A Tribute to Georgia Benkart

Women in Algebra and Combinatorics

SUNY-Albany, 28-30 April 2023

Research

- 135 publications
- >1750 citations
- 96 coauthors

Mentoring

- 22 PhDs supervised
- Hundreds of young mathematicians mentored

Service

- Past President of AWM
- AMS Central Section Secretary
- Editor of J. Algebra, Comm. Algebra, Alg. and Number Theory, AMS Surveys and Monographs, and others
- Many AMS and NSF committees, MSRI board of directors, NSF and NSERC grant panels

PI MU EPSILON Journal

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CONTENTS

A GENERALIZATION OF SUBNET WITH SOME RESULTING IMPROVEMENTS IN MOORE-SMITH CONVERGENCE THEORY

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Ohio State University

Section 1. Introduction.

This paper is intended to improve the theory of Moore-Smith Convergence by generalizing the definition of subnet. We begin by examining some short-comings of the present Moore-Smith theory of convergence. Given a net S, it is possible to construct in a natural way a filter dependent on S. From this filter a second net T may be constructed. While S may be shown to be a subnet of T, T in general is not a subnet of S, even though S and T generate the same filter (See example 3). Also, given nets S and T defined on the same directed set, T may equal S on all but one element of the directed set and still not be a subnet of S (See example 1). These limitations in the theory illustrate the need for a new definition of subnet.

The new definition will generalize the classical definition of subnet. It will have the advantage of preserving the classical theorems, while eliminating the above disadvantages. It will also yield the following powerful result:

Given nets S and T, and filters Φ_S and Φ_T constructed from them, $\Phi_S \subseteq \Phi_T$ implies T is a subnet of S under the new definition. In addition, this result will provide an easy method for finding a common supernet for nets S and T.

Section 2. Definition and generalization of subnet.

Newton

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Hopkins

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Maxwell

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Crystal

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Wedderburn

|

Jacobson

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Benkart

Jordan Algebras

A Jordan algebra (in char $\neq 2$) is a nonassociative unital algebra J satisfying

$$x \bullet y = y \bullet x$$
$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$$

for all $x, y \in J$.

Every unital associative algebra A admits a Jordan structure A^+, with product $a \bullet b = \frac{1}{2}(ab + ba)$. Jordan subalgebras of A^+ are called special Jordan algebras. Almost all simple Jordan algebras are special.

Tits-Kantor-Koecher Construction

$$\text{TKK functor} : \textit{Jordan algs} \longrightarrow \textit{Lie algs}$$
$$J \longmapsto (\mathfrak{sl}(2) \otimes J) \oplus \{J, J\},$$

where $\{J, J\} := (J \otimes J)/\langle a \otimes b + b \otimes a,\ a \otimes bc + b \otimes ca + c \otimes ab \rangle$, with Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x|y)\{a, b\}$$
$$[\{a, b\}, x \otimes c] = x \otimes \tfrac{1}{2}(b, c, a)$$
$$[\{a, b\}, \{c, d\}] = \tfrac{1}{2}\{(b, c, a), d\} + \tfrac{1}{2}\{c, (b, d, a)\},$$

Every A_1-graded Lie algebra is a central quotient of some TKK(J). Jacobson used variants of this idea to construct E_6, E_7, E_8.

Crossing the Jordan

Jordan definition

A subspace B of a special Jordan algebra $J \subseteq A^+$ is an inner ideal if $xax \in B$ for all $a \in J$ and $x \in B$. For example, any 1-sided ideal of A is an inner ideal of A^+.

Benkart thesis

A subspace B of a Lie algebra L is an inner ideal if $[B, [B, L]] \subseteq B$. For example, Span $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \subseteq \mathfrak{sl}(2)$ is an inner ideal.

