Kohnert diagrams & their not well-behaved poset structure

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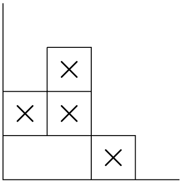
Women in Algebra and Combinatorics Northeast Conference Celebrating the AWM: 50 Years and Counting

(w/ Felix Hutchins, Nick Mayers, and Etienne Phillips)

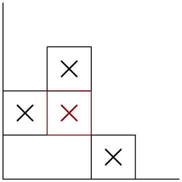
Main ingredients

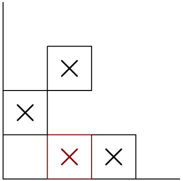
Diagrams



Main ingredients

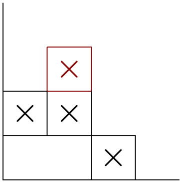
Kohnert Moves

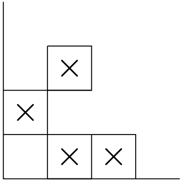


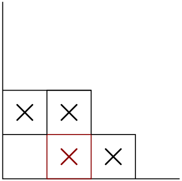


Main ingredients

Kohnert Moves



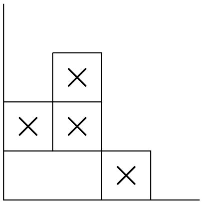




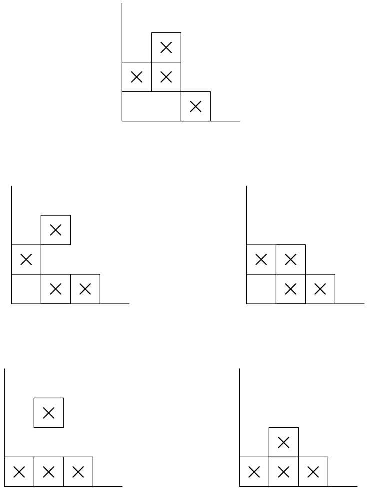
Main ingredients

Set of Kohnert diagrams of D: $\mathbb{KD}(D)$

D

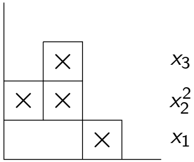


$\mathbb{KD}(D)$



Kohnert Polynomials

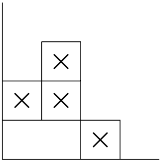
D



$$wt(D) = x_1 x_2^2 x_3$$

Kohnert Polynomials

D

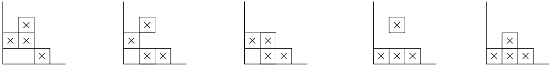


x_3

x_2^2

x_1

$$wt(D) = x_1 x_2^2 x_3$$



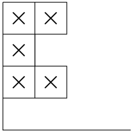
$$K_D = \sum_{T \in \mathbb{KD}(D)} wt(T) = x_1 x_2^2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2^2 + x_1^3 x_3 + x_1^3 x_2$$

Where are they coming from?

Composition

$a = (0, 2, 1, 2)$

$\mathbb{KD}(0, 2, 1, 2)$



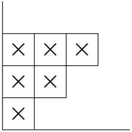
Demazure character

$\kappa(a)$

Partition

$\lambda = (3, 2, 1)$

$\mathbb{KD}(1, 2, 3)$



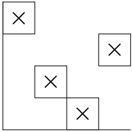
Schur polynomial

$s_\lambda(x_1, \ldots, x_n)$

Permutation

$w = 4213$

$\mathbb{KD}(4213)$



Schubert polynomial

$\mathfrak{S}_w$

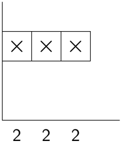
Enumerative Results

Question: Can we find a formula for $|\mathbb{KD}(D)|$?

