The geometry and combinatorics of Springer fibers

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April 29, 2023

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Springer fibers

The flag variety is G/B

If $G = GL_n(\mathbb{C})$ and B is upper-triangular matrices then each flag is

- ... a coset gB
- ... a nested subspace $V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n$
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$$\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

- … a matrix with zeros to the right and below a permutation

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Springer fibers

Each gB has a representative in exactly one of the following:

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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These are the **Schubert cells** BwB/B. They are parametrized by permutation matrices w.

Springer fibers

Fix a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$

The Springer fiber of X consists of flags $gB = V_\bullet$ for which

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- ... the image XV_i of each subspace is contained in V_i

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For example: The Springer fiber of $X = 0$ is the full flag variety.

Springer fibers

Springer fibers generalize to many groups G

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There are other descriptions.

... EG: the set of Borel subspaces $\widetilde{\mathfrak{b}} \subseteq \mathfrak{g}$ with $X \in \widetilde{\mathfrak{b}}$

Springer fibers

We focus on the case when X is nilpotent

- ....$X^m = 0$ for some m
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The conjugacy class of X has a unique representative in Jordan canonical form with blocks arranged in nondecreasing order.
Jordan blocks partition n into $\lambda(X)$.

If $\lambda(X)$ has 2 rows then the Springer fiber is a 2-row Springer fiber.

Springer fibers

There are many related varieties that we will not discuss today:

- **Orbital varieties:** An irreducible component of the preimage of the Springer fiber under the projection $G \longrightarrow G/B$. (Melnikov, Fresse, ...)

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- **Springer fibers for partial flag varieties:** The construction of Springer fibers works for G/P where P is parabolic. (Borho & MacPherson, Fresse, Precup & Tymoczko, ...)
- **Hessenberg varieties:** The same construction works when $H \subseteq \mathfrak{g}$ is any linear subspace with $\mathfrak{b} \subseteq H$ and $[H, \mathfrak{b}] \subseteq H$

  $$\{ \text{ Flags } gB : g^{-1}Xg \in H\}$$

  (De Mari & Procesi & Shayman, Tymoczko, Precup, ...)

Representation theory

Theorem

- *S_n acts naturally on the cohomology $H^*(\mathcal{S}_X)$*
- *The top-dimensional cohomology of $H^*(\mathcal{S}_X)$ is irreducible*
- *In fact $H^{top}(\mathcal{S}_X)$ is irreducible of type $\lambda(X)$*
- *The set $\{H^{top}(\mathcal{S}_\lambda)\}$ is precisely the collection of irreducible representations of S_n*

  (λ ranges over nilpotent conjugacy classes, or partitions of n)

  Springer '76, Kazhdan-Lusztig '80, Borho-MacPherson '83, Hotta '82, Lusztig '84, Garsia-Procesi '92, and others...

Representation theory

One construction of this representation uses Borel's presentation

$$H^*(GL_n(\mathbb{C})/B, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal of symmetric functions with no constant term.

The symmetric group S_n acts by permuting variables.

When X is nilpotent and $\lambda = \lambda(X)$ is its partition into Jordan blocks, then there is an ideal of symmetric functions $\mathcal{I}_\lambda$ so the cohomology ring of the Springer fiber of X is

$$\mathbb{Z}[x_1, \ldots, x_n]/\mathcal{I}_\lambda$$

(Kraft, DeConcini & Procesi, Tanisaki, Garsia & Procesi, Biagioli & Faridi & Rosas)

Representation theory

$\mathcal{N} \subset \mathfrak{g}$ are nilpotents and $\mathfrak{g}^{rs}$ are the regular semisimple elements.

$\widetilde{\mathfrak{g}}$ are the pairs $(X, gB) \in \mathfrak{g} \times G/B$ with $g^{-1}Xg \in \mathfrak{b}$.

The map $\varphi : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is projection.

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 & \mathfrak{g} \times G/B & \\
\widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d] & \widetilde{\mathfrak{g}} \arrow[u, hook] \arrow[d, "\varphi"] & \widetilde{\mathfrak{g}^{rs}} \arrow[l, hook'] \arrow[d] \\
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The symmetric group acts as deck transformations on $\widetilde{\mathfrak{g}^{rs}}$.