Benkart, expanding on thesis

Benkart '77

Several original ideas:

1. Classification of minimal inner ideals in Lie algebras in terms of ad-nilpotent elements.

2. Use of ad-nilpotent elements to find Jordan algs and their modules inside inner artinian Lie algs.

3. First steps toward classification of fin dim simple Lie algs in char > 5.

Mary Ellen Rudin

G.B., Mary Ellen, Jessica Millar

Root systems

$\mathfrak{g}$ fin dim simple Lie alg over $\mathbb{C}$ (e.g. $\mathfrak{g} = \mathfrak{sl}(n)$)
$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalg (e.g. $\mathfrak{h}$=diagonal matrices in $\mathfrak{sl}(n)$)

Then $\mathfrak{g} = \mathfrak{h} \oplus \displaystyle\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, with

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

$\Delta = \{\alpha \in \mathfrak{h}^* \setminus 0 : \mathfrak{g}_\alpha \neq 0\}$ the **root system** of $(\mathfrak{g}, \mathfrak{h})$

$\dim \mathfrak{h}$ = the **rank** of $\mathfrak{g}$.

$\{$Iso classes of f.d. simple Lie algs/$\mathbb{C}\}$ $\xleftrightarrow{1-1}$ $\{$irred root systems$\}$

Classification of irred root systems: $A_n, B_n, C_n, D_n + 5$ exceptionals

Kostrikin-Shafarevich Conjecture

K-S Conjecture '66

Restricted fin-dim simple Lie algs / alg closed field $\mathbb{F}$ of char $p > 3$ are **classical type** or **Cartan type**.

Classical type: mod p reduction of A_n, B_n, C_n, D_n, exceptionals

Cartan type: $W(m) = \mathrm{Der}(\mathbb{F}[x_1, \ldots, x_m]/\langle x_1^p, \ldots, x_m^p \rangle)$ and its simple subalgs $S(m), H(m), K(m)$

Block-Wilson '84

K-S is true in char > 7.

Classification in characteristic p

Benkart-Osborne '84

K-S holds for **unrestricted** rank 1 Lie algs in char > 7 if we enlarge the truncated current alg using more divided powers

Premet-Strade-Wilson '04

For alg closed $\mathbb{F}$ of char > 3, the f.d. simple Lie algs are those of classical, Cartan, or Melikyan type (only in char 5).

Missing piece: Recognition Thm

Classification of associated gradeds based on filtrations on simple Lie algs. Kac conjectured and gave partial proof (1970); complete proof written down by Benkart-Gregory-Premet '09.

Weight Lifting

Definition

Let $\mathfrak{h}$ be a Cartan subalgebra of a fin dim simple Lie alg $\mathfrak{g}$ over $\mathbb{C}$. A $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{h})$-weight module if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, and $M_\lambda = \{m \in M : h.m = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}$ is fin dim for all λ.

Fernando thesis '83, TAMS '90

All simple weight modules M are obtained by parabolic induction: M is the (unique) simple quotient of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} N$ for some parabolic subalg $\mathfrak{p} \supseteq \mathfrak{h}$ and simple weight $\mathfrak{p}$-module N. Can assume N is cuspidal, i.e. $\dim N_\lambda$ is constant for all $\lambda \in \operatorname{supp} N$.

Classification of Weight Modules

Benkart-Britten-Lemire '97

Infinite-dimensional cuspidal modules occur only for types A and C. When $N_\lambda \leq 1$ for all λ, these modules come from modules for Weyl algebras.

Mathieu '00

Classification of simple weight modules for fin dim reductive Lie algebras by classifying cuspidals in types A and C.

Georgia and Paula Benkart, Dan Britten, Frank Lemire & families

Classification of root-graded Lie algebras

L Lie alg / $\mathbb{C}$ containing f.d. simple Lie alg $\mathfrak{g}$, Cartan $\mathfrak{h} \subset \mathfrak{g}$, and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$