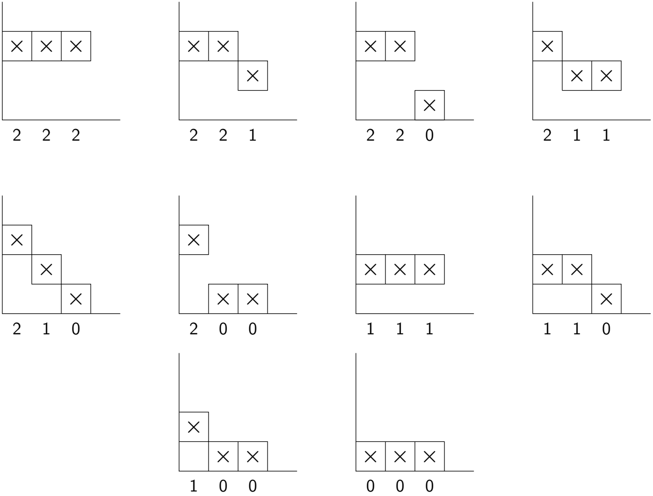
First observations:

- Finding a formula for all D seems impossible.
- Even figuring out the *parameters* is complicated.
- *First semester plan*: Let's play around and see what we find.
  - Felix had never done research before and has a very interesting background.
  - Etienne had done research before and was in my combinatorics grad course.
  - Fun for both of them because they ran a lot of experiments with a program we wrote.
  - They also enjoyed finding out some of the proofs, and struggled with others.
  - We got many ideas that we would like to explore.

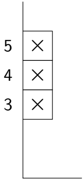
Row of Boxes



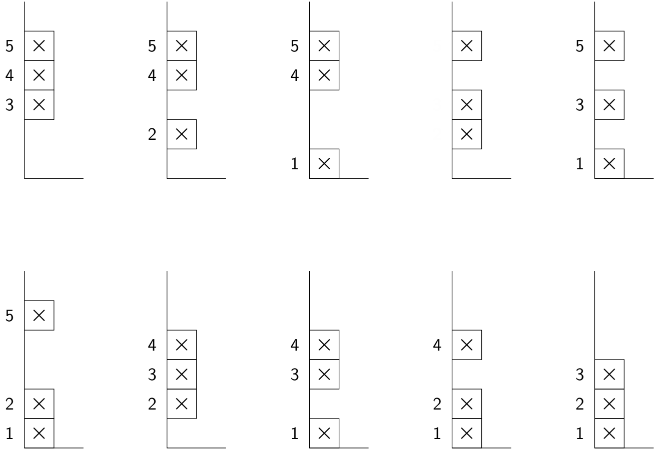
Row of Boxes



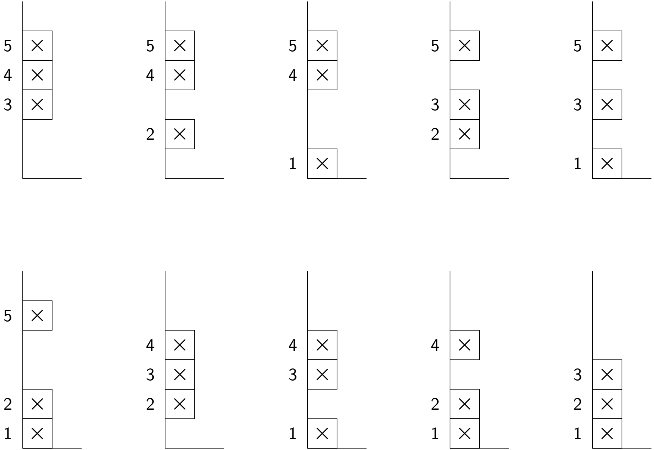
Column of Boxes



Column of Boxes



Column of Boxes



Binomial results

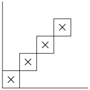
- Let D be a diagram with n boxes in total, all in row r. Then,