Representation theory

An $\mathfrak{sl}_k$ web is a plane graph that is a morphism in a diagrammatic category encoding the representation theory of $U_q(\mathfrak{sl}_k)$.

Webs satisfy skein-theoretic relations arising from the algebra they encode. Consider the vector space they generate as formal vectors, up to the equivalence relations.

$k = 2$**:** Noncrossing matchings form a basis for webs. Kuperberg describes the combinatorial representation theory of webs (including for other Lie types). Khovanov uses webs for $\mathfrak{sl}_2$ to construct the cohomology of certain Springer fibers.

$k = 3$**:** There is a more complicated basis for webs, bijective with standard tableaux (Khovanov & Kuperberg).

Representation theory

Suppose λ is a partition of n.

Definition

The Young diagram of shape λ is an arrangement of boxes (left- and top-aligned) with λ_i boxes in the i^{th} row. A Young tableau is a Young diagram that has been filled with numbers according to some rule:

- **row-strict** means the numbers increase in each row (L to R)
- **standard** means row-strict and that numbers increase in each column (top to bottom)

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline 5 \\ \cline{1-1}\end{array}$$

Representation theory

Theorem

Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. The cells

$C_w \cap \mathcal{S}_X$

are enumerated by tableaux of shape $\lambda(X)$.

Representation theory

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Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. The cells $C_w \cap \mathcal{S}_X$ that are nonempty are enumerated by row-strict tableaux of shape $\lambda(X)$.

- Fillings tell you when to add each basis vector to flag

Representation theory

Theorem

Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. The cells $C_w \cap \mathcal{S}_X$ that

are nonempty
have maximal dimension

are enumerated by

row-strict
standard

tableaux of shape $\lambda(X)$.

- Fillings tell you when to add each basis vector to flag
- Dimension comes from certain inversions

Representation theory

$k = 3$**:** There is a more complicated basis for webs, bijective with standard tableaux (Khovanov & Kuperberg).

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Theorem

As $\mathfrak{sl}_n$ representations, the vector space of webs is isomorphic to the top homology of the Springer fiber. This isomorphism corresponds to an upper-triangular change of basis between a particular basis for webs and the basis induced by the components of the Springer fiber.

(Fontaine, generalizing Westbury, Kuperberg)

Geometry and combinatorics of Springer fibers: motivation

The closure of each Schubert cell $\overline{BwB/B}$ in the flag variety is called a *Schubert variety*. The geometry of Schubert varieties is encoded by combinatorics.

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- The closure $\overline{BwB/B}$ is the union of Schubert cells BvB/B for all permutations $v \prec w$ in *Bruhat order*[1]

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- The closure $\overline{BwB/B}$ is the union of Schubert cells BvB/B for all permutations $v \prec w$ in *Bruhat order*[1]
- The Schubert variety $\overline{BwB/B}$ is smooth if and only if the permutation w avoids certain patterns[2]

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- The closure $\overline{BwB/B}$ is the union of Schubert cells BvB/B for all permutations $v \prec w$ in *Bruhat order*[1]
- The Schubert variety $\overline{BwB/B}$ is smooth if and only if the permutation w avoids certain patterns[2]
- The type of singularity in $\overline{BwB/B}$ is determined by the kind of pattern w contains, more generally combinatorics of w[3]

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[1]Chevalley; [2]Lakshmibai-Sandhya; [3]Kumar, Carrell-Kuttler, Woo-Yong

Geometry and combinatorics of Springer fibers

- Which Springer fibers are singular?
- Which components of Springer fibers are singular?
- Which components intersect each other?

Geometry and combinatorics of Springer fibers

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- Which components of Springer fibers are singular?
- Which components intersect each other?

We said that Schubert cells pave Springer fibers by affines.

- What are the closures of Springer Schubert cells $C_w \cap \mathcal{S}_X$?

