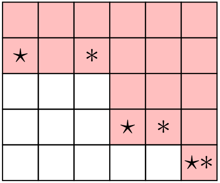
$\mathcal{I}_{x,h} = \sum_i \mathcal{I}_{x,h,i}$

$\mathcal{I}_{x,h,i}$ equations obtained from the rank condition

$$rank \begin{bmatrix} | & | & & | & | & | & & | \\ xv_1 & xv_2 & \cdots & xv_i & v_1 & v_2 & \cdots & v_{h(i)} \\ | & | & & | & | & | & & | \end{bmatrix} \leq h(i).$$

It is sufficient to sum over i indexing the *corners* of the Hessenberg space:

$h = (2,2,2,5,5,6)$ $\qquad \mathcal{I}_{x,h} = \mathcal{I}_{x,h,1^*} + \mathcal{I}_{x,h,4^*} + \mathcal{I}_{x,h,6^*}$



Minimal Hessenberg varieties: nilpotent case

$\overline{\mathcal{H}ess}(\mathsf{x}, h) = Spec(\mathbb{C}[\mathbf{z}]/\mathcal{I}_{\mathsf{x},h})$.

$$\mathsf{x} = \mathsf{n} = E^{1n} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Theorem (Abe-Crooks)

For $\mathsf{n} = E^{1n}$, *the Hessenberg* variety *in* G/B *is a union of Schubert varieties. Furthermore, the variety* $\mathcal{H}ess(\mathsf{n}, h)$ *is the union of* $\mathcal{X}_w$ *for* w *satisfying* $w^{-1}E^{1n} \in \phi_H^+$. *If* h *is indecomposable, it is reduced.*

Theorem (G.-Precup)

For $\mathsf{n} = E^{1n}$, *the matrix Hessenberg scheme* $\overline{\mathcal{H}ess}(\mathsf{n}, h)$ *is a union of matrix Schubert varieties. Furthermore... It is reduced if and only if* h *is indecomposable.*

$\overline{\mathcal{H}ess}(\mathrm{n}, h) = Spec(\mathbb{C}[\mathbf{z}]/\mathcal{I}_{\mathrm{n},h})$

THEOREM (G.-PRECUP)

For $\mathrm{n} = E^{1^n}$*, the matrix Hessenberg scheme* $\overline{\mathcal{H}ess}(\mathrm{n}, h)$ *is a union of matrix Schubert varieties. Furthermore,*

$$\overline{\mathcal{H}ess}(\mathrm{n}, h) = \bigcup_{i \in \mathcal{C}(h)} \mathcal{X}_{w_0 u_i v_{h(i)}}, \text{ with}$$

$$u_i := s_1 s_2 \cdots s_{i-1} \text{ for all } i > 1 \text{ and } u_1 := e$$

$$v_i := s_{n-1} s_{n-2} \cdots s_i \text{ for all } i < n \text{ and } v_n := e.$$

It is reduced if and only if h is indecomposable.

Furthermore,

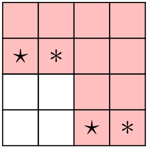
$$\mathcal{I}_{\mathrm{n},h} = \sum_{i \in \mathcal{C}(h)} \mathcal{I}_{\mathrm{n},h,i}$$

$$\mathcal{I}_{\mathrm{n},h,i} = \langle z_{n1}, \ldots, z_{ni} \rangle \cdot \langle p_J : J \subseteq \{2, 3, \ldots, n\}, |J| = h(i) \rangle$$

p_J are $h(i) \times h(i)$ minors of (z_{ij}) with rows specified by J, columns $1, \ldots, h(i)$.

An example: minimal nilpotent Hessenberg varieties

Consider $h = (2, 2, 4, 4)$. Observe that h is decomposable.



Two corners: only one contributes to the defining ideal.

$\mathcal{I}_{n,h} = \mathcal{I}_{n,h,2} = \langle z_{41}, z_{42} \rangle \cdot \langle p_J \mid J \subset \{2, 3, 4\}, |J| = 2 \rangle$.

The associated primes of $\mathcal{I}_{n,h}$ (calculated with Macaulay2) are

$$\begin{aligned} P_1 &= \langle z_{41}, z_{42} \rangle, \\ P_2 &= \langle z_{31}z_{42} - z_{32}z_{41}, z_{21}z_{42} - z_{22}z_{41}, z_{21}z_{32} - z_{22}z_{31} \rangle, \\ P_3 &= \langle z_{41}, z_{42}, z_{21}z_{32} - z_{22}z_{31} \rangle. \end{aligned}$$

$P_1 \subsetneq P_3$, so that P_3 corresponds to embedded points: $V(P_3) = \mathcal{X}_{[3142]}$
$V(P_1) = \mathcal{X}_{[3241]}$, and $V(P_2) = \mathcal{X}_{[4132]}$.
The Hessenberg's underlying variety is $\mathcal{X}_{[3241]} \cup \mathcal{X}_{[4132]}$, with an embedded Schubert variety $\mathcal{X}_{[3142]}$ (contained in both $\mathcal{X}_{[4132]}$ and $\mathcal{X}_{[3241]}$)

Minimal Hessenberg varieties: semisimple case

$\overline{\mathcal{H}ess}(\mathsf{x}, h) = Spec(\mathbb{C}[\mathbf{z}]/\mathcal{I}_{\mathsf{x},h})$. For $a_1, a_2 \in \mathbb{C}$, $a_1 \neq a_2$, let

$$\mathsf{s} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_2 \end{bmatrix}.$$

Theorem (G.-Precup)

For s semisimple, the matrix Hessenberg scheme $\overline{\mathcal{H}ess}(\mathsf{s}, h)$ is reduced for all h. It is a union of matrix Richardson varieties,

$$\overline{\mathcal{H}ess}(\mathsf{s}, h) = Spec(\mathbb{C}[\mathbf{z}]/\mathcal{I}_{\mathsf{s},h}) = \bigcup_{i \in \mathcal{C}(h)} \left(\mathcal{X}^{u_i} \cap \mathcal{X}_{w_0 v_{h(i)}}\right).$$

where $\mathcal{C}(h)$ are corners of the box diagram for h and

$$u_i := s_1 s_2 \cdots s_{i-1} \text{ for all } i > 1 \text{ and } u_1 := e$$
$$v_i := s_{n-1} s_{n-2} \cdots s_i \text{ for all } i < n \text{ and } v_n := e.$$

BACK TO POSITIVITY!

- Positivity occurs for the cohomology of the Hessenberg varieties over the minimal semisimple and nilpotent orbits; what about equivariantly?
- What can be said about the K-theory of Hessenberg vareities?
- How do we more generally find geometric representatives for cohomology classes of $\mathcal{H}ess(\mathrm{n}, h)$?
- More generally, what are the relationships of these geometric representatives to those in $\mathcal{H}ess(\mathrm{s}, h)$?

THANK YOU!

Back to positivity!

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