Then L is called a Δ-graded Lie algebra if

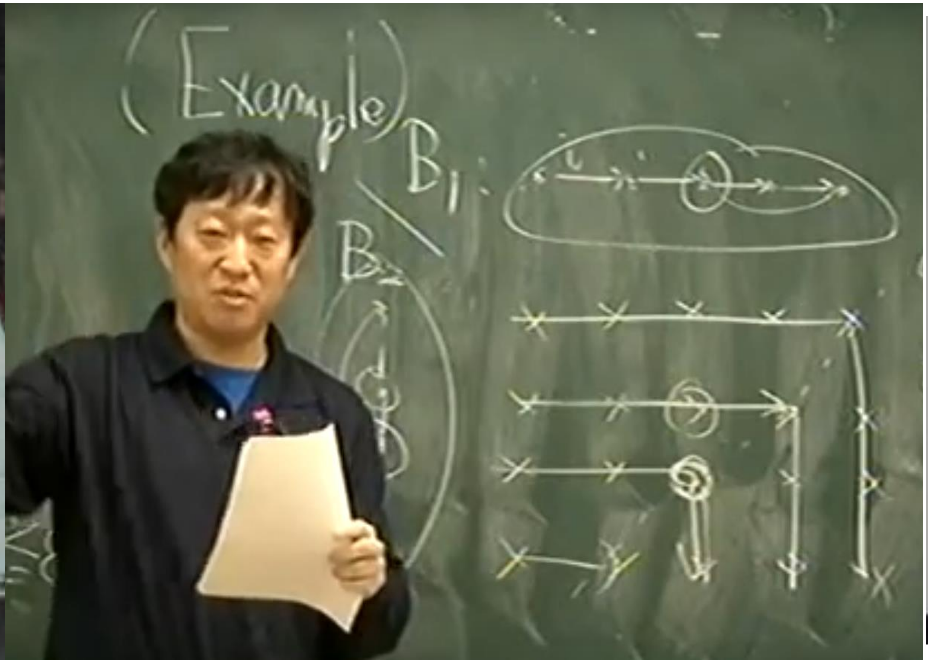
1. $L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$, where L_μ is $\mathfrak{h}$-weight space;
2. $L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$.

If L is Δ-graded, it decomposes into fin dim simple $\mathfrak{g}$-submodules:

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D,$$

where $\mathfrak{g}$ is adjoint module, V is little adjoint module (hi wt is highest short root), D is a trivial module.
The possibilities for A, B, D and multiplication between components generalize TKK to other nonassoc algs.
Proof. Type-by-type: Berman-Moody '92, Benkart-Zelmanov '96, Neher '96, Allison-Benkart-Gao '02, Benkart-Smirnov '03.

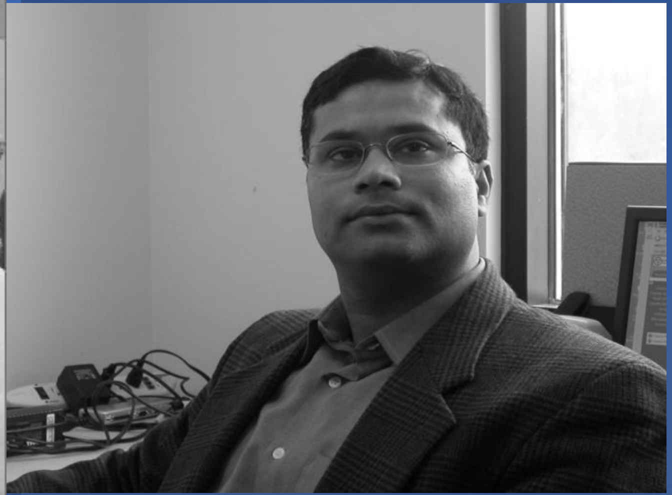
Alberto Elduque, Consuelo Martinez, G.B., Santos Gonzalez,
Seok-Jin Kang, Dongho Moon