Geometry and combinatorics of Springer fibers

Some results are known about singularities of Springer fibers:

- Fung: all components of 2-row Springer fibers are smooth; Ehrig & Stroppel: for all classical Lie types
- Fresse & Melnikov: a complete (short) list of all partitions for which every component of the Springer fiber is smooth
- Fresse & Melnikov, Fresse & Melnikov & Sakas-Obeid: analyzed singular and smooth components for 2-column case
- Mansour & Melnikov: combinatorial description of singular components of the 2-column case
- Fresse, Melnikov: analyze some other Springer fibers

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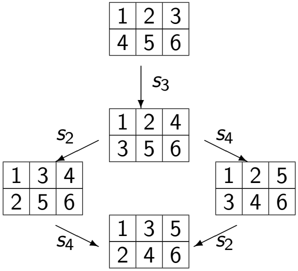
- Fresse, Melnikov: analyze some other Springer fibers
- Perrin & Smirnov: Springer components are normal (also D_n)

Geometry and combinatorics of Springer fibers

Melnikov and Pagnon analyze intersections of irreducible components of Springer fibers (using orbital varieties).

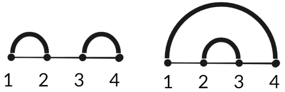
Spaltenstein described components for the Springer fiber using dense smooth subsets corresponding to standard tableaux.

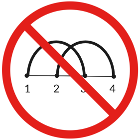
Pagnon and Ressayre showed the boundaries of these sets are governed by the following partial order.



Geometry and combinatorics of Springer fibers

2-row row-strict tableaux are bijective with *noncrossing matchings* or *Temperley-Lieb diagrams*:



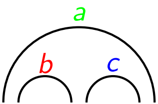


Standard tableaux correspond to perfect matchings.

Geometry and combinatorics of Springer fibers

Theorem (Goldwasser, Sun, T)

The Springer Schubert cells $C_w \cap \mathcal{S}_X$ for 2-row Springer fibers are naturally indexed by noncrossing matchings.



$$\begin{pmatrix} a & b & 1 & 0 & 0 & 0 \\ 0 & a & 0 & c & 1 & 0 \\ 0 & 0 & 0 & a & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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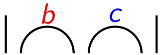


$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \color{red}{b} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \color{blue}{c} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

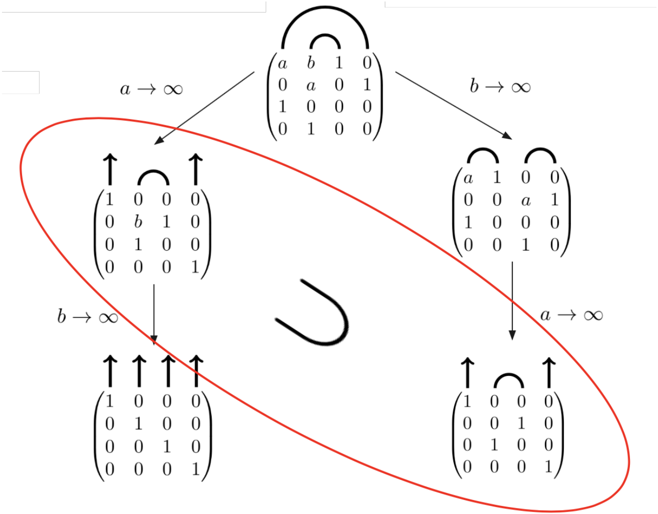
Geometry and combinatorics of Springer fibers

Theorem (Goldwasser, Sun, T)

The closures of the cells are given by nesting/unnesting.

These cells form a paving by affines, not (eg) a CW-complex. This means that sometimes only part of a cell is in the closure of another cell.