$$|\mathbb{KD}(D)| = \binom{n+r-1}{n}$$

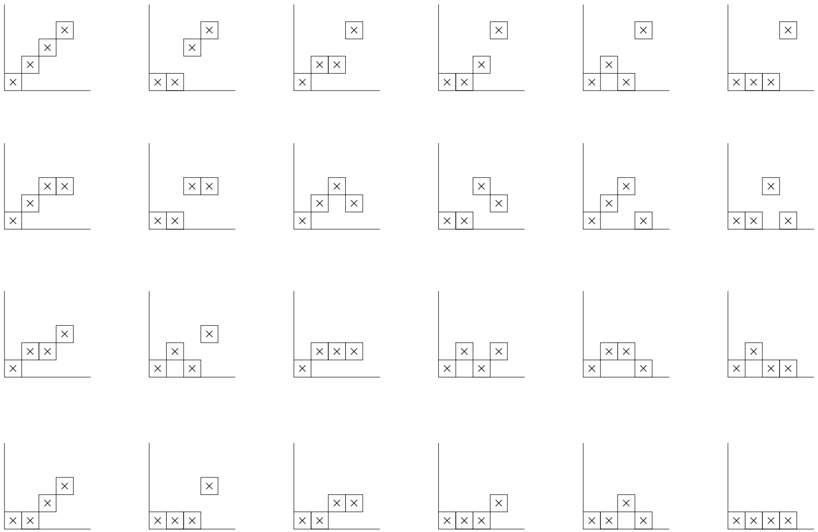
- Let D be a diagram with only boxes in one column from row r_1 to row r_2, with $r_1 < r_2$. Then,