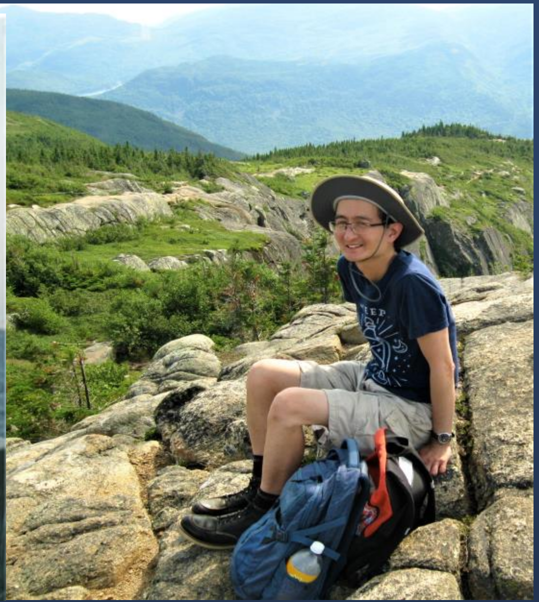
PhD Students

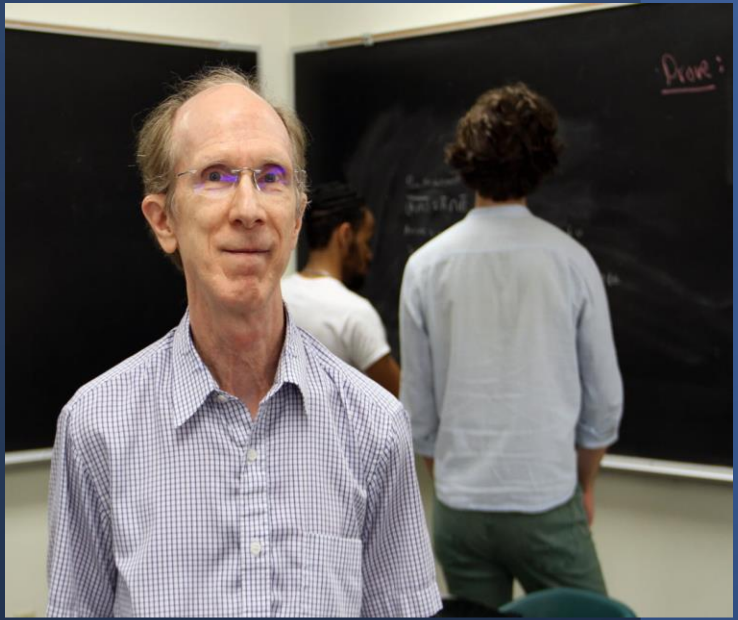
Fernando, Suren	1983
Kass, Steven	1984
Neidhardt, Wayne	1985
Hall, Mark	1987
Stroomer, Jeffrey	1991
Lee, Chanyoung	1992
Peters, Karl	1992
Wang, Qing	1992
Halverson, Thomas	1993
Leduc, Robert	1994
Eng, Oliver	1996

Grood, Cheryl	1998
Moon, Dongho	1998
Hildebrand, Jeffrey	2000
Bloss, Matthew	2002
Chakrabarti, Manish	2003
Lopes, Samuel	2003
Lau, Michael	2004
Mukherjee, Shantala	2004
Ondrus, Matthew	2004
Christodoulopoulou, Konstantina	2007
Madariaga, Sara	2012




Hoboken, NJ
September 1, 2001




Prove:








Classical Schur-Weyl Duality

$V = \mathbb{C}^r$
$GL_r(\mathbb{C}) \curvearrowright V^{\otimes n}$ diagonal action
$V^{\otimes n} \curvearrowleft S_n$ permutation of tensor factors

The S_n-action generates $\mathrm{End}_{GL_r(\mathbb{C})}(V^{\otimes n})$;
the $GL_r(\mathbb{C})$-action generates $\mathrm{End}_{S_n}(V^{\otimes n})$.

Commuting actions decompose $V^{\otimes n}$ into sums of simples for S_n and $GL_r(\mathbb{C})$.

Generalized Schur-Weyl Duals

$$\begin{array}{ccll} GL_r(\mathbb{C}) & \longleftrightarrow & \mathbb{C}[S_n] & \text{Symmetric gp alg } (\sim 1900) \\ \bigcup & & \bigcap & \\ O_r(\mathbb{C}) & \longleftrightarrow & B_n(r) & \text{Brauer algebra } (\sim 1930) \\ \bigcup & & \bigcap & \\ S_r & \longleftrightarrow & P_n(r) & \text{Partition algebra } (\sim 1990) \end{array}$$

Georgia+Chakrabarti, Halverson, Leduc, Lee, Stroomer '94

Determined the Schur-Weyl dual of $GL_r(\mathbb{C}) \curvearrowright V^{\otimes m} \otimes (V^*)^{\otimes n}$, now called the **walled Brauer algebra**.

Numerous further variations/generalizations by Georgia, together with many other younger collaborators, notably with Tom Halverson.

BIRS 2011

Leeds 2019

BIRS 2016

We will miss you Georgia!