Geometry and combinatorics of Springer fibers



Geometry and combinatorics of Springer fibers

For $k = 3$ webs are planar, directed graphs with boundary such that the following hold:

- boundary vertices have degree one (for us, always sources)
- interior vertices are trivalent
- vertices are either sources or sinks

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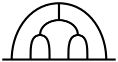

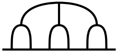

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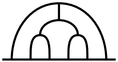
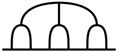
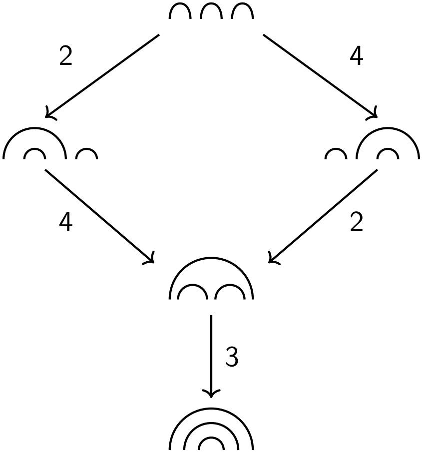




Reduced webs have no interior faces with fewer than six edges.
Reduced webs form a basis for the web vector space.

Geometry and combinatorics of Springer fibers

S_n acts on webs by braiding their strands and then resolving according to the skein-theoretic reductions.



Geometry and combinatorics of Springer fibers

Theorem (Russell, T)

The natural map between webs and 2-row tableaux is not S_n-equivariant. However, the transition matrix is upper-triangular with ones along the diagonal.

Geometry and combinatorics of Springer fibers

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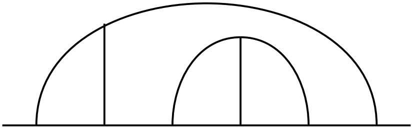
The natural map between webs and 2-row tableaux is not S_n-equivariant. However, the transition matrix is upper-triangular with ones along the diagonal.

- Rhoades proved that the entries are nonnegative.
- Im and Zhu proved that the vanishing terms we identified are in fact the only zero entries in the transition matrix.
- One key observation: the (representation-theoretic) partial order on webs corresponds to nesting and unnesting of arcs.

What does nesting/unnesting mean for $\mathfrak{sl}_3$ webs?



Geometry and combinatorics of Springer fibers

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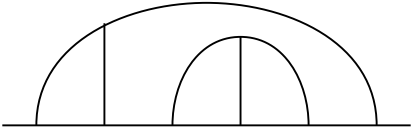

Khovanov and Kuperberg considered the *depth* of the regions, as measured from the unbounded regions.

Geometry and combinatorics of Springer fibers

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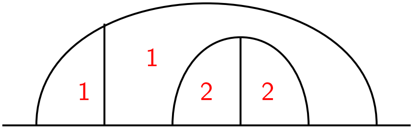

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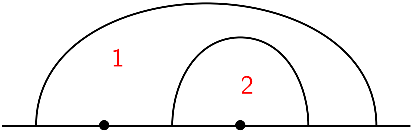
Geometry and combinatorics of Springer fibers



- The middle lines aren't necessary
- This partitions the regions into same-depth bands
- We get a noncrossing matching with isolated dots
- Nesting/unnesting these bands gives a *different* partial order, the *shadow containment* partial order

Geometry and combinatorics of Springer fibers

Theorem (Russell, T)

The transition matrix between the 3-row tableaux basis and the web basis with respect to this shadow containment partial order is upper-triangular with ones along the diagonal.

Geometry and combinatorics of Springer fibers

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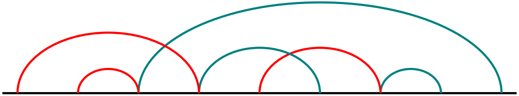
Theorem (Russell, T)

Shadow containment partial order is a refinement of the tableaux partial order, and the transition matrix between the 3-row tableaux basis and the web basis with respect to the usual partial orders is upper-triangular with ones along the diagonal.

Fact: There are natural bijections between reduced webs, 3-row standard tableaux, Yamanouchi words on 3 symbols, and *m-diagrams* (pairs of 2-colored arcs so that same-colored arcs don't cross).