$$|\mathbb{KD}(D)| = \binom{r_2}{r_2 - r_1 + 1}.$$

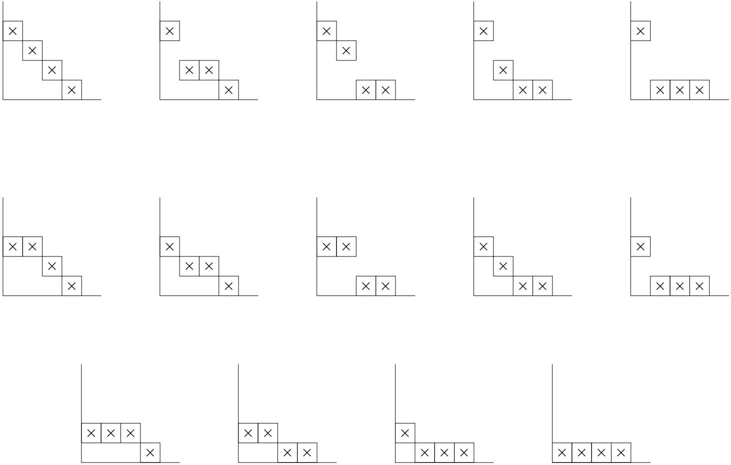
Diagonal



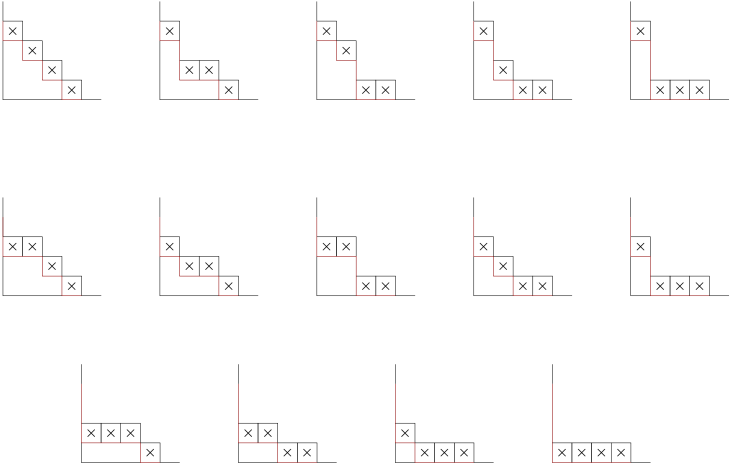
Diagonal



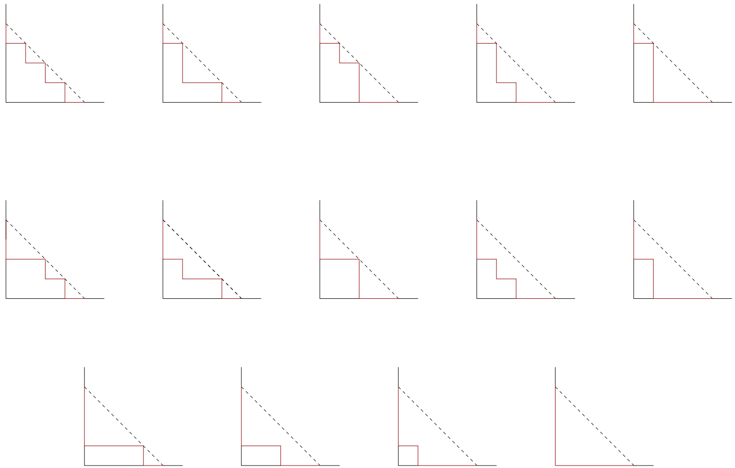
Anti-Diagonal



Anti-Diagonal



Anti-Diagonal



Results

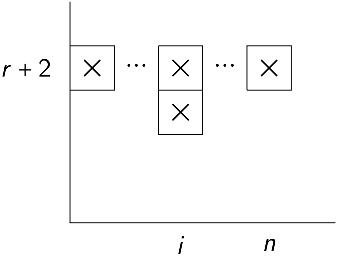
- Let D be a diagonal diagram with n boxes. Then

$$|\mathbb{KD}(D)| = n!.$$

- Let D be an anti-diagonal diagram with n boxes. Then

$$|\mathbb{KD}(D)| = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

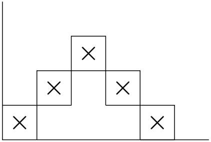
Row with appendage



$$|\mathbb{KD}(D)| = \sum_{j=2}^{r+2} \binom{r+i-j+1}{i-1}\binom{n-i+j-1}{j-1}(j-1)$$

Other experiments

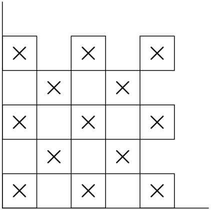
Triangle



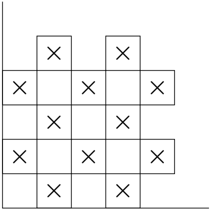
$2n+1$ boxes in total

$$|\mathbb{KD}(D)| = 2^n n!$$

Checkered



$1,\ 2,\ 6,\ 36,\ 373,\ 6389, \ldots$

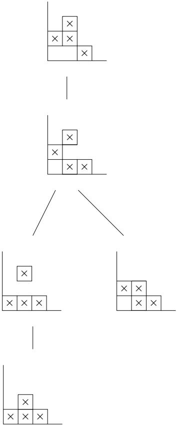


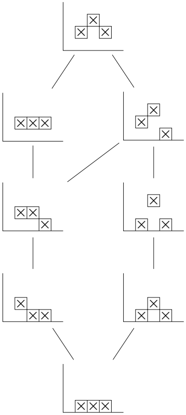
$1,\ 2,\ 6,\ 41,\ 373,\ 8236, \ldots$

Kohnert Poset $(\mathbb{KD}(D), \prec)$

Kohnert Poset $(\mathbb{KD}(D), \prec)$

$D_2 \prec D_1$, for $D_1, D_2 \in \mathbb{KD}(D)$, if D_2 can be formed from D_1 by applying Kohnert moves





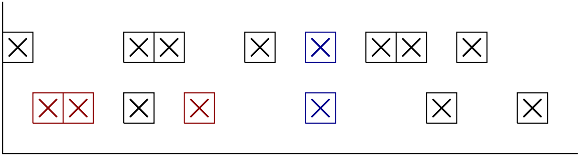
Poset results

Question: When does the poset $\mathbb{KD}(D)$ have a unique minimal element?

$|Min(\mathbb{D}(a))| = 1$ at least in the following cases:

- When D is a diagram for which all boxes are contained in columns $c_1 < c_2 < \cdots < c_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, where b_i denotes the number of boxes in column c_i.
- When D is a diagram which has at most one box in each column.
- When D is a diagram associated to a composition a.
- In some cases when the diagram D has boxes in only two rows.

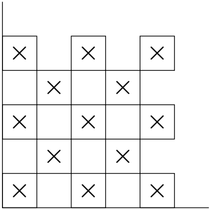
Two Row case



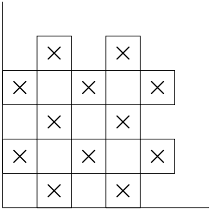
- Let D be a diagram with all boxes in rows r_1 and r_2, where $r_1 < r_2$. Then

$$|Min(D)| = \begin{cases} |Col^{\leftarrow}(D; r_1)| + 1, & r_1 > 1; \\ 1, & r_1 = 1. \end{cases}$$

Checkered



$Ch^1(5)$



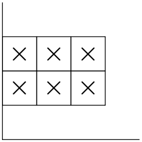
$Ch^2(5)$

- For the checkered diagrams,

$$|Min(Ch^1(n))| = |Min(Ch^2(n))| = \begin{cases} 1, & n \text{ even;} \\ \binom{n-1}{(n-1)/2}, & n \text{ odd.} \end{cases}$$

Ranked posets

Question: When is the poset corresponding to D ranked?



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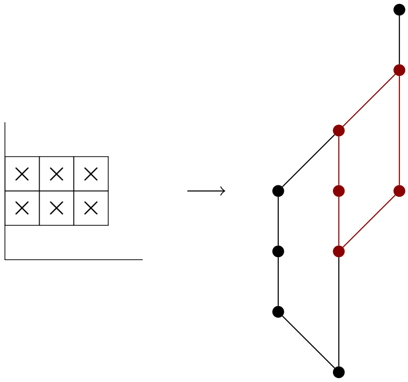

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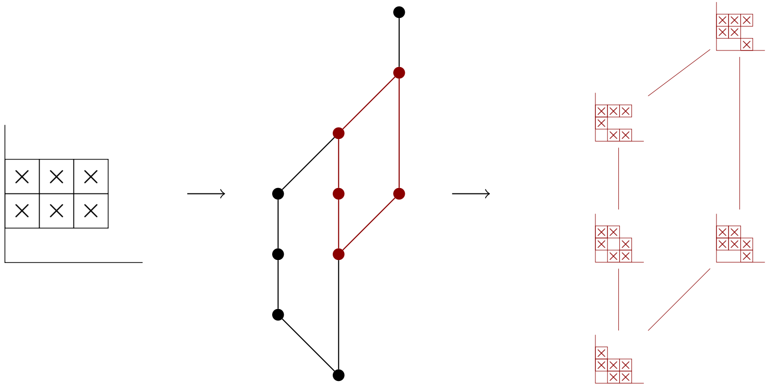

Ranked results

- We have identified several subdiagrams for which we would get the situation above, and so $P(D)$ is not ranked.
- Let D be a row/column diagram. Then $P(D)$ is ranked only if
  - D has all boxes in one column or
  - all boxes except one are bottom justified in each column.
- For diagrams with at most one box per column, $P(D)$ is always ranked.
- For the two-row case, we have characterized when the poset is ranked.
- The poset corresponding to the checkered diagrams is ranked if and only if $1 \leq n < 4$.

Thank you!

¡Muchas gracias!

References

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