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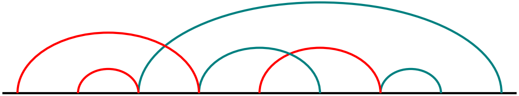
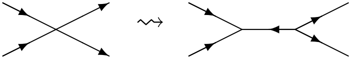
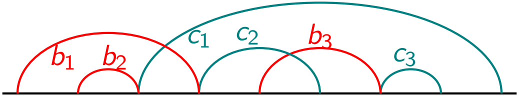


To obtain a reduced web from an m-diagram, replace the point at each crossing with a small edge:



3-row Springer fibers



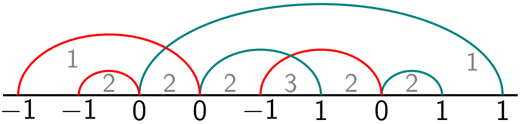
$$\left(\begin{array}{ccccccccc}
a_1 & a_2 & c_1 & c_2 & a_3 & 1 & 0 & 0 & 0 \\
0 & a_1 & 0 & c_1 & a_2+* & 0 & c_3 & 1 & 0 \\
0 & 0 & 0 & 0 & a_1+\dagger & 0 & c_1 & 0 & 1 \\
\hline
b_1 & b_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right)$$

$* = -b_2c_1 + (b_3 - b_1)c_2$ and $\dagger = +(b_3 - b_1)c_1$

Geometry and combinatorics of Springer fibers

Theorem (Hafken, Lang, T, Vandegrift)

Suppose T is a balanced Yamanouchi word T on $\{-1, 0, 1\}$. Let $\mathcal{M}_T$ be the m-diagram of T, namely the noncrossing matching of -1 and 0 together with the noncrossing matching of 0 and 1. Let $\mathcal{A}_T$ be the arcs created as the unique noncrossing matching of -1 and 1. Then $\mathcal{A}_T$ represents the depth bands of $\mathcal{M}_T$.



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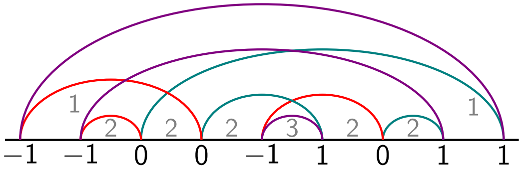
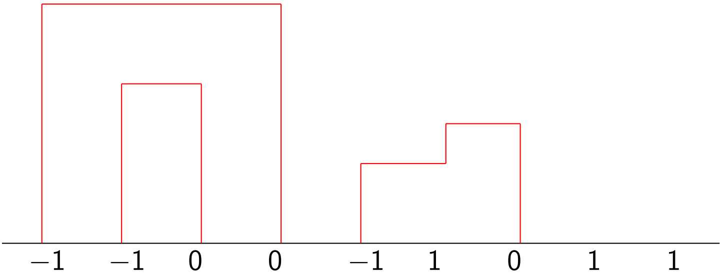


Geometry and combinatorics of Springer fibers

Square off the arc construction to get (reduced) webs:



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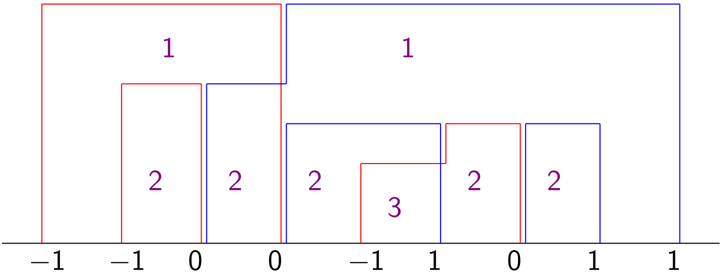
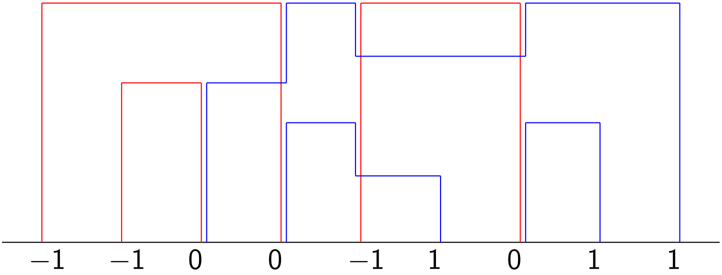


Pull reds above blues to get entries of Springer cells:



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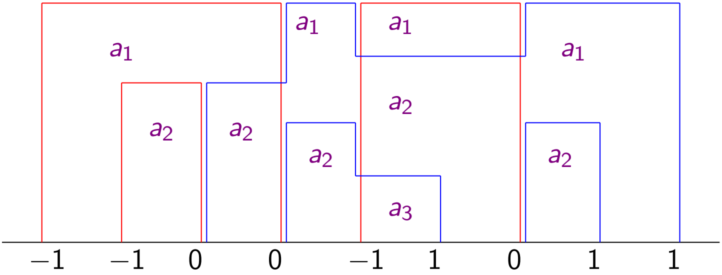

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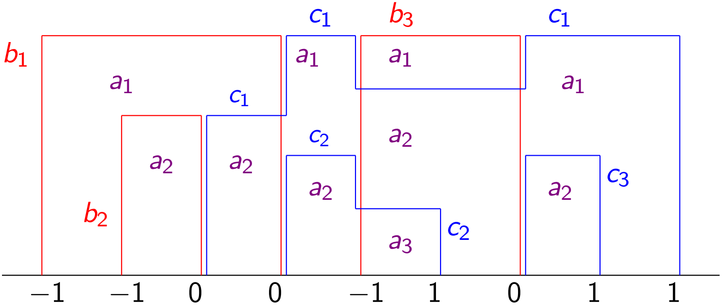

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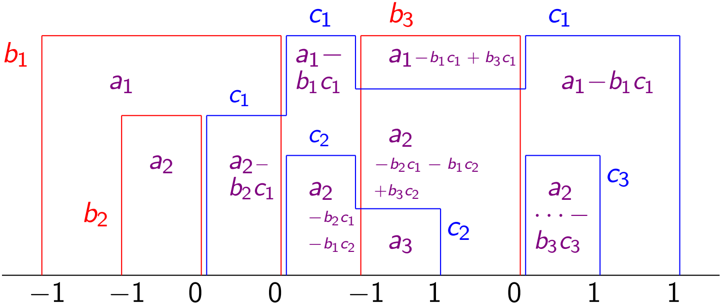
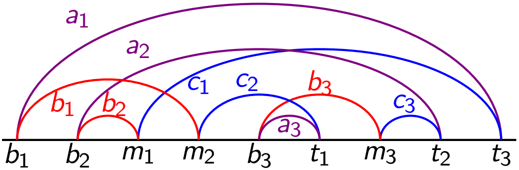


Geometry and combinatorics of Springer fibers



$$\left(\begin{array}{ccccccccc}
a_1 & a_2 & c_1 & c_2 & a_3 & 1 & 0 & 0 & 0 \\
0 & a_1 & 0 & c_1 & a_2+* & 0 & c_3 & 1 & 0 \\
0 & 0 & 0 & 0 & a_1+\dagger & 0 & c_1 & 0 & 1 \\
\hline
b_1 & b_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Geometry and combinatorics of Springer fibers

Theorem (Hafken, Lang, Tashman, T, et al.)

Consider the labeled Springer circuit diagram $\mathcal{M}_T$ obtained from a balanced Yamanouchi word T on $\{-1, 0, 1\}$.

The blue and red entries of the Springer Schubert cells appear according to nesting exactly as in the 2-row case.

The purple entries are the labels in the nested depth bands just to the right of each 1.

Geometry and combinatorics of Springer fibers

Takeaway:

- The basic theorem should extend to all Schubert cells, not just top-dimensional Schubert cells.

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Geometry and combinatorics of Springer fibers

Takeaway:

- The basic theorem should extend to all Schubert cells, not just top-dimensional Schubert cells. But lower-dimensional cells are not always affine.
- Much of the information about closures for 2-row Springer Schubert cells also extends directly, using a generalized notion of nesting/unnesting.
- **The shadow containment partial order from representation theory governs the geometry in a natural way.**

Questions

- Can we classify the singular locus of Springer fiber components?
- What are reduced webs for $k \geq 4$?
- What are the entries of the transition matrices between web and tableaux bases? What do they count?

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Thank you!